MATHEMATICAL PAPERS

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THE COLLECTED

MATHEMATICAL PAPERS

OF

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PREFATORY NOTE.

THE present volume deals very largely with the Author's enumerative method of obtaining the complete system of concomitants of a system of quantics, with the help of generating functions; the brief but very luminous papers here reprinted, at the end of the volume, from the *Johns Hopkins University Circulars* shew the Author preparing his memoir on the Constructive Theory of Partitions, which begins the next, and last, volume of his Mathematical Works. The previous volume included the period of the Author's activity at the Military Academy, Woolwich; this volume nearly covers the time of that surely most interesting experiment in educational method when, at Baltimore, unhindered by traditional routine, and encouraged to give full rein to his invention, he was able, nay obliged, as he tells us (p. 76), to yield to the inquisitive student who would have the New Algebra, that or nothing; with results that are imperishable. The matter is seen so well from the Author's point of view in his Commemoration day Address at Johns Hopkins University (1877), that, after some hesitation, a reprint of this is included in the present volume (No. 10). The Remarks on Research, in *Nature*, vol. XVI. (1877), are from this Address. The present volume also includes the Author's investigations on Chemistry and Algebra (No. 24), the paper on Certain Ternary Cubic-Form Equations (No. 39), and the paper on Subinvariants and Perpetuants (No. 67). In connection with the enumerative methods in this volume the reader's attention may be directed to a paper, by F. Franklin, "On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics," in the *American Journal of Mathematics*, III. (1880), pp. 128—153, to which, as to one or two other memoirs referring to matters dealt with in the text, I have ventured to add a reference at the appropriate place.

H. F. BAKER.

ST JOHN'S COLLEGE, CAMBRIDGE.
24 *November* 1909.

TABLE OF CONTENTS

1.

ON THE ROTATION OF A RIGID BODY.

[Three letters to *Nature*, Vol. I. (1870), pp. 482, 532, 582.]

The Motion of a Free rotating Body.

I SHALL feel obliged if, through the medium of your widely-circulated journal, you will allow me to point out an extraordinary mistake into which M. Radau has fallen, in a memoir inserted in the *Annales Scientifiques de l'École Normale Supérieure*, tom. VI. 1869, in which he criticises certain of my conclusions about the representation of the motion of a free rotating body contained in a paper published by me in the *Philosophical Transactions* for 1866*. In his preamble, M. Radau says, speaking of the theory of rotation in connection with the names of Poinsot, Rueb, Jacobi, and Richelot:—"Tout récemment M. Sylvester a essayé d'appliquer au même sujet des considérations nouvelles qui l'ont conduite à des résultats intéressants, à côté d'autres dont l'exactitude peut être contestée."

Later on in his memoir M. Radau points out, and accompanies with very biting (albeit toothless) criticism, the nature of his objection, which is, in short, that I suppose Poinsot's ellipsoid, under the influence of an original impulse, to roll without slipping by virtue of its friction against the plane with which it is in contact. My answer is, that of course I do. And why not? when I suppose the plane "indefinitely rough" (see p. 761 of *Philosophical Transactions*, 1866†), and have actually determined the friction and pressure at each point of the motion, so that by solving a maximum and minimum problem of one variable, the extreme value of the ratio of one of these forces to the other, or if we please to say so, the limiting angle of friction, or, in other words, the necessary degree of roughness of the plane, may be analytically determined for every given case. M. Radau falls into the school-boy blunder of making the *ratio between the friction and pressure constant throughout the motion*, confounding the actual friction with its limiting maximum value! It is, indeed, surprising that such a perversion

[* Vol. II. of this Reprint, p. 577.] [† *ibid.* p. 582.]

of the facts of the case should have found insertion in a serious journal, such as that published by the École Normale Supérieure, and I might fairly have expected from M. Radau the courtesy habitual with his adopted countrymen, of applying to me for information on anything in my paper which might have appeared to him obscure or erroneous, before rushing into print with such a *mare's nest.*

But out of evil cometh good. M. Radau says:—"Mais M. Sylvester va plus loin; il pense que le problème pourrait se résoudre par l'observation directe du mouvement d'un ellipsoide matériel tournant sur un plan fixe en même temps qu'il tournerait autour de son centre également fixe. On ne se figure pas facilement par quel artifice on fixerait le centre d'un ellipsoide matériel."

In a future number of your esteemed journal (as time at present fails me) I propose to show how, by the simplest contrivance in the world, a downright material top of ellipsoidal form may be actually made to roll, with its centre fixed, on a fixed plane and so exhibit to the eye the surprising spectacle of a motion precisely identical *in time*, as well as in its successive displacements of *position*, with that of a body, turning round a fixed centre, but otherwise absolutely unconstrained.

This mode of representation, which flashed upon my mind almost instantaneously when my eye first lighted upon M. Radau's objections, is the compensating good to the evil of being made the victim (to the temporary disturbance of my beloved tranquillity) of so hasty and futile a criticism as has been allowed insertion in the "Scientific Annals" of so great an institution as the École Normale of Paris.

The *bureau de rédaction* must surely have been nodding when they allowed such observations, so easily refuted by turning to the original memoir, to pass unchallenged. It was only within the last few days that I received M. Radau's paper.

Rotation of a Rigid Body.

My previous communication about the rotating ellipsoid to this journal, has attracted the attention of M. Radau. "One touch of *Nature* makes the whole world kin." In a note addressed to me full of true dignity, this gentleman has made much more than sufficient reparation for his previous trifling act of inadvertence, and states that to his great regret he had misunderstood my meaning, in the passage of my memoir in question, and that "sa critique n'est pas fondée." I, on my part, deeply lament the unnecessary tone of acerbity in which my reference to this criticism was couched, and wish I could recall every ungracious expression which it contains. "When I spoke that, I was ill-tempered too."

I will pass over this, to me, painful topic, to say two or three words on the mode in which the rotating ellipsoid may be supposed to roll or *wobble* on a rough plane, with its centre fixed. My solution may remind the reader of Columbus' mode of supporting an egg on its point—or, rather, of a fairer mode which Columbus might have employed, and which would not have necessitated the breaking of the shell, namely, by resting the blade of a knife or rough plate on the upper end of his egg.

So, to make an ellipsoidal or spheroidal top roll, with its centre fixed—say, upon a rough horizontal plane—imagine a second horizontal plane in contact with the upper portion of its surface; then the line joining the two points of contact will pass through the centre of the top. We may conceive a slight perforation in either or each plane at its initial point of contact with the top, and a screw wire introduced through this, and inserted into a female screw in the body to be set rolling (a mode of spinning which Sir C. Wheatstone recommends as the most elegant in any case, and in this case evidently the most eligible). On withdrawing the wire with a jerk, the top may be set in motion about its centre, in such a direction as to remain in contact with the two planes, and if these be sufficiently rough the motion will eventually be reduced to one of pure rolling between them, the axis (that is, the line joining the two points of contact), continually shifting, but the centre remaining absolutely stationary: for, vertical motion this point cannot have, so long as the top continues to touch both planes, and any slight horizontal motion (if it should chance to take on such at the outset) would be checked and ultimately destroyed by the friction, which would also keep the two points of contact stationary (like the single point of contact of a wheel rolling on a rail), in each successive atom of time. Thus the motion upon the lower plane would in the end be precisely the same as if the upper plane were withdrawn, and the centre of the top kept fixed by some mechanical adjustment. If the spin were not sufficiently vigorous, after a time the rolling top might quit the upper plane, and of course sooner or later by the diminution of the *vis-viva* due to adhesion, resistance of the air, imperfection or deformation of the surfaces, and other disturbing causes, this would take place, but abstracting from these circumstances the principal axes of the spheroidal or ellipsoidal top would move precisely in place and time like the "axes of spontaneous rotation" of any free body of which the top was the "Kinematic Exponent."

I do not pretend to offer an opinion what materials for the planes and rolling body (ground glass and ebony or roughened ebonite have been suggested to me) it would be best to employ, or whether the "wobbling top" could easily be made to exhibit its evolutions. It is enough for a non-effective, unpractical man (as unfortunately I must confess to being) to have shown that there is no intrinsic impossibility in the execution of the conception.

With regard to the friction and pressure: if W be the weight of the body, F and P the friction and pressure in the case of a single plane (the values of which are set out in my memoir, pp. 764—766, *Philosophical Transactions*, 1866*), it may easily be proved that eventually the friction at each point of contact will be $\frac{F}{2}$, the pressure upwards at the lower point $\frac{P+W}{2}$, and downwards at the upper one $\frac{P-W}{2}$, so that if P should become equal to W the top would quit the upper plane and the experiment come to an end. At p. 766 of my memoir the factor $\sqrt{M\Lambda}$ has accidently dropped out of the expression for P which I mention here, in case any one should feel inclined to consult the memoir in consequence of this note. Mr Ferrers has taken up my investigations, and given more compendious expressions than mine for F and P; with the aid of these it would probably be not difficult to determine the maximum value of $\frac{F}{P}$ so as to assign the necessary degree of roughness of the confining planes, and also to ascertain under what circumstances $P-W$ would become zero, but I do not feel sufficient interest in the question, nor have I the courage to undertake these calculations with the complicated forms of P and F contained in my memoir. Mr Ferrers' results are contained in a memoir ordered to be printed in the *Philosophical Transactions*, and will shortly appear.

In my memoir will be found an exact kinematical method of reckoning the time of rotation by Poinsot's ellipsoid when the lower surface is made to roll on one fixed plane at the same time that its upper surface is sharpened off in a particular way (therein described) so as to roll upon a parallel plane which turns round a fixed axis; this upper plane is compelled to turn by the friction, and acts the part of a moveable dial in marking the time of the free body imaginarily associated with the ellipsoid. I have also shown there that the motion of any free body about a fixed centre may be regarded as compounded of a uniform motion of rotation and the motion of a disc, or, if one pleases, a pair of mutually bisecting cross-wires left to turn freely about their centre. But I fear that *Nature*, used to a more succulent diet, has had as much as it can bear upon so dry a topic, and, although having more to say, deem it wiser to bring these remarks to an end.

An after-dinner experiment.

Suppose in the experiment of an ellipsoid or spheroid, referred to in my last letter, rolling between two parallel horizontal planes, we were to scratch on the rolling body the two equal similar and opposite closed curves (the *polhods* so-called), traced upon it by the successive axes of instantaneous rotation; and suppose, further, that we were to cut away the two extreme

[* Vol. II., above, pp. 585, 587.]

segments marked off by those tracings, retaining only the barrel or middle portion, and were then to make this barrel roll under the action of friction upon its bounding curved edges between the two fixed planes as before, or more generally, imagine a body of any form whatever bounded by and rolling under the action of friction upon these two edges between two parallel fixed planes; it is easy to see that, provided the centre of gravity and direction of the principal axis be not displaced, the law of the motion will depend only on the relative values of the principal moments of inertia of the body so rolling, in comparison with the relative values of the axes of the ellipsoid or spheroid to which the *polhods* or rolling edges appertain; and consequently, that, when a certain condition is satisfied between these two sets of ratios, the motion will be similar in all respects to that of a free body about its centre of gravity.

That condition (as shown in my memoir in the *Philosophical Transactions**) is, that the nine-membered determinant formed by the principal moments of inertia of the rolling body, the inverse squares and the inverse fourth powers of the axes of the ellipsoid or spheroid shall be equal to zero—a condition manifestly satisfied in the case of the spheroid, provided that two out of the three principal moments of inertia of the rolling solid are equal to one another.

My friend Mr Froude, the well-known hydraulic engineer, with his wonted sagacity, lately drew my attention to the familiar experiment of making a wine-glass spin round and round on a table or table-cloth upon its base in a circle without slipping, believing that this phenomenon must have some connection with the motion referred to in my preceding letter to *Nature*: an intuitive anticipation perfectly well-founded on fact; for we need only to prevent the initial tendency of the centre of gravity to rise by pressing with a second fixed plane (say a rough plate or book-cover) on the top of the wine-glass, and we shall have an excellent representation of the free motion about their centre of gravity of that class of solids which have, so to say, a natural momental axis, that is (in the language of the schools) two of their principal moments of inertia equal. For greater brevity let me call solids of this class uniaxal solids. I suppose that the centre of gravity of the glass is midway between the top and bottom, and that the periphery of the base and of the rims are circles of equal radius. These circles will then correspond to *polhods* of a spheroid, conditioned by the angular magnitude and dip of the spinning glass; to determine from which two elements the ratio of the axes of the originally supposed but now superseded representative spheroid is a simple problem in conic sections; this being ascertained, the proportional values of the moments of inertia of the represented solid may be immediately inferred. The wine-glass

[* Vol. II., above, p. 583.]

itself belonging to the class of uniaxal bodies, the condition that ought to connect its moments of inertia with the axis of the representative spheroid (in order that the motion may proceed *pari passu* with that of a free body) is necessarily satisfied.

The conclusion which I draw from what precedes is briefly this—that a wine-glass equally wide at top and bottom, and with its centre of gravity midway down, spinning round upon its base and rim in an inclined position between two rough but level fixed horizontal surfaces, yields, so long as its *vis-viva* remains sensibly unaffected by disturbing causes, a perfect representation, both in space and time, of the motion of a free uniaxal solid, as for example, a prolate or oblate spheroid, or a square or equilateral prism or pyramid about its centre of gravity, and conversely that every possible free motion about its centre of gravity of every such solid admits of being so represented.

To revert for an instant to the general question of the representative rolling ellipsoid, I think it must be admitted that the addition of the time element to the theory and the substitution of a second fixed plane in lieu of a fixed centre, considerably enhance the value and give an unexpected roundness and completeness to Poinsot's image of the free motion of rotation of a rigid body, of which so much and not altogether undeservedly has been made. From an idea or shadow Poinsot's representation has now become a corporeal fact and reality, as if, so to say, Ixion's cloud, in a moment of fruition, had substantified into a living Juno. I heard the late Professor Donkin, of revered and ever-to-be-cherished memory, state that when as a referee of the Royal Society he first took in hand my paper on rotation, he did so with a conviction that all had already been said that could be said on the subject, and that it was a closed question; but that when he laid down the memoir he saw reason to change his opinion. I owe my thanks to M. Radau and the editors of the Annals of the École Normale Supérieure for having been at the pains to disentomb the little-known conclusions therein contained from their honourable place of sepulture in the *Philosophical Transactions.*

2.

ON RECENT DISCOVERIES IN MECHANICAL CONVERSION OF MOTION.

[*Proceedings of the Royal Institution of Great Britain*, VII. (1873—75), pp. 179—198. Also *La Revue Scientifique*, 1874—75, pp. 490—498, and *Van Nostrand's Engineering Magazine* (New York) XII. (1875), pp. 313—321.]

THE speaker stated that the subject he proposed to bring under the notice of the meeting related mainly to the discovery of a perfect parallel motion,—that is to say, of a mode of producing motion in a straight line by a system of pure link-work without the aid of grooves or wheel-work, or any other means of constraint than that due to fixed centres, and joints for attaching or connecting rigid bars. This important discovery was made by M. Peaucellier, an officer of Engineers in the French army*—and first published by him, in the form of a question, in the *Annales de Mathématique* in the year 1864, and subsequently formed the subject of two communications to the *Société Philomathique* of Paris by Captain Manheim, but seems not to have received the attention it deserved from that learned body, and may be said to have passed into oblivion; so much so, that when rediscovered by a young student of the University of St Petersburg, of the name of Lipkin, several years subsequently, the discovery was attributed to Lipkin instead of to Peaucellier even in works published in the French language, and so recently as 1873 by M. Colignan, in his *Traité de Cinématique*. The eminent Professor Tchebicheff had long occupied himself with the question, but with less than his usual success in overcoming difficulties insuperable to the rest of the world. Lipkin was a student in his class, and may thus have had his attention turned to the question; at all events, Professor Tchebicheff's warm interest in the subject was displayed by his bringing Lipkin's name before the Russian Government, and securing for him a substantial reward for his

* Now Colonel Peaucellier, and in command of the fortress of Toul; at the time of his discovery lieutenant and officier d'ordonnance on the staff of the "illustrious Marshal Niel."

supposed original discovery. Before Peaucellier's time all so-called parallel motions were imperfect, and gave merely approximate rectilinear motion*; in substance they will be without exception found to be merely modifications of Watt's original construction, and to depend on the motion of a point in, or rigidly connected with, a bar joining the extremities of two other bars rotating round fixed centres, which may be described briefly as three-bar motion. Peaucellier's exact parallel motion depends on a link-work of seven

* The late lamented Professor Rankine, in his treatise on Millwork, and elsewhere, mentions a so-called "exact parallel motion," the invention of which he dubiously assigns to Mr Scott Russell. In its *exact* form this is no parallel motion at all, for it works by means of a slide, and in its modified form it ceases to be *exact*, the motion produced being no longer truly rectilinear.

Mr Kaulbach, a mechanical draughtsman, resident in London, has shown the speaker a sketch of a very ingenious *quasi*-parallel motion, which he took the first steps to patent a year or two ago, but has not thought it worth his while to proceed with further. Its principle depends upon finding a curve made to rotate about a fixed point, and enjoying the property that the tangent to each point of it, as that point passes a given vertical line, shall take up a horizontal position. A piston-rod is guided in the direction of such vertical line, and the beam, which always presses on a friction wheel attached to the rod, is so shaped in its outward contour as to satisfy the above condition; the consequence is that the reaction on the piston-rod can only take effect vertically, that is, in the direction of its motion, and no lateral pressure is produced.

Peaucellier's invention effects the perfect conversion of circular into linear motion. An easy practical deduction from this is the conversion of spherical into plane motion, by aid of universal joints and other familiar modes of effecting free motion in space, of a shaft about a fixed point or round another shaft. The announcement of these facts has occasioned many persons unacquainted with the technical language of mechanism to suppose that the discovery of Peaucellier is connected with the quadrature of the circle or cubature of the sphere, and led to the idea that the speaker was in possession of some secret for flattening spheres and turning circles into right lines. Such a misconception was one (as indeed the wide extent of its prevalence demonstrates) quite likely to occur even to intelligent persons untrained in mathematical science. Technical names are a frequent occasion of traps to the uninitiated. A lady present at one of Mr Norman Lockyer's course of lectures on Spectral Analysis, near the close of it was overheard inquiring with some anxiety as to "when the spectres might be expected to make their appearance." Names are of course all-important to the progress of thought, and the invention of a really good name, of which the want, not previously perceived, is recognized, when supplied, as having ought to be felt, is entitled to rank on a level in importance with the discovery of a new scientific theory. Imagine *plane*, *straight*, *circle*, and you are potentially a geometer. Think the meaning of the one word *Syzygy*, and the logic of algebra has become part of your being. But, on the other hand, there are cases where over-naming does harm. The speaker has no doubt that if reading music on the piano with the fingers were taught without the intervention of learning the names of the notes, twice the velocity of execution (and quick reading is here the *sine-quâ-non* for the existence of every other kind of excellence) might be acquired in half the time required under the present system. The names of the notes of course would have to be learned at a later stage as a medium for discourse; but they should not be used as a vehicle for obtaining command of digitation, as such use amounts to throwing upon the brain the labour of going through two steps when one would suffice, and the passage of a direct nervous current from the eye to the touch in the act of reading, even at an advanced stage, becomes by force of habit interrupted and diverted into a broken channel. The new method for learning to read on the pianoforte here suggested may be distinguished as the abnominal or undenominational or tactile method. The writer is prepared to show in detail how it can be carried out in practice.

bars moving like Watt's, and the other imperfect parallel motions of the same class, round two fixed centres*.

To understand the principle of Peaucellier's link-work, it is convenient to consider previously certain properties of a linkage† (to coin a new and useful

* The perfect parallel motion of Peaucellier looks so simple and moves so easily that people who see it at work almost universally express astonishment that it waited so long to be discovered. The idea of the facility of the result by a natural mental illusion gets transferred to the process of conception, as if a healthy babe were to be accepted as proof of an easy act of parturition. No impression can be more erroneous. The speaker, on the contrary, the more he reflects upon the problem that was to be solved, and the nature of the solution (essentially a process of transformation operating on polar co-ordinates), wonders the more that it was ever found out, and can see no reason why it should have been discovered for a hundred years to come. Viewed *à priori* there was nothing to lead up to it. It bears not the remotest analogy (except in the fact of a double centring) to Watt's parallel motion or any of its progeny. In the three-bar motion the two fixed points are so to say one as good as the other, there is no distinction to be drawn between them; whereas the two fixed centres (hereafter designated as the fulcrum and pivot) in Peaucellier's seven-bar arrangement are absolutely dissimilar in position and function. Peaucellier's apparatus naturally resolves itself into a cell and a spare link; no such decomposition presents itself in the three-bar motion. Again, looking at the matter *à posteriori*, it occurs to many well-grounded mathematicians to suppose that, as the most general motion of a link-work of seven or any number of bars for each possible mode of conjunction and centring must be capable of being expressed by a general algebraical equation, the particular combination for rectilinear motion, when such motion is possible, ought to be contained therein and inferrible therefrom by studying under what conditions the characteristic of the general equation can degenerate into a power of a linear function or, as might perhaps happen (and would be sufficient if it did), into such power multiplied by a function incapable of changing its sign. But the answer to this is that *practically* there could be little or no hope of ever obtaining the general equation. In one-bar motion the general curve (that is, a circle) is of the 2nd order; in three-bar motion, as is well known, of the 6th order; very likely, therefore, in five-bar motion it would be of the 24th order at least; and in seven-bar motion, of the 120th order at least. The equation or system of equations of the 120th order, supposed to be applicable to seven-bar motion, one could hardly dream of obtaining, or of being able to manipulate if obtained. Written out at full length in a handwriting of moderate size, the area of a very large room might be insufficient to contain the whole of its terms, which would consist of 7381 groups, and might be tens or hundreds of thousands in number. No; it must either have been fallen upon in a chance or experimental way, and subsequently verified theoretically, or else hit off in some sudden glow of insight akin to but of a much intenser degree of illumination than that under which Professor Stokes was able to see that the hydrodynamical theorem of Lagrange before him, proved imperfectly by its author and others, and correctly but with great difficulty by Cauchy, was an immediate inference from the pretty nearly self-obvious fact of the complete time-derivatives of the three quantities to be proved *if ever then always zero*, being by virtue of the well-known general hydrodynamical equations, syzygetic functions of these quantities themselves. Dr Tchebicheff has informed the writer that he has succeeded in proving the non-existence of a five-bar link-work capable of producing a perfect parallel motion; he is probably therefore in possession of the actual numerical order of the general equation or system of equations applicable to this case. It is not proved, and may not be true, that Peaucellier's is the only seven-bar link-work that will solve the problem of a perfect parallel motion. Who shall say whether there may not exist some other combination of seven bars in which the same or an analogous zig-zag symmetry to that which exists in the three-bar arrangement may reappear! This is a point which should not be allowed to remain subject to doubt.

† A link-work consists of an odd number of bars, a linkage of an even number. A linkage may be converted into a link-work *additively* by fixing one point of it as a fulcrum and attaching

word of general application), consisting of an arrangement of six links, obtained in the following manner:—first conceive a rhomb or diamond formed by four equal links joined to one another; and now suppose a pair of equal links to be joined on to two opposite angles of such figure and to each other. All six links are supposed to lie (and to be constrained by the nature of their attachments to remain) in the same plane. The point of junction of the last-named pair of links (which it will be found convenient to call the fulcrum), according as they are greater or smaller than the sides of the diamond, will lie outside or inside the diamond. The linkage consisting of the six links may be termed a positive *cell* in the one case and a negative *cell* in the other*. It is easily seen, as a geometrical necessity, that the fulcrum,

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a second point disconnected from the first by a new link to another fulcrum, or *ablatively* by fixing two ends of a link, which may then be removed. When one point only of a linkage is fixed, any other point may be made to describe an arbitrary curve, but then the path of every other point becomes prescribed. In order for a combination of links to fulfil this so to say fatalistic condition, and to entitle it to the name of a linkage in the speaker's sense, which when greater precision is required may be distinguished as a *perfect* linkage, equivalent to the French *système de tiges à liaison complète*, a numerical relation must be satisfied between the number of links and the number of joints, namely, three times the number of links must be four greater than twice the number of joints. In applying this rule it must be understood that, if three links are jointed together, the junction counts for two joints; if four are jointed together, for three joints; and so on. A compass or a pair of scissors is the simplest kind of linkage; a set of lazy-tongs is another; a Peaucellier cell, subsequently described in the text, a third. If no three joints lie on the same link, the above numerical relation between joints and links may be stated in another form, namely, twice the number of joints is four greater than the number of links. But in applying the rule in this form all joints count alike as units, and for a simple compass the ends must be reckoned as joints.

* Mr Penrose, the eminent architect and surveyor to St Paul's Cathedral, the scientific expositor and elucidator in succession to Mr Pennethorne of the surprising law of curvilinearity in the temples of the Greeks, has put up a house-pump worked by a negative Peaucellier cell, to the great wonderment of the plumber employed, who could hardly believe his senses when he saw the sling attached to the piston-rod moving in a true vertical line, instead of wobbling as usual from side to side. There seems to be no reason why the perfect parallel motion should not be employed with equal advantage in the construction of ordinary water-closets. The author has been admitted to see the geometrical pump at work in Mr Penrose's kitchen at Wimbledon. A sister pump of the ordinary construction stands beside it. The former, although quite as compact as its neighbour, throws up a considerably larger head of water with the same sweep of the handle. Its elegance, and the frictionless ease with which it can be worked (beauty as usual the stamp and seal of perfection) have made it the pet of the household. Some circular steps outside St Paul's Cathedral very lately requiring repair, Mr Penrose employed a circulo-circularly-adjusted Peaucellier cell to cut out templets in zinc for the purpose. The radius of the steps is about 40 feet, but to the great comfort and delectation of his clerk of the works, they were able to operate with a radius of not more than 6 or 7 feet in length. General Sir H. James, R.E., lately gave a lecture on the subject at Southampton, and informs the writer that this has been the means of inducing a gentleman of fortune residing there, well known in the yachting world, to fit up a marine engine with a Peaucellier parallel motion to use on board a steam yacht.

A very good idea of the form and operation of a negative cell may be gained by putting together the fore-fingers and ring-fingers of the two hands, and placing one middle finger a little over the other so as to keep all six fingers in the same plane. The first Peaucellier cell constructed in this country was a positive one, made by the speaker's friend, the eminent musician

in whatever way the linkage is moved about, will always lie in a straight line with the two free angles of the diamond, which may be called its poles, and the distances of these poles from the fulcrum, or the ideal lines which represent those distances, may be called the arms of the cell. It is upon the geometrical relation between these arms that the remarkable mechanical properties of Peaucellier's cell depend. The cell may be made to change its

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and inventor of the laryngoscope, Mr Manuel Garcia, Ph.D., who happened to visit him shortly after his memorable interview with Dr Tchebicheff, in which that great mathematician announced in answer to his inquiries after the progress of the disproof of the impossibility of the exact conversion of circular into rectilinear motion, which had so long occupied the attention of his illustrious guest, that it, the thing itself, not the proof of its impossibility, had been actually effected in France, and subsequently in Russia, by a freshman student in his own class. He showed Mr Garcia the drawing of the cell and mounting left by Tchebicheff, and the next day was gratified by receiving from him a model constructed with a few pieces of wood, fastened together with nails as pivots, which, rough as it was, worked perfectly, and drew forth the most lively expressions of admiration from some of the most distinguished members of the Philosophical Club of the Royal Society (not mathematicians, but naturalists, geologists, chemists, and physicists), when it was brought in with the dessert, to be seen by them after dinner, as is the laudable custom among the members of that eminent body in making known to each other the latest scientific novelties. Presently after the speaker exhibited the same model in the hall of the Athenæum Club to his brilliant friend Sir William Thomson, of Glasgow, who nursed it as if it had been his own child, and when a motion was made to relieve him of it, replied, "No! I have not had nearly enough of it—it is the most beautiful thing I have ever seen in my life." This rude but invaluable model ought to be preserved in some physical laboratory as a historical relic. It served as an instrument by which the speaker in every case where it was seen gained immediate converts to the belief of the importance of Peaucellier's great discovery, whereas a mere geometrical diagram would have been as little regarded as a figure of the celebrated asses' bridge in Euclid at last, so great is the difference of the impression produced on the practical English mind by the *esse* and the *posse*—being told how a thing ought to act, and seeing it actually going. Considering the extraordinary conversions worked with Mr Garcia's model, it would not be unsuitable to write in letters of gold on the board attached to it which gives support to the two frail centres, the famous motto of Constantine—"In hoc signo vinces."

Apropos of the mistaken impressions of great men. Did not Newton live and die in the belief of the incurability of chromatic dispersion; Cayley affirm the infinitude of the number of the asyzygetic invariants of binary quantics beyond the sixth order, thereby arresting for many years the progress of the triumphal car which he had played a principal part in setting in motion; Pontecoulant the possibility of the existence of a rotating fluid ellipsoid of equilibrium for other than forms of revolution?

And as regards the speaker himself, twenty years ago he emitted* in the *Philosophical Magazine* a conjectural criterion for distinguishing *à priori*, geometrical propositions capable only of indirect demonstration from those susceptible of direct, when, lo and behold! but a few days ago came over a seemingly incontrovertible refutation of his supposed law, addressed to the Vice-Chancellor of our University of Cambridge (as a sort of Patriarch of the West, and recognized Official Defender of the Faith (as it is in Euclid) for the British isles), by Miss Chart, of Oakland, California, U.S., which it is to be hoped will speedily appear in the same journal where the erroneous hypothetical dogma first saw the light. His sin, after so long a delay, and travelling half round the world in the interim, has found him out. It ought to be added that Miss Chart does not claim for herself the merit of the refutation, but represents herself as having received it some years ago from a gentleman bearing the, to geometrical ears, auspicious-sounding name of Hesse.

[* Vol. I. of this Reprint, p. 395.]

form like a set of lazy-tongs or any other kind of linkage, by closing or opening the diamond: as this is done evidently the lengths of the arms alter; but it will be found, and is capable of easy geometrical proof, that they remain subject to a very simple condition, namely, one increases just as much as the other decreases, so that their product remains invariable; this product is equal to the difference between the square of either of the links (called the connectors) proceeding to the fulcrum and the square of any side of the diamond, to which we may give the name of the modulus of the cell. The speaker illustrated this property experimentally, using a negative cell for the purpose. When the fulcrum was midway between the two poles each arm was 12 inches in length. When one arm was made 18 inches the other was found to be 8; when again it was stretched to the length of 24 inches the other was 6, and so on, the product of the two remaining always 144; or, reckoning in feet, to the lengths 1, $\frac{3}{2}$, 2, 3 of one arm corresponded the lengths 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$ of the other; showing that the length of one arm was so governed by the length of the other as that the numbers denoting the two were always inverse or reciprocal to each other when the modulus was taken as unity. Hence a Peaucellier's cell may be conveniently termed a Reciprocator or Inverter. If we were to suppose the connectors at their free ends, instead of being attached to the side angles of the diamond, to be joined on to two adjoining sides in such a manner as to become parallel to the other pair of sides, this parallelism would continue to subsist for all positions of the linkage, and the arms or distances of the fulcrum from the opposite angles or poles of the diamond would still remain in the same right line, but the relation between them would now be one of direct instead of inverse proportion. Conceive the fulcrum in such an arrangement to become fixed. Since we can not only alter the angles of the diamond, but make the whole arrangement turn round the fixed point, we can make either pole describe any plane curve whatever: the other pole will then describe a curve precisely similar in shape, but drawn on a different scale, as in any ordinary pantigraph*.

But if we revert to the Peaucellier cell or Reciprocator, whether of the positive or negative form, and treat it in the same manner as the supposed pantigraphic arrangement, fixing the fulcrum, and making one of the poles—that is, an extremity of one of the arms—describe any plane curve, the other pole will no longer describe a similar curve, but what in the language of geometry

* Sometime, according to the authority of a questionist in the *Educational Times* for the current month, called a Pentegram. Theoretically two Peaucellier cells are equivalent to a Pentegram (for we may change r into $\frac{K}{r}$ by one, and that into Kr by the other), but whilst combinations of the former are adequate to the transformation of r into any algebraic function of r, the latter are absolutely sterile, leading only to the one single sort of transformation (if it may be called so), r into Kr. It seems then going too far to say (as does the writer alluded to above) that the germ of Peaucellier's invention is contained in the Pentegram.

is termed an inverse of the curve in question, the fulcrum being the origin of the inversion.

Suppose now one of the poles is made to describe a circle, the other will describe the inverse of a circle, which geometricians are well aware will in general be another circle, subject to the exception that if the arc described by one pole is part of a circle passing through the fulcrum, which is here the origin of the inversion, the path of the second pole will be no longer a circle, but a perfect straight line, which, under a mathematical point of view, may be regarded as a circle with an infinite radius. If then, in addition to fixing the fulcrum, we still further constrain the motion of the Peaucellier cell by attaching one of the poles to a centre (which for the sake of distinction from the other fixed point above defined we may term the *pivot*) round which it can revolve, situated at an equal distance from that pole and the fulcrum, the other pole will describe a perfect straight line perpendicular to the line joining the fulcrum and the pivot. We have thus a combination of seven radiating bars attached to two fixed centres, one point of which describes a true rectilinear path, and thus the long-sought-for problem of a perfect parallel motion meets for the first time its complete solution*.

* The centre above spoken of may be taken in the line itself, which joins the poles and the fulcrum. If it be taken not too far out of this position of symmetry it will in the course of the motion be brought into such position; but if it be taken at starting (as it may be), at a sufficiently great distance from the cell, the position of symmetry may never be attained throughout the whole possible course of the motion. This circumstance has been generally overlooked, and accordingly too narrow a rule has been given for the construction of a Peaucellier parallel motion, namely, it is laid down that the pivot is to be taken midway between the fulcrum and one of the poles for some certain position of the instrument. The position of the fulcrum relative to the two poles gives rise to the distinction between a negative and positive cell; but the preceding remark shows that there is a further subdivision of Peaucellier parallel motions depending on the length of the mounting radius, and that positive and negative mounted cells each of them embrace two radically different forms or genera, which may be distinguished as the symmetrical and non-symmetrical respectively; in the one form there exists a position where the *first* lies in the line containing the *fulcrum* and the two poles, in the other, no such position can be found. In the ordinary rule given for the construction of a P.P.M. only the former of these two genera is included which, as machines, differ between themselves as much as do the ellipse and hyperbola as curves.

It ought to be added that the motion of the *parallel-point* is always perpendicular to the line of centres, and in every position makes, with the line containing the fulcrum and the poles, an angle equal to the angle contained in the segment of the circle (of which one pole describes an arc), which lies between it and the fulcrum. If we join the two fixed centres by a new link, and then unfix them, we obtain a linkage of eight bars, possessed of very remarkable properties, of one of which Peaucellier has availed himself to obtain a mechanical description of the Limaçon of Pascal, which is the inverse of a conic in respect to the focus as the origin of inversion.

By a combination of such linkages it is possible to cause any number of points, otherwise free, to remain always in a straight line with each other. The speaker believes that he is in possession of a *bonâ fide* valid proof of the proposition assumed on totally insufficient grounds by Peaucellier, namely, that every algebraical curve may be best described by link-work. The proof is founded on the union of the above statement (or still better, one founded on his own Kine-

The speaker illustrated these results by various models constructed in wood. By changing the length of the radial bar connecting one pole of the cell with a fixed point, the free pole was shown to describe arcs of circles convex or concave to the fulcrum, according as the ideal circle, in an arc of which the first-named pole moved, fell short of the fulcrum or contained that point within it; in the limiting case, when it passed through the fulcrum, the path was shown to be neither convex nor concave, but a straight line free from all curvature in either direction. This was further verified mechanically by connecting together at their free poles two perfectly equal and similarly mounted cells. If the tendency of either of these was to deviate from the straight path, the tendency of the other would be to deviate in the contrary direction; so that either the pair of mounted cells would become an

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matical Paradox subsequently referred to) with Grassmann's method of describing algebraical curves by means of an apparatus of fixed points and lines; this proposition, as far as concerns curves of the first nine genera (that is, of a *cursality*, or, so to say, circuit-*complexity* not transcending the 9th degree), and also for curves of the first six orders, or for any order where the degree of one of the variables in the representing equation is 5 or less, he had already demonstrated by a direct method. In using this method he found it necessary to prove that a general algebraical equation of the fifth degree could always be reduced to a trinomial form by *real* transformations, which, by Tschirnhausen's (the only method hitherto applied), as often as not, is incapable of being done. By an extension of the principle of Tschirnhausen's method he succeeded in establishing this important algebraical proposition. A very much more important conclusion relating to the representation of every algebraical function (that is, the function that one quantity is of another connected with it by any algebraical equation), under a quasi-explicit form, he believes he can show may be deduced from the transformed Grassmannian construction above alluded to: by quasi-explicit, meaning a form capable of being obtained by the elementary processes of addition, multiplication, change of sign, and reciprocation with that of general form-inversion superadded. Thus Peaucellier's discovery seems likely to throw open a new chapter in the highest summits of Analysis, no less important in the theoretical direction than its numerous applications to the mechanical arts in the direction of practice.

In the lineo-circular or parallel-motion adjustment imagine the connectors to be detached from the angles of the diamond, and joined on to the two sides of the diamond, which meet at the "parallel point," at equal distances from it. Then the motion of that point will no longer be in a straight line, but in a circle.

This method of producing one circular motion from another (which was first given by the speaker in the *Educational Times*) may probably be found to possess important practical advantages over the circulo-circular adjustment of the Peaucellier cell described in the text above.

The speaker exhibited another modification of the Peaucellier cell; like it consisting of six links, but having the property that the sum of the squares of the two arms (instead of their product) remains constant. This he calls a quadratic-binomial extractor.

By means of this cell, mounted with a suitable radius, a perfect lemniscate may be described; and what is very interesting, and flows from this construction (but was first observed by Dr Henrici), the same curve may be described by means of a binomial-extractor, of a certain kind, reduced to a link-work by the *ablative* method of fixing one of the links: in other words, a perfect lemniscate may be described throughout its complete extent by means of 5-bar motion. Peaucellier refers to, without specifying, a combination, "assez compliquée," of cells (or, as he terms them, compound compasses) by means of which a lemniscate may be traced; whereas, in the method above described the number of links employed is less by a pair than in the single mounted Peaucellier cell.

absolute fixture, or the two would crush or tear each other to pieces; but in the experiment exhibited the pair of mounted cells were seen to move together (as if in happy wedlock), without let or hindrance to each other's motion. The circular motion of the free pole of a single mounted cell in the general case was also verified experimentally, and even more simply than in the rectilinear case, by the addition of a second radial bar, taken of a suitable length, determined by previous mathematical calculation. As a general rule, the total number of bars in a link-work machine must be odd, but here there were eight bars, and yet the combination admitted of being set in free motion,—any one of the eight being, in fact, what may be termed a lazy-bar, and capable of being removed without disturbing the motion, very much in the same way as any one of the four legs of a table may be removed without disturbing the equilibrium*.

The speaker pointed out the important applications of the two kinds of motion above referred to (which he proposed to call the circulo-linear and the circulo-circular respectively) to various constructions in machinery, such as the steam-engine, planing and grinding machines, the construction of maps on the stereographic projection, millwrights' work, laying out of railway curves, dioptric apparatus for lighthouses, ornamental tracery, pendulum suspension to effect motion in a practically exact cycloidal arc, &c., &c., and referred to the use which, as he was informed by the authorities at Woolwich, might have been made of the circulo-circular adjustment in saving several weeks' work, inconvenience, and expense in cutting out the fish-bellied torpedo casings recently constructed in the laboratory department at the Royal Arsenal there, and the use contemplated to be made of the circulo-linear, or perfect parallel motion, for guiding a piston-rod in certain machinery connected with some new apparatus for the ventilation and filtration of the air of the Houses of Parliament, now under course of construction.

He next referred to the unlimited command over the motion of a point furnished by a combination of cells. Returning to the simple Peaucellier

* Suppose four circles to be given, and that it is proposed to inscribe upon them a quadrilateral whose four sides are given in length.

This is a determinate problem which will in general admit of a definite number of solutions. (The method of correspondence and of bipartite equations founded thereon seeming to indicate thirty-two as the total number of such solutions, some or all of which may be imaginary.) But now the question may be put, "Under what circumstances can the number of such solutions become infinite and the problem undeterminate?" It follows from what is stated in the text above that this may happen (other conditions being satisfied) when two of the circles coincide and the four given lengths are all equal. It remains to be ascertained whether with any new set of conditions a like undeterminateness can be brought about for the case of four circles all distinct. If so, a solution would be obtained of the problem of converting by link-work circular into circular and conceivably (as an extreme case) into linear motion by an arrangement radically distinct from Peaucellier's, and involving the use of three instead of two fixed centres, but with the same number of links.

cell, its use may be modified in a very remarkable manner by setting free the point of junction of the two connectors (termed, in what precedes, the fulcrum), and fixing one of the poles as a centre of rotation in its place. If now the liberated fulcrum be made to describe any curve, the free pole will describe a curve corresponding to it, according to a certain easily-statable mathematical law. Imagine the first-named curve to be part of a circle passing through the fixed point—it may be shown that in that case the free pole will describe the inverse of a conic section in respect to a vertex of the conic as the origin of the inversion; consequently, by combining with this cell a second, used as a Reciprocator, we may, mounting with a suitable radius a pair of Peaucellier cells duly adjusted, cause a point to move in a parabola, ellipse, or hyperbola.

The speaker exhibited a combination of this kind, and caused a point to describe portions of an ellipse, a parabola, and of the two branches of a hyperbola in succession; the traversing pole of the first cell, which might be termed the first follower, being seen to describe beautiful nodal cubics (or the inverses of the conics), whilst the free pole of the second cell or second follower described the conics themselves*.

* The nodal cubics or conic-inverses above described are for the parabola, the common cissoid, and for the ellipse and hyperbola curves which may be termed trans-cissoid, and cis-cissoid, or less barbarously and more euphoniously the hyper-cissoid and hypo-cissoid respectively. The common cissoid, as is well known, has a cusp which here coincides with the fulcrum. In the hyper-cissoid this becomes a detached, or, as it is ordinarily termed, a conjugate point, and in the hypo-cissoid a node on the curve, which in this case possesses a loop in addition to an infinite branch. When the first follower moves in this infinite branch, the second follower describes a portion of that branch of the hyperbola in which the fulcrum lies—but of course can never reach the vertex, which coincides with the fulcrum; when the first follower moves in the loop the second follower describes the opposite branch of the hyperbola, and can be made to pass through the vertex of that branch.

The geometrical construction for the common cissoid, or cissoid proper, is well known to be as follows. Imagine a pencil of rays proceeding from one extremity of a diameter of a circle, and meeting a tangent to the circle drawn at the other extremity. Then if the portion of each ray intercepted between the circle and tangent be shifted along the ray until one point of it coincides with the centre of the pencil, the other point will mark out the cissoid. Now imagine everything to remain as above, with the exception that the tangent is moved parallel to itself and becomes fixed in a new position nearer to or further from the centre of the pencil than it was at first, then the curve marked out becomes the hypo-cissoid or hyper-cissoid respectively, a remark due to Mr Howard Elphinstone. The smoothness of the motion, and the facility with which the cissoidal curves and the corresponding curves were drawn was matter of general surprise and admiration to the audience. This circumstance, due in part to the skill of Dr Henrici in choosing the proportions of the parts, ably seconded by the mechanical experience and ingenuity of Mr Grant, modeller to University College, at the same time served to evince the extraordinary superiority of pure link-work motion, that is, motion due exclusively to the action of radiating bars about centres, over motion effected in whole or in part through the intervention of grooves and slides. It was the analogous superiority enjoyed by circular over linear construction for the purpose of graduating instruments of precision that actuated Mascheroni (the favourite geometer of the first Napoleon) in devising his admirable, most valuable, and most tedious exposition of the geometry of the compass. The superiority in question was still more

He next went on to state that by a combination of cells properly proportioned and suitably attached to each other in succession in a manner similar or analogous to that in which simple machines, as for example a number of levers, may be combined to produce a complex one, we are able to bring about any mathematical relation that may be desired between the

strongly evinced in the triple-cell combinations employed in the instrument for the extraction of cube-roots and the trisection of an angle.

Clairaut has given a method of constructing an instrument for extracting the roots of an equation by means of linear measurements described in Borgni's *Traité de Mécanique Appliquée* (volume on *Machines Imitatives*, p. 226); but the author's method, founded on Peaucellier's discovery, is beyond all comparison superior in the range of algebraical operations which come within its scope, in the simplicity, homogeneity, and smaller number of its parts, in the facility of its application, and the smoothness of the resulting motion. His instrument for solving cubic equations is far less complicated than that of Clairaut for quadratics (which he does not suppose has ever been realized) and infinitely easier of application. For instance, in working his cube-root machine, one point of the instrument is fixed as the zero-point; a second point, called the setter, is drawn out to a division on a scale corresponding to any proposed number; a third point, called the finder, will then automatically place itself over the division on the same scale, corresponding to the cube root of that number. The zero, setter and finder points in the calculating linkage are identical (or, as in the transformation scene of a pantomime, may be said to change characters) with the fulcrum, power and weight (or driver and follower), points in the corresponding link-work used as a machine. In Clairaut's and other similar machines the calculations are made by means of measurements made upon curves described by the machines. The author's method is direct and does not involve the use of any such intermediary process.

Returning to the subject, which has led to this digression, it will be noticed that by the method referred to in the text a mounted double reciprocating cell, that is, an apparatus of thirteen links, serves to describe a conic. Peaucellier's method, founded on the combination of what may be termed a collineator or radial protractor, with a mounted reciprocator, involves the use of fifteen links, besides a cross-piece rigidly attached to one of them, and, so far, is less simple, as well as less symmetrical than the author's method: but this must not be supposed to be said in derogation from the merit of the admirable invention of the collineator itself, by which Peaucellier has solved the beautiful and most important kinematical problem of devising a perfect linkage, enjoying the property, that however it is turned about, or drawn in and out, one point of it shall always remain upon or in the direction produced of a physical line rigidly attached to the linkage, but in different positions upon such line. It is believed that a conic-describing instrument (may one say conicograph?) on Peaucellier's plan has not been actually executed, and that a pure link-work for effecting conical motion was witnessed for the first time since the creation of the world in the lecture-room of the Royal Institution on the 23rd of January, 1874. Although it may be presumed that the Peaucellier conicograph would not work so simply as the one exhibited, it possesses a superiority in one respect, namely, that the fulcrum on this arrangement lying off the curve at the focus, the part of the curve described may be made to include the vertex of the parabola, which cannot be reached by the other method. It has been thought by competent judges, conversant with practical mechanism, that this (the writer's) method might be applied with advantage to constructing parabolic light-house reflectors; and as these, from the nature of the case, are made *without backs*, consisting of two paraboloids of revolution, situated *dos-à-dos*, having a common focus, at which the source of light is placed, from which the rays stream through the opening upon the surfaces of the two reflectors, the fact of the tracing or cutting or grinding instrument not being able to reach the vertex, would be no disadvantage in this case, since the portion of the surface in the neighbourhood of that point is not required, and, indeed, if formed would have to be subsequently cut away. But it should be added that by a generalized single-mounted cell an approximation to the parabolic form can be attained to a degree of precision far in excess of all practical needs.

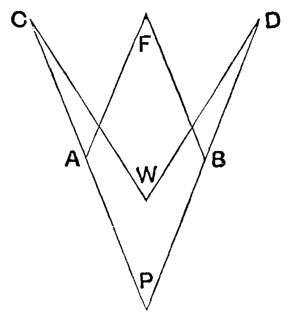
distances of two of the poles of a linkage from a third, and are thus potentially in possession of a universal calculating machine. He exhibited and worked a cube-root extracting machine constructed on this principle, and claimed to have given the first really practical solution of the famous problem proposed by the ancients of the duplication or multiplication of the cube. This machine consisted of a combination of three cells; by changing the modulus of one of the three, he explained that it was also quite easy to solve the cubic equation involved in the analytical solution of the problem of the trisection of the angle; and a working model of an instrument of this kind executed in zinc was exhibited by Professor Henrici after the lecture. He concluded by expressing his great obligations to this gentleman, without whose aid he would have been able to do little more than adumbrate in general terms the results which, thanks to his friend's practical knowledge and skill, he had had the pleasure of exhibiting in a tangible form, and submitting before his audience to the test of actual experiment; and expressed his conviction that Peaucellier's unhoped-for discovery (even if viewed merely on its practical side as a new vital element of mechanism) was destined to produce lasting and important results through innumerable applications to the useful and ornamental arts, and would hand down the name of its inventor to posterity as one of the benefactors of mankind.

Postscript.

In some possibly forthcoming number of *Nature* a detailed account, which was expected to appear two months ago, will be given, illustrated with the necessary diagrams, of the cube-root extractor and angle-trisector: the materials for this purpose are in the hands of the editor of that journal, and have been entrusted by him to the most competent person to draw them out into form—the writer not feeling within himself the necessary energy for accomplishing this task. He thinks it, however, desirable (indeed almost a moral duty on his part) to supplement those materials by the desultory remarks which follow, in order that some results, which he believes to be important to the progress of mechanical and algebraical science, may be rescued from the chances of total oblivion and virtual annihilation.

The first question which presents itself relates to the square-root extractor. It is a remarkable fact that a cellular system for extracting square roots is much more complicated than what is required for the cube root; and so in general all even-degreed extractors require a more extensive apparatus of link-work than is required for the odd degrees. Such extractions may be performed in all cases by a system consisting of Peaucellier cells exclusively; but the process may be abridged in the case of even degrees by the interpolation of another form of cell, alluded to in a previous foot-note under the name of the quadratic-binomial extractor, which deserves a somewhat more

detailed description. It is figured in the diagram below. $FAPB$ is a jointed rhomb or diamond; PC and PD are each doubles of the sides of the rhomb and CW, DW are two equal links. The difference between the squares of CP and CW is the modulus. FP, FW are the arms, and the difference between their squares is equal to the modulus. This is the instrument which, when F is fixed and P moves in a circle passing through W, describes a curve which may be called the Lemniscatoid, having the same general kind of relation to the Lemniscate that the Hypercissoid and Hypocissoid bear to the Cissoid proper. This Lemniscatoid becomes the Lemniscate when a certain simple arithmetical relation subsists between the modulus and the diameter of the circle described by P. If A as well as F be fixed, P will move in a circle passing through F, of which AP will be the radius, and consequently the five-bar link-work, consisting of the links CW, CP, DW, DP, FB (centred at F and A), will serve to describe the Lemniscate when the arithmetical relation above referred to subsists between CP and the modulus; that is, between CP and the difference of the squares of CP and CW; consequently, when the lengths CP, CW have a certain simple arithmetical proportion to each other, W will describe the Lemniscate: this proportion, it will be found, is such that when W comes to F the angle at P is a right angle. So much for the binomial-root extractor: obviously by aid of this kind of linkage when one arm is the tangent of any angle, the other arm may be made equal to the secant, and *vice versâ*. Again, it should be observed that, as in the Peaucellier cell (used as a reciprocator) the arms may be taken as x and $\frac{1}{x}$, by interchanging the fulcrum with one of the poles that is, reckoning the two arms as the distance between the fulcrum and one pole and from the other pole to the arm x, the new arm may be made to become $\frac{1}{x} - x$, which may be reciprocated into $\frac{2x}{1-x^2}$ by the use of a second Peaucellier cell. Hence by two Peaucellier cells an arm denoted by $\tan\theta$ may be, so to say, transformed into an arm $\tan 2\theta$. Thus we see that we may pass through the following series of transformations



$$\cos\theta,\ \sec\theta,\ \tan\theta,\ \tan 2\theta,\ \sec 2\theta,\ \tfrac{1}{2}\cos 2\theta$$

by means of a P.C., a Q.B.E., a pair of P.C.'s, a Q.B.E., and a P.C.,—that is, by an apparatus containing four Peaucellier cells and two cells of the new kind—making a linkage of six cells or 36 links in all. In other words, by means of such a linkage the arm x may be, so to say, converted into $x^2 - \frac{1}{2}$.

If, therefore, by a Q.B.E. we first convert x into the square root of $x^2 + \frac{1}{2}$ by superadding to this the linkage last named, that is, by a linkage of seven cells

or 42 links, x becomes converted into x^2. Thus, then, seven cells are required for a squaring or square-root extractor instrument analogous to the cubing or cube-root instrument for which only three cells are required*.

The above investigation leads to a further construction of extraordinary interest, which the speaker is wont to describe as the Kinematical Paradox: every new flight in physics and mathematics, and the same seems equally true of politics, ethics, and philosophy†, is apt to commence with a paradox. Two perfect linkages have been described above, one of six, the other of seven cells. Let these linkages both be constructed simultaneously; they will have two detached points of the one (namely, the two extremities of the arm x) coincident with two of the other: their union will itself (according to a general principle) form a perfect linkage. In this linkage of 13 cells two points will lie in the same straight line with the original zero point from which the arms are measured, one at the distance x^2, the other at the distance $x^2 - \frac{1}{2}$ therefrom. Hence there will be two points in this linkage which are disconnected, but in whatever way the other links are drawn in and out, retain an invariable distance from each other! Any other two points of the apparatus may be made to vary their distances from each other, but no force that can be applied at these two points to force them nearer to or separate them further from each other can be of any effect. There is no immediate rigid connection between them, and yet they are as good as rigidly connected. Imagine now that they become connected by a material link: the linkage will not be a fixture, but a perfect linkage as before, consisting, however, of an odd number, namely, 79 links; any one of these may be regarded as a lazy-bar, and may be removed without affecting the motion of which the apparatus is susceptible. Returning to the original state of things, where there are 13 cells, if we fix the two points of invariable distance the instrument will not become a fixture (as would be the case if any two other disconnected points in it were fixed), but a free link-work with a superfluous or lazy-bar,

* The much simpler scheme for converting x into x^3, which explains the principle of the cube root machine, is as follows:

$$\text{First conversion, } x - \frac{1}{x}, \text{ that is, } \frac{x^2-1}{x}.$$

$$\text{Second conversion, } \frac{x}{x^2-1} - \frac{1}{x}, \text{ that is, } \frac{1}{x^3-x}.$$

$$\text{Third conversion, } (x^3-x)+x, \text{ that is, } x^3.$$

For the trisection of the angle it is necessary to solve kinematically the equation between $\cos 3\theta$ and $\cos\theta$, to effect which it is only necessary to replace the third conversion above by

$$4(x^3-x)+x, \text{ that is, } 4x^3-3x.$$

† As for example Cramer's paradox (the foundation of the highest modern geometry) the ποῦ στῶ of Archimedes and the hydrostatic paradox, "The king can do no wrong," "It is better to suffer than to do wrong," "All proof is reducible to syllogisms, and the syllogism can prove nothing," "A heavy body begins to fall with no velocity." The Kantian antinomies. Helmholtz's vortices. A variable function which never varies, that is, an Invariant as distinguished from a Constant.

represented by any of the links at will; for by fixing these particular two points, not *four*, but only *three* degrees of liberty are abstracted. By fixing one of them two such degrees are taken away; but as the other is then not free, but compelled to move in a circle, fixing *it* takes away only one additional degree of liberty of motion.

By this link-work of 78 bars (one supererogatory) a remarkable Kinematical problem has been solved (and it is probably the simplest solution of which it admits), which may be stated as follows:—"Required to construct a link-work fixed or centred at two of its points, such that (when the machine is set in motion) some other point or points therein shall be compelled to move in the line of centres."

There are some similar questions to this, which ought, in a strict logical order, to have preceded it, which we may now take into consideration. By a single mounted Peaucellier cell fixed at two centres, one point is made to move perpendicular to the line of centres. Suppose now it were required to devise a link-work such that a point should move parallel to such line.

The motion perpendicular to the line of centres is due to the fact that by the Peaucellier cell the radius vector $C \cos \theta$ is transformed into $C \sec \theta$; in like manner to get the parallel direction a means must be found of passing from the cosine to the cosecant. Now although a single cell serves to change the tangent into the secant, *or vice versâ*, and consequently a single *imaginary* cell will serve to change the cosine into the sine (which of course could then be immediately Peaucellierized into the cosecant), he is not aware of any direct real process simpler than that about to be stated by which this can be effected. His actual law of deduction is as follows: Cosine; secant; tangent; cotangent; cosecant, involving the use of two Peaucellier cells and two quadratic-binomial extractors.

With one cell more, that is, with five in all, the cosine becomes converted into the sine, and consequently by introducing a pantigraphic cell $\cos \theta$ may be converted into $\cos(\theta + \alpha)$, and this reciprocated into $\sec(\theta + \alpha)$. Thus it seems (at all events after the present method) that four cells are required to obtain by link-work rectilinear motion parallel to the line of centres, and seven cells to convert it into motion oblique to the line of centres; or taking into account the mounting radius 7, 25, 43 links are required to obtain motions respectively perpendicular, parallel, and oblique to that line. In the Kinematical Paradox it will have been seen that there are 13 cells employed, that is, 78 links, of which any one is liable to removal at will, so that for motion in the very line of centres 77 links are requisite. Consider this system in its entirety. In a straight line with the two fixed points there will be 13 other medial points; and two parallel ranks on both sides, each also containing 13 points. The whole apparatus admits of being moved with a sort of see-saw motion backwards and forwards; and it may assist the imagination of

the reader if he will conceive such an instrument armed with 13 picks in the line of centres, each at work to remove the asphalt of a pavement under repair; an idea suggested by a member or visitor at a soirée of the Amateur Mechanical Society of London, of which the ingenious and accomplished "Senior Member for Greenwich" acts as honorary secretary. Or we might describe the Kinematical Paradox as a kind of compound saw. If the "two points of invariable distance" be set free, and some other of the medial points be fixed as a fulcrum, the instrument may be used like Peaucellier's second invention referred to in a previous foot-note as a radial protractor to change the curve

$$\rho = \text{a given function of } \theta$$

into the curve $\rho + c = \text{the same function of } \theta$;

as, for instance, to pass from the circle to the limaçon of Pascal, or from a straight line to a conchoid. For while one of the two points of constant distance described any curve, the other would describe the same curve with all its radii vectores reckoned from the fixed point lengthened or shortened by a constant quantity. The Kinematical Paradox ought not to be regarded in the light of a mere luxury of speculation; it serves to represent a constant as a Kinematical function of the independent variable (corresponding to the use of the zero power of x to represent unity in algebra), without which the general analytical theory of linkages, and the very important theory of algebraical functions founded thereon, would fall to the ground, or rather be incapable of being constructed.

It would be difficult to quote any other discovery which opens out such vast and varied horizons as this of Peaucellier—in one direction, as has been shown, descending to the wants of the workshop, the simplification of the steam-engine, the revolutionizing of the millwright's trade, the amelioration of garden-pumps, and other domestic conveniences (the sun of science glorifies all it shines upon), and in the other soaring to the sublimest heights of the most advanced doctrines of modern analysis, lending aid to, and throwing light from a totally unsuspected quarter on the researches of such men as Abel, Riemann, Clebsch, Grassmann, and Cayley. Its head towers above the clouds, while its feet plunge into the bowels of the earth.

Prophetic and well-timed were the parting words to the speaker of the illustrious Tchebicheff: "Take to Kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space." So also said Lagrange.

In the course of the foregoing exposition, incidental reference has been made to the addition of perfect linkages to each other*. This gives rise to

* Namely, by pivoting together two disconnected points of the one with two disconnected points of the other, each with each. The sum of two perfect linkages so connected will satisfy the same numerical linear equation between joints and links as its two constituents, and thus will itself constitute a perfect linkage.

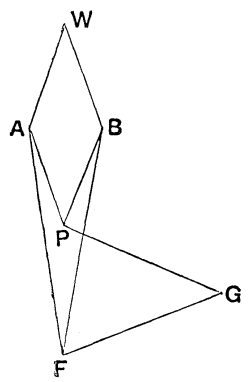
the important distinction of all perfect linkages into prime and composite—prime ones being such as can be resolved into the sum of two others, and composite those for which no two such components can be found. As an example of one kind, imagine an octagon with its four pairs of opposite angles (or, which will do as well, its four pairs of opposite sides) connected by links. There will then be 12 links and 16 joints; and since $3 \times 12 - 2 \times 16 = 4$, the linkage will be perfect. Such a linkage is prime, for it will be found impossible to resolve it into two others. Whereas, every cell previously described is capable of being formed by the successive accretions of single pairs of links, thereby justifying in a new and specialized sense the title of Compound Compass, used by Peaucellier to designate his cell. Moreover, cells belong to a very special class of compound linkages, those namely which by successive processes of decomposition can eventually be reduced to depend on sets of link-pairs, and which may accordingly be termed Dyadisms. Dyadisms, again, require to be classed according to their order. A dyadism of the first order is one that can be obtained by successive additions of single duads at a time. A dyadism of the second order is one that can be formed by successive additions of single dyadisms of the first order at a time, and so on; and it is very essential to notice that the addition together of two dyadisms of a given order will not in general be a dyadism of the same order. Thus we see that a pure tactical theory of colligation underlies the subject of linkages, a theory of the same nature as that which is known to underlie the doctrine of crystallography and polyhedra; and as that which, under the name of ramification (proposed by the speaker), gives the clearest notion of the modern chemical doctrine of the atom-groupings of the hydro-carbons, and in a manner supplies an *à priori* ground for the formula of the saturated hydro-carbons $C_n H_{2n+2}$, which, for the simpler case of the hydro-borons (if such series existed), would become $C_n B_{n+2}$.

It may be shown that every ramification may be subjected to a process of reduction (a sort of divulsion process, the number of steps of which fixes its genus, or order), which leads eventually to a single intrinsic centre or a pair of intrinsic centres, and consequently may be referred to one or the other of two great classes of forms which may be termed central and axial respectively; and it seems only reasonable to anticipate that the physical properties of such chemical compounds as the hydrocarbons will eventually be found to correspond to this distinction between their representative ramifications; and that they will accordingly range themselves under one or the other of two great families distinguished by properties at least as important and specific as those which serve to distinguish the crystalloidal and colloidal states of matter. The theory of ramification is one of pure colligation, for it takes no account of magnitude or position; geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation.

The sphere within which any theory of colligation works is not spatial but logical—such theory is concerned exclusively with the necessary laws of antecedence and consequence, or in one word of *connection* in the abstract, or in other terms is a development of the doctrine of the compound parenthesis. M. Camille Jordan, independently of and anteriorly to the author, discovered and published in a memoir, the title of which would never suggest the notion of ramification, the existence of the intrinsic centre and centres here referred to—without having any suspicion of its bearing on modern chemical doctrine. He has moreover discovered the existence of another kind of intrinsic centre of ramification which was unknown to the author of these lines.

A ramification, it ought to be added, is a rootless tree, that is, one in which the root only ranks the same as the terminal of a branch, and saturated hydrocarbons are typified by ramifications in which every joint is trifurcated, meaning thereby that in tracing the wood outwards from any terminal assumed as the root, it splits and splits again, so that trifurcation takes place at each joint, or in other words, *four* lines radiate out from each joint*; the joints are supposed to adumbrate the carbon atoms and the terminal points the hydrogens.

To conclude, as he has begun, with the principal personage of his story, the author thinks it will be useful to several of his readers to have before their eyes the figure which contains the property of the admirable linkage which lies at the root of Peaucellier's conicograph.



In the given figure $APBW$ is a rhomb. PA is equal to PB, GP to GF, and G' is a point lying on FG, or FG produced such that $FG'W$ is a right angle. Then, however the links are moved about, the motion of W *relative* to FG will be always perpendicular to FG, from which it follows that $FG'W$ will always continue to be a right angle, and consequently an upright piece attached at G' perpendicular to FG will always continue to point to W. When W is fixed, the instrument serves as a radial protractor. One point of the upright can describe any curve, and any other point a radial protraction (or retraction) of that curve. When one point of the upright perpendicular is fixed, the combination becomes ideally equivalent to a revolving slot, in which W is free to traverse. The inverse of a conic in respect to a focus (that is, the Limaçon of Pascal) is a protraction or retraction of the circle. Hence the use of the instrument for describing conics.

* Observe that if there were *no* splitting, as in a bamboo cane, *two* lines would issue from each joint.

In the above linkage let a pair of equal links GP, GW be *substituted* for the pair GP, GF. It is easy to prove that if O be the intersection of the diagonals of the rhomb, GO and FO will then be at right angles to each other, and the sum of their squares will be a constant. If now any one link of the rhomb is transferred parallel to itself so as to pass through O, and is jointed on to the sides at the points where it meets them, and O is fixed, and F made to move in a circle containing O, the path of G will be the *inverse in respect to O of a conic* of which O is the centre, so that by the aid of a radius and a reciprocator in addition to the transformed linkage above described, a point may be made to move in any conic round its *centre* as a fixed point*. This is rather a simpler construction than Peaucellier's for motion in a conic round the *focus* as a fixed point, for the number of links is no greater, and the ungainly cross-piece disappears. Moreover, it possesses all the advantages of Peaucellier's method arising from the fulcrum lying off the curve to be described. Finally, as regards the most general motion that can be produced by a Peaucellier-mounted cell in its generalized form, if F be the junction of two links on which FA, FB are two equal segments, and FC, FD two other equal segments, and PA, PB and WC, WD be two pairs of equal links in the same plane with the first pair, such combination of three pairs is the generalized form of cell in question. In applying it to draw curves, F may be fixed, and a mounting radius of any length attached to P or W, or P or W may be fixed, and the mounting radius attached to W or P, or P or W be fixed, and the mounting radius attached to F. In a résumé of this general kind it would be out of place to enter into a discussion of the forms thus generated†.

* It follows as a particular case of the above, that an apparatus of nine links moving round two fixed centres will serve to generate motion in a circle whose centre is in a right line drawn through one of the given two, perpendicular to the line joining it to the other.

† It is too late to make any change in the many places where the term perfect linkage appears in the text, but the author regrets to have used the word *perfect* when *complete* would have expressed the meaning more clearly, and suggests this change of nomenclature to any writer who may hereafter have occasion to employ the term—besides being better in itself, it comes nearer to Peaucellier's "système de tiges à liaison complète"; two words (and those much more expressive) supplying the place of six. The existence of such words as surplusages, curtilage, equipage, assemblage, and many similar ones in the English language, appears quite sufficient to justify the innovation in the use of the final syllable in linkage. A question of great interest remains over, namely, "how to extend the above inquiry to linkages in space"; any two links being supposed free to move by means of universal joints in all directions round each other. As regards surfaces of revolution, the solution of the problem is virtually contained in the theory of plane linkages, and consequently as a plane may be regarded as a surface of revolution, the difficulty does not begin to be felt until the problem of producing motion in an ellipsoid or other surfaces of the second order, by means of solid link-work, comes under consideration. It seems to be a problem well worthy of being investigated and thought out, especially for the sake of its analytical consequences and the light it might be expected to throw upon the theory of algebraical functions of two variables.

3.

ON THE PLAGIOGRAPH *aliter* THE SKEW PANTIGRAPH.

[*Nature,* Vol. XII. (1875), pp. 168, 214—216;
also, *Archivo de Mat.* I. pp. 112—114.]

I HAVE been led by the study of linkages to the conception of a new instrument, or rather a simple modification of an old and familiar one, the Pantigraph, by means of which a figure in the act of being magnified or reduced may at the same time be slewed round the centre of similitude. Some of the readers of *Nature,* such possibly as my able and most ingenious friends, Messrs George Cayley and Francis Galton, may be able to pronounce with authority how far the invention is new and whether it is likely to be found in any way useful in practice as applied to the art of the designer or engine-turner. Already my invention of the Isagoniostat, or equal angle setter, which I shall take some other opportunity to communicate to this journal, has been deemed available in practice for working automatically the train of prisms of a spectroscope.

In Fig. 1, $AOBCQ$ represents an ordinary pantigraph. O is the fixed point, P is the tracer, and Q the corresponding follower; then, as everybody

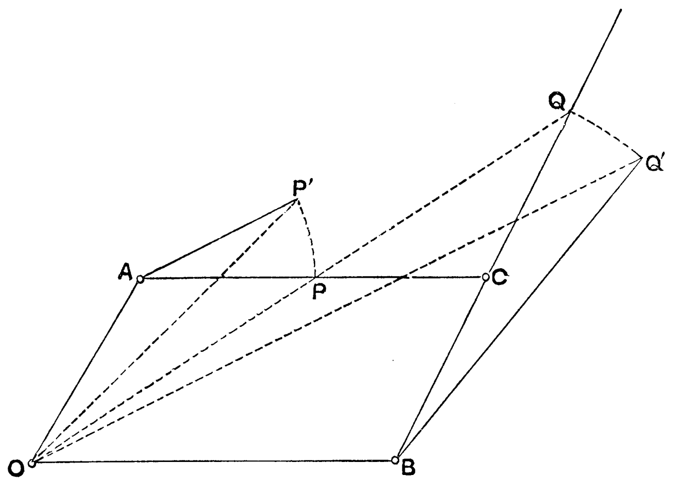


Fig. 1.

knows, any curve traced out by P will be imitated by Q, and the two curves

will be similarly situated in respect to O. The point of addition is the following:—

Let P be moved through any angle, $P'AP$ round A, and Q through *an equal angle* QBQ' in the *opposite* direction round B, and let P' and Q' be supposed to be in any manner rigidly connected with the bars AC, BC respectively. Then it admits of an easy proof that in whatever way the jointed parallelogram $AOBC$ is deformed, OQ' will bear to OP' the constant ratio of AC to AP, and moreover the angle $P'OQ'$ will always remain equal to the angles $P'AP$, $Q'BQ$.

It follows that whilst P' is made to move upon any curve the follower Q' will trace out a similar curve altered in magnitude, and at the same time turned round the first point O.

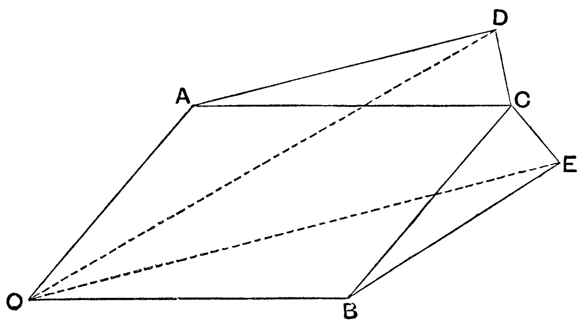


Fig. 2.

If, as in Fig. 2, we take AD equal to AC, BE equal to BC, and the angles CAD, CBE equal to each other, then the rays OD, OE will always remain equal and be inclined to each other at a constant angle. With this adjustment the instrument may be used to transfer a figure from one position in a sheet of drawing paper to any other position upon it, leaving its form and magnitude unaltered, but its position slewed round through any desired angle.

History of the Plagiograph.

I should like to add a few words to my description of the instrument called the Plagiograph* (the g to be pronounced soft, like j, as in Genesis

* It may be questioned whether a new-born child can have a history. Perhaps it might have been more correct to have used for my title, "History of the Birth of the Plagiograph," but this would have been long; moreover, the Plagiograph proves to be an unusually precocious child, having in its very cradle given birth to a greater than itself, the Quadruplane, a full-grown invention described in the sequel of the text.

Plagiarist, Oxygen) in *Nature*, Vol. XII. p. 168, for the purpose of explaining the order of ideas in which it took its rise, and also a very beautiful extension of another recent kinematical invention to which it naturally leads the way, and which, thus generalised, I propose to term the Quadruplane.

The true view of the theory of *linkages** is to consider every link as carrying with it an indefinitely extended plane, and to look upon the question as one of relative† motion which may be put under this form: When a *complete* linkage (meaning thereby a combination of jointed planes capable of only a definite series of relative movements) is set in motion, *what is the curve which any point in one of these planes will describe upon any other?*

In this mode of stating the question, the lines joining the pivots round which the planes can turn correspond to the jointed rods of the common theory. Fix any one of the planes, and the linkage becomes a link-work, or, to speak with more precision, a piece-work.

The curve described by a point in one plane upon any other plane has been termed by me with general acquiescence a Graph, and to keep the

* It is quite conceivable that the whole universe may constitute one great linkage, that is, a system of points bound to maintain invariable distances, certain of them from certain others, and that the law of gravitation and similar physical rules for reading off natural phenomena may be the consequences of this condition of things. If the Cosmic linkage is of the kind I have called complete, then determinism is the law of Nature; but, if there be more than one degree of liberty in the system, there will be room reserved for the play of free-will. We should thus revert to the Aristotelian view under a somewhat wider aspect of circular (the most perfect because the simplest form of motion) being the primary (however recondite) law of cosmical dynamics. Speaking of cosmical laws brings to my mind a reflection I have made upon the new chemical theory of atomicity. Suppose it should turn out that the doctrine of *Valence* should be confirmed by experience, and that the consequent logico-mathematical theory of colligation containing the necessary laws of consecution, or if one pleases so to say of cause and effect, should plant its foot and introduce a firm basis of predictive science into chemistry, how beautiful will be the analogy between this and the physical law of inertia! which really merely affirms the fact of each atom or point of matter carrying about with it a certain number, denoting its communicative and inverse receptive faculty of motion; for in such case Valency, also affirming a numerical capacity for colligation, will be the exact analogue in chemistry to Inertia in the theory of mass motion, and might properly assume the name of chemical inertia. Social individuals differ as egregiously as Isomers in their capacity for forming multifarious attachments.

† I believe it is to Mr Samuel Roberts that we are indebted for the idea of passing from mere copulated links to planes associated with the links, and for the observation that the order of the corresponding Graphs is not thereby augmented. The substitution of the more general idea of linkage for link-work, and of isolating completely the conception of relative in lieu of absolute motion, is due to the author of these lines. Take the case of a Quadruplane in which the four joints in their natural order of sequence form a contra-parallelogram. It is well known (and the fact has been applied to machinery under the name of "the parallelogram of Reulleux") that the relative motion of an opposite pair of planes may be represented by causing two curves to roll upon each other; but I add that this may be done simultaneously for both pairs of planes, giving rise to a beautiful and previously unthought-of double motion of rolling (without slip) between two ellipses for one pair and two hyperbolas for the other pair of planes. This is an immediate deduction from the conception of purely relative motion.

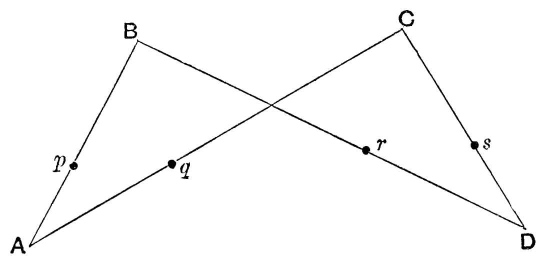
correlation closely in view, I have proposed to call the describing point the Gram*. We may further understand by canonigrams describing points taken in the lines connecting the joints and their corresponding curves, canonigraphs; Grams lying outside these lines and their appurtenant Graphs may be termed Epipedograms and Epipedographs; or, if these names are found too long for use, Planigrams and Planigraphs.

Now consider more particularly the generalised form of the linkage which corresponds to three-bar motion, of which Watt's parallel motion (so-called) offers a simple instance. If we were to revert to the old notion of link-work we should say that a three-bar motion is obtained by fixing one of the sides of a jointed quadrilateral of any form; but adhering to the more general conception of the matter here set forth, we may describe it as resulting from the fixation of any one of the planes of a quadriplane, that is, a system of four planes connected together by four joints. Mr A. B. Kempe, who has brought to light magnificent additions to Peaucellier's ever memorable discovery of an exact parallel motion in a paper which I have had the pleasure of presenting to the Royal Society of London, in the course of conversation with me made the very acute and interesting remark that in an ordinary three-bar motion, supposing the distance between the two fixed centres to be given, and the lengths of the two radial arms and the connecting rod to be also given, the order in which these three latter elements are arranged will not affect the nature of the canonigraphs described. Whichever of the three lengths is adopted as the length of the connector and the remaining two as the lengths of the radial arms, the very same system of curves will be described in all three cases so far as regards their form: every canonigram in the arrangement will have a canonigram corresponding to it in each of the other arrangements such that the corresponding curves described will be similar and similarly placed—a most remarkable, and, for the purposes of theory, an exceedingly important observation; but, as Prof. Cayley observed, when once stated, a self-evident deduction from the principle of the ordinary pantigraph†. It

* Gram is intended to suggest the notion of a *letter* discharging the duty of a point. In inventing new verbal tools of mathemathical thought, the following are the rules which I bear in mind:—1. The word must be transferable into the common currency of the mathematical centres of Europe, France, Germany, and Italy. 2. It must enter readily into combinations and be susceptible of inflexion fore and aft. 3. It should contain some suggestion of the function of the idea intended to be conveyed. 4. It should by similarity in quality or weight of sound conjure up association with the allied ideas. 5. When all these conditions are incapable of being simultaneously fulfilled, they should be observed as far as possible, and their relative importance estimated according to the order in which they are written above.

† Suppose AB, BC, CD to be three jointed rods fixed at A and D. Choose either of the fixed points, say A, and complete the parallelogram $ABCB'A$, regarding CB', $B'A$ as two additional jointed rods; through A draw any transversal, cutting the two indefinite straight lines CB, CB' in P and P' respectively; then whatever curve P describes when the system is set in motion, P' by the principle of the common Pantigraph will describe a curve similar and similarly situated

therefore occurred to me that a corresponding theorem ought to hold for all graphs whatever—for plagiographs just as well as for canonigraphs; and by a very simple application of the general double-algebra method of *Versors*, I found that this would be the case, the only difference being that now the corresponding graphs, instead of being similar and similarly situated, would be similar but not similarly situated; in other words, that the lines joining the centre of similitude with the corresponding points, instead of coinciding in direction, would make for each particular graph a constant angle with each other. Thus I passed from the conception of the common Pantigraph to that of the Quergraph, or Plagiograph, or Skew Pantigraph, as the new instrument described in the previous article may indifferently be called. I now come to



the second part of my story, and proceed to explain the remarkable extension a theorem analogous to and naturally suggested by the Plagiograph gives of Mr Hart's remarkable discovery of a *cell* consisting of only four jointed rods which possesses the same property of reciprocation as Peaucellier's six-sided cell.

This cell is exhibited in the figure above. The four jointed rods AB, AC, CD, BD are equal in pairs, AB and CD being equal, also AC and BD. In fact, the figure is nothing else but a jointed parallelogram twisted out of its position of combined parallelisms, and may be termed a contra-parallelogram. When the cell is in any position whatever, imagine a geometrical line to be drawn parallel to the lines joining A and D or B and C

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thereto, A being the centre of similitude. Now, it will be noticed that $AB'CD$ is a system of four jointed rods in which the lengths AB', $B'C$ are the same as the lengths AB, BC in inverted order, namely, $AB'=BC$, and $B'C=AB$, and as we may proceed from the point D equally well as from A, it follows that all the six interchanges may be rung between the three lengths AB, BC, CD. This is the proof of Mr Kempe's admirable theorem; but does the simplicity of the principle involved take away in any degree from the beauty of the result, or rather, is not the interest of the conclusion enhanced by the simplicity of the means by which it is arrived at? In fact, as Kant has observed, the groundwork of all mathematical proof consists in putting things together by a free act of the imagination; and the essence of Euclid is to be sought in the constructions which antecede the formal proofs. The real proof is the construction, and no one has the right to call Mr Kempe's discovery "a truism."

(for these lines will obviously always remain parallel to each other), cutting the four links in the points p, q, r, s.

Now take up the cell and manipulate it in any manner you please so as to change its form, the same four points p, q, r, s will always remain in the same straight line, the distances pq and rs will always remain equal to one another, and the product of pq by pr, or, which is the same thing, of sr by sq, will never vary, so that pr remains (so to say) a constant inverse of pq, and sr of sq—the actual value of the constant product (called the modulus of the cell) being the difference between the squares of the unequal sides of the contra-parallelogram, multiplied by the product of the segments into which any one of the links is separated by the points p, q, r, or s, and divided by the square of such link. Now Mr Kempe and myself—he by the free play of his vivacious geometrical imagination, I by the sure and fatal march of algebraical analysis—have arrived at the following beautiful generalisation of Mr Hart's discovery. On AB, CA, BD, DC describe a chain of four similar triangles, the angles of which are arbitrary, but looking towards the same parts, and so placed that the equal angles in any two contiguous triangles are adjacent—call the vertices of these triangles P, Q, R, S (which will be in fact the analogues of the points p, q, r, s before mentioned): then it will be found that the figure $PQRS$ will be a parallelogram whose angles are invariable, and the product of whose unequal sides is constant; in a word, a parallelogram of constant area and constant obliquity*.

The modulus, or constant product of the sides, follows the same rule as in the special case, except that for the product of the segment of a link divided by the square of its entire length, must be substituted the product of the sines of the angles adjacent to any link divided by the square of the sine of the angle subtended by it.

* I early noticed the analogy between M. Peaucellier's six-linked reciprocator and the primitive form of the pantigraphic proportionator formed by two parallelograms having an angle and the directions of its two containing sides in common, also therefore consisting of six links; and indeed pointed out that, starting (to fix the ideas) from a negative Peaucellier-cell (such as is in successful use in the Houses of Parliament for ventilating the brains of our representative and hereditary legislators), we have only to unfix the two interior links from the angles to which they are attached, and attach them instead to two sides of the containing lozenge, so as to be parallel to the other two sides; and we pass from a Reciprocator to the comparatively barren Proportionator. Now as a Proportionator (the Pantigraph in common use) exists with only four sides, it ought to have been inferred as fairly probable that a Reciprocator also might be discovered having only four sides, that is, by analogy, the probable existence might have been inferred of a Hart cell—the contra-parallelogram first imagined by Mr Samuel Roberts, but rediscovered and hugged with the affection of a supposed original discoverer, and warmed into new and unsuspected uses by its foster-parent Mr Hart. I shall have no difficulty in finding a generalisation of the Peaucellier-cell standing to it in the same relation as the Quadruplane does to the Hart-cell, and similarly for the Proportionator, so that we shall have the fourfold proportion—Peaucellier-cell : Hart-cell : Quadruplane : New Peaucellier-cell :: Old Pantigraph : Common Pantigraph : Plagiograph : Oblique Old Pantigraph; but, except as completing a chain of analogies, the last terms in each quatrain will probably not prove of any practical importance.

Just as in the first case $pq \,.\, pr$ and $sr \,.\, sq$ are constant, so now $PQ \,.\, PR$ and $SR \,.\, SQ$ are constant; but whereas pq coincided in direction with pr and sr with sq, PQ and PR, like SR and SQ, remain inclined to each other at a constant angle; in a word, as the Plagiograph is to the Pantigraph, so is the Sylvester-Kempe Inverter or Reciprocator to Mr Hart's*. Do not let it be supposed that this new reciprocator is to be consigned to the limbo of barren mathematical generalities—very far from it; it is very likely indeed to find a most valuable application to mechanical practice, and to subserve the purposes of that immediate "Utilitarianism†" so dear to the Philistine mind; for, as by means of Mr Hart's Quadrilateral, when one of the four named points, say p, is absolutely fixed, and one of its non-conjugate points,

* In the case of a three-piece motion whose fundamental linkage (that is, the quadrilateral formed by the lines joining the pivots and the fixed points in their natural order of succession) is subject to the condition that either the two pairs of opposite sides or two pairs of contiguous sides are equal for each pair, the Planigraph (leaving out of account its circular portion) is the inverse of a conic. In the first case (that of the contra-parallelogram) the position of this node is seen immediately to be the opposite to the Planigram in respect to the centre of the figure in its untwisted (that is, parallelogrammatic) form. In the second case, that of the so-called kite-form, it was found to be far from easy to determine its position. Even our Cayley did not quite succeed in determining it from the analytical equations, and it was reserved for M. Manheim to deduce it geometrically by a most elegant but very elaborate construction given in a paper inserted in the *Proceedings of the Mathematical Society of London.* By the aid of the reciprocity established by me above we pass at once from the case of the contra-parallelogram to that of the kite-form, and the problem literally solves itself as easily as a musical passage can be transposed from one key to another. It is to that profound mathematician, Mr Samuel Roberts, that we are indebted for bringing to light these two cases of three-bar motion, in which the general three-bar sextic Graph breaks up into a circle, and the inside of a conic, and I have proved that no other such cases exist. Mr Roberts's papers are inserted in the *Proceedings of the London Mathematical Society*, which is indebted for its existence, at least in its present form (being originally little more than a juvenile mathematical debating society among the students of University College), to the organising talents of Mr Hirst, who has reason to be proud of his progeny. Similar societies on a precisely similar basis, and adopting the rules of its elder sister, have been subsequently founded in Paris, Warsaw, and, I believe, other capitals in Europe, and, it is safe to predict, are destined to play no unimportant part in the further evolution of our time-honoured yet ever young, ever fresh, and self-renovating science—Othello, Hamlet, and Romeo all in one. Meanwhile, in the University supposed to be peculiarly dedicated to the advance of mathematical science, a young and very promising mathematician (whose name shall not be divulged) *àpropos* of a movement kindly attempted, without my being previously consulted, to place me in a position where, in the vicinity of our central luminary, I might have been in my proper place, and helped to reflect some portion of his rays upon surrounding bodies, wrote to me lately: "You cannot imagine the bitter prejudice that prevails here against pure mathematics, &c." I freely forgive those, "the bigots of a narrow creed," who entertain such sentiments, knowing that "they know not what they do."

† What would our English statesmen say to the conduct of the proverbially parsimonious Prussian Government and the nineteenth century Richelieu, "der tolle Bismarck," in appropriating a million and a half of marks (75,000*l.* sterling) placed at the free disposal of the modern Aristotle, Helmholtz, for constructing the bare shell alone of the new Physical Laboratory at Berlin! If such an appropriation were proposed at home, would there not run through the land a frantic shriek or muttered growl of disapprobation at such a wanton waste of the public funds on mere speculative science?

say r, is attached to the end of a radius so centred and of such a length that the path of r is a circle which, *geometrically* completed, would pass through p, the remaining conjugate point q will be forced to describe a straight line perpendicular to the line joining the two fixed points—so by means of our Quadruplane, when P is fixed and R made to move in the arc of a circle passing through P, the point Q may be made to describe a straight line having any *desired obliquity* to the line of centres, the amount of such obliquity depending on the magnitude of the supplemental equal angles P, Q, R, S. Thus the Plagiograph (and in the first instance Mr Kempe's notice of the homœographic commutability of the lengths of the connecting rod and the radial bars in ordinary three-bar motion) has led by a devious path to the construction of a three-piece-work giving the most general and available solution of the problem of exact parallel motion that has been discovered or that can exist—I say the most available, for it is evident, in general, that piece-work must possess the advantage of greater firmness and steadiness from the more equal distribution of its strain over ordinary link-work.

The Peaucellier and Hart cells, duly mounted, afford the means by obvious methods of adjustment to cut straight lines at any distance from either of the fixed centres, but confined to lying perpendicular to the line of centres; whereas the Quadruplane puts it into our power with one and the same instrument affected with simple means of adjustment to make straight cuts (and, if desired, two parallel ones simultaneously) in all directions as well as at all distances in the plane of motion. So again the Plagiograph enables us to apply the principle of angular repetition (as, for instance, in making an ellipse with dimensions either fixed or varying at will, successively turn its axis to all points of the compass) to produce designs of complicated and captivating symmetry from any simple pattern or natural form, such as a flower or sprig; and as the head of a house at Oxford in the good old port-wine days was heard to complain about the electro-magnetic machine, that "he feared it would place a new weapon in the hands of the incendiary" (the power of Swing being then rampant in the land), so, but with better reason and upon the highest authority, it may be predicted that this simple invention will be found to place a new and powerful experimentative and executory implement in the hand of the engine-turner, the pattern-designer, and the architectural decorator.

P.S.—I rejoice to be able to state that the Institute of France has quite recently adjudged its great mechanical prize, the "Prix Montyon," to Col. Peaucellier for his discovery of an exact parallel motion when a lieutenant in 1864. The first practical application of this discovery, made by Mr Prim under the sanction of Dr Percy, may be seen daily at work in the Ventilating

Department of our Houses of Parliament. The workmen there, who never tire of admiring its graceful and silent action, have given it the pet name of the "Octopus," from some fancied resemblance between its backward and forward motion and that of the above-named distinguished Cephalopod. I feel a strong persuasion that when the inertia of our operative classes shall have been overcome, this application will prove to be but the signal, the first stroke of the tocsin, of an entire revolution to be wrought in every branch of construction; and that machinery is destined eventually to merge into a branch of the science of Linkage in the sense in which that word is used in the text above.

4.

ON A LADY'S FAN, ON PARALLEL MOTION, AND ON AN ORTHOGONAL WEB OF JOINTED RODS.

[*Proceedings of London Mathematical Society*, VI. (1875), pp. 196, 197.]

By means of Prof. Sylvester's Fan, it is possible to divide any angle into any assigned equal number of parts; and the trajectories of points taken in the several links connecting together the sticks of the fan have finite nodes, whose numbers are successively 1, 2^4, 3^4, 4^4, ...

Prof. Sylvester stated, in his second communication, That parallel motions exist at all is a paradox more wonderful than ever, now that his method gives the means of determining the conditions to be satisfied, and comparing their number with that of the disposable constants. The orders for 3, 5, 7, ... bars are 6, 20, 72, ... Formerly the existence of *one* was doubted; now a finite number for *every* order of linkwork is rendered highly probable. In particular, Prof. Sylvester showed how to determine whether *Parallel Motions* exist, and, if so, how to find them for any given number of bars and mode of colligation. He showed how to form a determinant involving only the lengths of the bars and other quantities which fix their direction; this determinant, if a parallel motion exists, must vanish identically for all values of the latter set of quantities. This is called the *Determinant of Parallel Motion*. The determinant is formed as a Jacobian of Equations, involving only linear functions of the lengths, and of a determinant corresponding to a set of equations of the same form as the above. Its evanescence gives a system of conditions to be satisfied, all expressed as rational functions of the lengths; and, by known algebraical methods, these enable us to find *necessary* relations of the lengths, if a Parallel Motion exists. It must then be ascertained whether these solutions are sufficient, and the problem is solved.

Prof. Sylvester's remarks on "An Orthogonal Web of Jointed Rods" were to the following effect: If two sets of joints be taken respectively

in two lines perpendicular to each other, either in a plane or in space, and a *linkage* be formed by connecting each point in one set with each point in the other by jointed rods, this constitutes an orthogonal web. It is *not* a fixture, and its motion is subject to this curious condition, that either each set of points must always continue to lie in the same right line, which may be called a neutral position, or else one set will lie in a right line, and the other in a plane at right angles to such line. Starting from the neutral position (a position of *double-lock*), the system may be said to be subject to an optional locking about one or the other of two perpendicular lines, and an unlocking about the others; but, when once put in motion, the system must again be brought into the same, or a new neutral position, before the one axis of lock can be got rid of, and another at right angles thereto substituted in its stead. If the whole motion be confined to a plane, the paradox consists in the link-combination forming one degree of liberty of deformation (*ἀλλοίωσις*, as distinguished by Plato from *κίνησις*), although a calculation of the amount of restraint by the general method applicable to such questions would seem to indicate that it ought to form an absolutely rigid system except in the case where there are only two joints in one at least of two sets. Taken in space, there is the further and more striking paradox, that the number of degrees of liberty of deformation, according to the choice made of one or the other of the two sets of points to be unlocked out of the rectilinear into the planar position, will be the *alternative of two numbers,* viz., the number of points in the one set or in the other set (which need not be the same), a kind of indeterminateness in the "Index of Freedom" without precedent in mathematical speculations. As lightning clears the air of impalpable noxious vapours, so an incisive paradox frees the human intelligence from the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of Prejudice.

5.

NOTE ON SPHERICAL HARMONICS.

[*Philosophical Magazine,* II. (1876), pp. 291—307, and p. 400.]

If for a moment we confine our attention to so-called "zonal" harmonics, and affect each element of a uniform spherical shell with a density varying as the product of two such harmonics of unequal degrees, we know that the mass of such shell is zero. A very slight consideration will serve to show that this is tantamount to affirming that if a given spherical surface be charged with a density inversely proportional to the product of the distances of each element from two fixed internal points lying in the same radius produced, then the mass of such shell will be a complete function of the product of the distances of the two points from the centre; and in fact, if we write dS for an element of a spherical surface, it is easy to find, by direct integration, that

$$\iint \frac{dS}{\sqrt{(c^2 - 2hx + h^2)}\sqrt{(c^2 - 2h'x + h'^2)}},$$

for the entire surface, is proportional to

$$\frac{1}{\sqrt{(hh')}} \log \frac{c^2 - \sqrt{(hh')}}{c^2 + \sqrt{(hh')}}.$$

In like manner, the truth of the more general theorem relating to the surface-integral of the product of any two harmonics of unequal degrees involves, and is involved in, the fact that the surface-integral $\iint \frac{dS}{R \,.\, R'}$, where

$$R^2 = (x-h)^2 + (y-k)^2 + (z-l)^2,$$
$$R'^2 = (x-h')^2 + (y-k')^2 + (z-l')^2$$

and $h^2 + k^2 + l^2$ and $h'^2 + k'^2 + l'^2$ are each less or each greater than the square of the radius of the sphere, is not merely a function (as we see *à priori* from the symmetry of the sphere must be the case) of the three quantities

$$h^2 + k^2 + l^2, \quad h'^2 + k'^2 + l'^2, \quad hh' + kk' + ll',$$

but, more definitely, is a complete function of the product of two of them, namely, $(h^2+k^2+l^2)(h'^2+k'^2+l'^2)$, and of the third. In other words, the fundamental law of spherical harmonics is exactly tantamount to the assertion that if each element of a sphere is charged with a density inversely proportional to the product of its distances from two internal or two external points, then the mass of the sphere will be a function only of the density at the centre and of the angle subtended at the centre by the line joining the given pair of points; or, venturing upon an irrepressible neologism, which explains its own meaning, the *Bipotential,* with respect to a given uniform sphere at any point-pair, is a function only of the Bipotential thereat with respect to a unit particle at the centre, and of the angle subtended at the centre by the line joining the two given points. Of course, if this is true for the volume of the sphere, it must be true for any shell of uniform thickness, or, in other words, for the surface, and *vice versâ.* In what immediately follows the volume of a spherical shell is to be understood. It is, I think, very noticeable that in that proof no process whatever of integration is employed; only the *idea* implied in integration is employed to acquire the fact that the integral in question cannot but be a function of three parts of the triangle, of which the centre of the sphere and the two given points are the apices. The rest of the proof follows as a matter of purely formal or algebraical necessity from the above fact, conjoined with that of each factor under the sign of integration being subject to Laplace's equation. In this feature of exemption from all use of integration as a process, this proof, I believe, stands alone.

It is further remarkable that its success depends on the proposition being stated as a whole; it would not be applicable, for example, to the simple case, taken *per se,* treated of at the beginning of this paper. It is by no means uncommon in mathematical investigation for this to happen, and (as regards the exigencies of reasoning) for the part to be in a sense greater than the whole—the groundwork of this wonder-striking intellectual phenomenon being that, for mathematical purposes, all quantities and relations ought to be considered (so experience teaches) as in a state of flux. In the particular case before us it is not difficult to see *à priori* why the general proposition should be more easily demonstrable than any special case of it, the reason being that more information as to the *form* of the function under consideration is made use of in dealing with the general than in dealing with any special case.

The integral under consideration is

$$\iiint \frac{dS}{RR'} \text{ (say } I\text{)},$$

where $$R^2 = x^2+y^2+z^2-2hx-2ky-2lz+h^2+k^2+l^2,$$
$$R'^2 = x^2+y^2+z^2-2h'x-2k'y-2l'z+h'^2+k'^2+l'^2.$$

Call $$h^2+k^2+l^2=r^2,\quad h'^2+k'^2+l'^2=t^2,\quad hh'+kk'+ll'=s.$$

Then $\frac{1}{RR'}$, expanded under the form of a converging series (x, y, z being for a moment regarded as constants), will be of the form $\frac{1}{rt}$ multiplied by a rational function of $\frac{1}{r}, \frac{h}{r^2}, \frac{k}{r^2}, \frac{l}{r^2}$ and of $\frac{1}{t}, \frac{h'}{t^2}, \frac{k'}{t^2}, \frac{l'}{t^2}$ when the two points are external, and (more simply) of h, k, l and of h', k', l' when they are both internal. I, we know, must turn out to be a complete function of r, s, t, and, when expressed in the form of a series derived from the above expansion, will be the sum of terms of the form $r^i . s^j . t^k$, where it is obvious that i and j must both be negative when the "pair-point" is exterior, both positive when it is interior to the shell, and one positive and one negative in the remaining case.

Now we have identically

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right) r = 0,$$

$$\left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right) t = 0,$$

and
$$\left\{\left(h\frac{d}{dk} - k\frac{d}{dh}\right) + \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right)\right\} s = 0.$$

Hence with respect to I as operand we have

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right) + \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right) = 0.$$

Operate on this identity with

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right) - \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right),$$

and we obtain

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right)^2 - \left(h\frac{d}{dh} + k\frac{d}{dk}\right) = \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right)^2 - \left(h'\frac{d}{dh'} + k'\frac{d}{dk'}\right);$$

and there will be two other equations of like form. Adding all these together, changing all the signs, and remembering that in regard to I as operand

$$\left(\frac{d}{dh}\right)^2 + \left(\frac{d}{dk}\right)^2 = -\left(\frac{d}{dl}\right)^2,$$

$$\left(\frac{d}{dh'}\right)^2 + \left(\frac{d}{dk'}\right)^2 = -\left(\frac{d}{dl'}\right)^2,$$

we obtain

$$\left(h\frac{d}{dh} + k\frac{d}{dk} + l\frac{d}{dl}\right)^2 + 2\left(h\frac{d}{dh} + k\frac{d}{dk} + l\frac{d}{dl}\right)$$
$$= \left(h'\frac{d}{dh'} + k'\frac{d}{dk'} + l'\frac{d}{dl'}\right)^2 + 2\left(h'\frac{d}{dh'} + k'\frac{d}{dk'} + l'\frac{d}{dl'}\right).$$

In this formula
$$\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2$$
stands for its algebraical value
$$h^2\left(\frac{d}{dh}\right)^2+2hk\frac{d}{dh}\frac{d}{dk}+\ldots;$$
but if we write
$$\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2$$
to denote the operation twice repeated, then
$$\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2$$
$$=\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2-\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right),$$
and so for the like expressions with the accented letters. The formula thus is
$$\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2+\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)$$
$$=\left\{\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right)*\right\}^2+\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right);$$
or say
$$\{(E*)^2+E-(E'*)^2-E'\}\,I=0,$$
or simply
$$(F-F')\,I=0.$$

Let now $r^is^jt^k$ be any term in I; then since
$$Er=r,\quad Es=s,\quad Et=0,$$
$$E't=t,\quad E's=s,\quad E'r=0,$$
we have
$$Fr^is^jt^k=\{(i+j)^2+(i+j)\}\,r^is^jt^k,$$
$$F'r^is^jt^k=\{(k+j)^2+(k+j)\}\,r^is^jt^k,$$
and thence
$$(F-F')\,r^is^jt^k=(i^2+i+2ij-k^2-k-2kj)\,r^is^jt^k.$$
Hence $\Sigma\,(i^2+i+2ij-k^2-k-2kj)\,r^is^jt^k$ must be identically zero; therefore $i-k=0$, or $i+k+2j+1=0$.

But when the two points to which the Bipotential is referred (and which I shall hereafter call the points of *prise*) are both external or both internal, i and k have the same sign; therefore $i=k$, and the integral is a function only of rs and t, or say of
$$(h^2+k^2+l^2)(h'^2+k'^2+l'^2),\quad (hh'+kk'+ll')\dagger.$$

† When the point corresponding to r is external and that corresponding to t is internal, the equation $i+k+2j+1=0$ applies, which shows that each term is of the form $\frac{1}{r}\left(\frac{t}{r}\right)^\lambda.\left(\frac{s}{rt}\right)^\mu$; that is to say, the Bipotential multiplied by r is a complete function of $\frac{t}{r}$ and the cosine of the angle which the line joining the two fixed points subtends at the centre.

Thus the desired theorem has been established by virtue of an algebraical necessity of form alone; and the proof is of course applicable to space in any number of dimensions, substituting for the sphere or spherical surface its analogue in such space, and for the reciprocal of distance the proper power necessary for the satisfaction of Laplace's equation, that is, the $(q-2)$th power of the reciprocal, where q is the number of dimensions (supposed to be greater than 2).

For the case of two dimensions, substituting the logarithm for the reciprocal, so that, for example, we are able to affirm that if each element of a circular ring be affected with a density proportional to the product of the logarithms of its distances from two fixed internal points, the mass of such ring will depend only on the product of their distances from the centre of the ring and the angle between these distances—for this case, writing $E = h\dfrac{d}{dh} + k\dfrac{d}{dk}$ and $E' = h'\dfrac{d}{dh'} + k'\dfrac{d}{dk'}$ in the equation $(F - F')\,I = 0$, $F = (E*)^2$ and $F' = (E'*)^2$; and if the two points are interior, every term in $\dfrac{1}{RR'}$ will be of the form $cr^i . s^j . t^k$, i and k being both positive, and we must have $i^2 + 2ij - k^2 - 2kj = 0$, and consequently $i = k$—the other solution, $i + k + 2j = 0$, being applicable to the case of one point being external and the other internal. If the points are both external there will be four sets of terms. One set will consist of the single term $A \log r \log t$; a second, of terms of the form $c \log r . r^i s^j t^k$; a third, of terms of the form $c \log t . r^i s^j t^k$; and the last set, of terms of the form $cr^i s^j t^k$: and it is easy to see that

$$F(\log r \log t) = 0, \quad F'(\log r \log t) = 0,$$

$$(F - F') \log r . r^i s^j t^k = \{(i+j)^2 \log r - (k+j)^2 \log r + 2(i-k)\}\, r^i s^j t^k,$$

and consequently $i = k$ for the second and third sets; as regards the fourth set, $i = k$ for the same reason as in the case of three dimensions. Hence

$$I = A \log r \log t + \log r \phi(rt, s) + \log t \psi(rt, s) + \omega(rt, s);$$

and as r and t are interchangeable, we must have $\phi = \psi$, and consequently

$$I - A \log r \log t = F(rt, s);$$

so that not now the mass of the ring, but the difference between it and the mass due to the density at the centre is invariable when rt and s are given.

For greater simplicity, and as bearing more immediately on the theory of spherical harmonics, I have hitherto regarded the points of the pair-point at which the "bipotential" is reckoned either both internal or both external. The results established in these two cases are not complementary, but mutually equivalent to each other, and to the theorem that the integral along a spherical surface of the product of two spherical harmonics of unequal degrees is zero. In the third case, where one point is internal and the other

external, then for the case of space of three dimensions the equation between i and k will have to be satisfied, not by $i=k$ but by $i+k+2j+1=0$, as previously stated in a footnote; and for two dimensions the equation would have to be satisfied, not by $i=k$ but by $i+k+2j=0$.

The advantage of the method here indicated is that it is immediately applicable to space of any number of dimensions. I shall now proceed to show that it leads at once to the determination of the values of the surface-integral of the product of any two given types of spherical harmonics of equal degrees, and *mutatis mutandis* to the corresponding surface-integral in space of any order.

To prove that the degrees must be equal or else the integral will vanish, we have combined the two Laplacian operators applicable to R and R' respectively; to find the value of the integral in a series, I use either of these operators to act singly on the result acquired by their use in combination. For greater simplicity suppose the point-pair to be internal; then, calling

$$a+b+c=\mu=\alpha+\beta+\gamma,$$

the problem to be solved is in effect that of finding the value of the numerical coefficient of $h^a k^b l^c \,.\, h'^\alpha k'^\beta l'^\gamma$ in the integral I. Now we know by what precedes that the value, say I_μ, of that part of I which is of the μth order in the two sets h, k, l; h', k', l' respectively is a rational function of rt and s; and we may accordingly write

$$I_\mu = As^\mu + Bs^{\mu-2} \,.\, \theta + Cs^{\mu-4} \,.\, \theta^2 + \ldots,$$

where $$s = hh' + kk' + ll',$$

and $$\theta = (h^2+k^2+l^2)(h'^2+k'^2+l'^2) = \rho\rho'.$$

When A, B, C, ... are determined, the problem is virtually solved, and we shall then know the coefficient of

$$h^a k^b l^c \,.\, h'^\alpha k'^\beta l'^\gamma$$

by mere binomial expansions.

Since $$\left(\frac{d}{dh}\right)^2 + \left(\frac{d}{dk}\right)^2 + \left(\frac{d}{dl}\right)^2,$$

say ∇, operating on the whole of I gives the result zero, the same must obviously be true for each part I_μ.

Now ∇s^p is obviously equal to

$$(p^2-p)\,\rho' s^{p-2},$$

and $$\nabla\theta^q = 2q\,(2q+1)\,\rho'\theta^{q-1};$$

for $$\begin{aligned}\frac{d^2}{dh^2}(h^2+k^2+l^2)^q &= \Sigma\,\frac{d}{dh}\{2qh\,(h^2+k^2+l^2)^{q-1}\}\\ &= \Sigma\,\{2q\rho^{q-1} + 4q\,(q-1)\,\rho \,.\, \rho^{q-2}\}\\ &= \{6q + 4q\,(q-1)\}\,\rho^{q-1}.\end{aligned}$$

Also $\nabla s^p \rho^q - \rho^q \nabla s^p - s^p \nabla \rho^q = 2pq\Sigma\left(\frac{d}{dh}s\,\frac{d}{dh}\rho\right)s^{p-1}\,.\,\rho^{q-1} = 4pqs^p\,.\,\rho^{q-1}.$

Therefore

$$\nabla s^p \theta^q = (p^2 - p)\,s^{p-2}\rho^q\rho'^{q+1} + (4pq + 4q^2 + 2q)\,s^p\,.\,\rho^{q-1}\,.\,\rho'^q,$$

or
$$\nabla s^{\mu-2j}\theta^j = (\mu - 2j)(\mu - 2j - 1)\,s^{p-2}\,(\rho\rho')^j\rho' \\ + 2j\,(2\mu - 2j + 1)\,s^p\,(\rho\rho')^{j-1}\,.\,\rho'.$$

Hence, equating to zero the coefficients of the different combinations of ρ, ρ', s, we easily obtain by writing for j successively 0, 1, 2, 3, ...,

$$\mu\,(\mu - 1)\,A + 2\,(2\mu - 1)\,B = 0,$$
$$(\mu - 2)(\mu - 3)\,B + 4\,(2\mu - 3)\,C = 0,$$
$$(\mu - 4)(\mu - 5)\,C + 6\,(2\mu - 5)\,D = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B = -\frac{\mu\,(\mu - 1)}{2\,(2\mu - 1)}A,$$

$$C = \frac{\mu\,(\mu - 1)(\mu - 2)(\mu - 3)}{2\,.\,4\,(2\mu - 1)(2\mu - 3)}A,$$

$$D = -\frac{\mu\,(\mu - 1)(\mu - 2)(\mu - 3)(\mu - 4)(\mu - 5)}{2\,.\,4\,.\,6\,(2\mu - 1)(2\mu - 3)(2\mu - 5)}A,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

To find the value of A, I observe that when $k = 0$, $l = 0$, $k' = 0$, $l' = 0$, and $h' = h$, I_μ becomes

$$(A + B + C + \ldots)\,h^{2\mu}.$$

But in that case, taking the radius of the sphere equal to unity, I becomes the surface-integral of $\dfrac{1}{1 - 2hx + h^2}$, and is equal to

$$2\pi\int_{-1}^{1}\frac{dx}{1 - 2hx + h^2} = \frac{2\pi}{h}\log\left(\frac{1+h}{1-h}\right) = 4\pi\left(1 + \frac{h^2}{3} + \ldots\frac{h^{2\mu}}{2\mu+1} + \ldots\right).$$

Therefore
$$A + B + C + \ldots = \frac{4\pi}{2\mu + 1},$$

or
$$S_\mu A = \frac{4\pi}{2\mu + 1},$$

where
$$S_\mu = 1 - \frac{\mu\,(\mu - 1)}{2\,(2\mu - 1)} + \frac{\mu\,(\mu - 1)(\mu - 2)(\mu - 3)}{2\,.\,4\,.(2\mu - 1)(2\mu - 3)} \\ - \frac{\mu\,(\mu - 1)(\mu - 2)(\mu - 3)(\mu - 4)(\mu - 5)}{2\,.\,4\,.\,6\,.(2\mu - 1)(2\mu - 3)(2\mu - 5)}\ldots.$$

This series admits of summation. And I find

$$S_1 = 1,\quad S_2 = \frac{2}{3},\quad S_3 = \frac{2}{5},\quad S_4 = \frac{8}{35},\quad S_5 = \frac{8}{63},\quad S_6 = \frac{16}{3\,.\,7\,.\,11},$$
$$S_7 = \frac{16}{3\,.\,11\,.\,13},\quad S_8 = \frac{128}{3\,.\,11\,.\,13\,.\,15},\quad S_9 = \frac{120}{5\,.\,11\,.\,13\,.\,17},$$
$$S_{10} = \frac{256}{11\,.\,13\,.\,17\,.\,19}\ldots,$$

and in general

$$S_{2m} = \frac{2 \cdot 4 \cdot 6 \ldots (2m)}{(2m+1)(2m+3)(2m+5)\ldots(4m-1)}$$

and
$$S_{2m+1} = \frac{2m+1}{4m+1} S_{2m};$$

that is to say, S_μ is the reciprocal of the coefficient of h^μ in $(1-2h)^{-\frac{1}{2}}$.

Hence the values of $A, B, C \ldots$ in I_μ are completely determined, and I_μ, and consequently the value of the complete integral of

$$\iint dS \left\{ \left(\frac{d}{dx}\right)^a \left(\frac{d}{dy}\right)^b \left(\frac{d}{dz}\right)^c \cdot \frac{1}{r} \right\} \left\{ \left(\frac{d}{dx}\right)^\alpha \left(\frac{d}{dy}\right)^\beta \left(\frac{d}{dz}\right)^\gamma \cdot \frac{1}{r} \right\},$$

is known for all values of a, b, c; α, β, γ—and this by a method which is applicable step by step to any number of variables, provided in place of $\frac{1}{r}$ we write $\frac{1}{r^{n-2}}$ when n exceeds 2, and $\log r$ when $n = 2$, and consider dS to be the element of what in n dimensions corresponds to a spherical surface in three-dimensional space.

The method employed, of first using two Laplacian operators in combination to determine one property of the form under investigation and then a single one of them to act on the form thus partially determined, reminds one very much of the method for obtaining invariants of given orders from their two general partial differential equations. Combined, these two equations express the law of isobarism; then, assuming the isobarism, a single one of the two serves to determine the special values of the coefficients. The analogy between that process and the one here employed seems to me to be exact, although the subject-matter is so very unlike in the two problems—and is the more interesting on that very account.

The bipotential in the case where the two points of *prise* are both internal being known under the form $F\left(\frac{rr'}{a^2}, \cos\alpha\right)$, where a is the radius of the sphere, its value for the case where these points are both external, and for the case where they are one internal and the other external, may be assigned without any further calculation as follows:—

1. Suppose r greater than the radius of the sphere, but r' less. We know *à priori* from the result previously obtained (and stated in a footnote), that the bipotential for this case is of the form $\frac{1}{r} G\left(\frac{r'}{r}, \cos\alpha\right)$. Now in place of r, r' substitute $a, \frac{ar'}{r}$; then the bipotential becomes $\frac{1}{a} G\left(\frac{r'}{r}, \cos\alpha\right)$.

But we may by an easily justifiable application of the principle of continuity now regard a $\left(\text{as well as } \frac{ar'}{r}\right)$ as the distance of an internal point from the centre. Hence we have

$$\frac{1}{a} G\left(\frac{r'}{r}, \cos\alpha\right) = F\left(\frac{r'}{r}, \cos\alpha\right),$$

or

$$\frac{1}{r} G\left(\frac{r'}{r}, \cos\alpha\right) = \frac{a}{r} F\left(\frac{r'}{r}, \cos\alpha\right),$$

which is the value of the bipotential of a spherical surface cut by the line of *prise*, r being the distance of the external point of *prise* from the centre.

2. Suppose r and r' to be each greater than the radius, and $r > r'$; we know the bipotential is of the form $H\left(\frac{rr'}{a^2}, \cos\alpha\right)$. For r, r' substitute respectively a, $\frac{rr'}{a}$. Then we may regard the case as that of an exterior and interior point of *prise*, and consequently from the last case we have

$$H\left(\frac{rr'}{a^2}, \cos\alpha\right) = \frac{a^2}{rr'} F\left(\frac{a^2}{rr'}, \cos\alpha\right).$$

If we compare the two expressions

$$F\left(\frac{rr'}{a^2}, \cos\alpha\right) \text{ and } \frac{a^2}{rr'} F\left(\frac{a^2}{rr'}, \cos\alpha\right)$$

respectively applicable to two internal and two external points of *prise*, it will easily be seen that it leads to the following theorem. Let there be two concentric spheres, and let any two radii cut the first and second surfaces in the points P, Q and P', Q' respectively; then the bipotential of the first surface with respect to P', Q' as the points of *prise*, is to the bipotential of the second surface with respect to P, Q as the points of *prise* in the ratio of the squares of the radii of the two surfaces to each other.

This is a theorem of precisely the same kind as Ivory's for the comparison of the attractions (or, if we please, the potentials) of two confocal ellipsoids in the particular case when they become two concentric spheres, and may be verified by precisely the same geometrical method. For we have only to divide the two spherical surfaces into corresponding elements m, m' by radii drawn in all directions to meet the two surfaces, and it is evident that we shall have the distances mP' and $m'P$ equal, as also mQ' and $m'Q$. And, moreover, the ratio of any two corresponding elements m, m' will be as the square of the radii, which evidently establishes the theorem in question. It may further be noticed that the relations between the bipotentials in

the three several cases considered may be deduced from the fact that each such radical as

$$\frac{1}{\sqrt{(1-2hx-2ky-2lz+h^2+k^2+l^2)}},$$

where $h^2+k^2+l^2$ is greater than unity, may be put under the form

$$\frac{1}{\sqrt{(h^2+k^2+l^2)}}\,\frac{1}{\sqrt{(1-2h_1x-2k_1y-2l_1z+h_1{}^2+k_1{}^2+l_1{}^2)}},$$

where h_1, k_1, l_1 and h, k, l are the coordinates of two points the inverses (or electrical images) of each other in regard to the origin, and consequently $h_1{}^2+k_1{}^2+l_1{}^2$ less than unity. This is going to the heart of the matter. So I may observe that if we would go to the root of the relation between positive- and negative-degreed solid spherical harmonics, the more logical mode of proceeding is not (as is usually done) to infer this by a lengthy *à posteriori* process, but immediately from the fact that since

$$\frac{1}{\sqrt{\{(x^2+y^2+z^2)-2(hx+ky+lz)+(h^2+k^2+l^2)\}}}$$

is nullified by the operator

$$\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2+\left(\frac{d}{dz}\right)^2,$$

so also must the same operator nullify the radical

$$\frac{1}{\sqrt{\{1-2(hx+ky+lz)+(h^2+k^2+l^2)(x^2+y^2+z^2)\}}}.$$

Before proceeding further, I ought to observe that I_μ in the above series for the bipotential may easily be shown to be $\dfrac{4\pi}{2\mu+1}$ multiplied into the coefficient of t^μ in the expansion of $\dfrac{1}{\sqrt{(1-2st+\theta t^2)}}$ *; or, in other words,

* s, it will be remembered, is $hh'+kk'+ll'$, and θ is the product $(h^2+k^2+l^2)(h'^2+k'^2+l'^2)$. The statement in the text follows as a consequence from the fact that $(1-2st+\theta t^2)^{-\frac{1}{2}}$ obeys Laplace's law, and, when expanded according to powers of t, is of the form found for I_μ, and must consequently be identical with it to a factor *près*, that factor being a function of μ, whose value is easily found by making $h=h'$ and k, l; k', l' each zero. In like manner it may be shown that in higher space of n dimensions the corresponding value of I_μ is a function of μ multiplied by the coefficient of t^μ in $\{1-2t\Sigma hh'+\Sigma(h^2)\,\Sigma(h'^2)\,t^2\}^{1-\frac{1}{2}n}$; and [writing m for μ] I find† that this function, say $\phi(m, n)$, (as will be shown in a sequel to this paper) is always a rational function in m, containing in the denominator, when n is odd, one factor of the form $2m+j$, all the others being of the form $m+i$—and when n is even, factors all of the form $m+i$. Whatever the form of these linear factors had been for even numbers, we could see *à priori* that the Bipotential for space of even dimensions could contain only algebraic and inverse circular or logarithmic functions. But as regards the case of space of odd dimensions, the fact of there being no factors except of the form $m+i$, $2m+j$, is prepotent in determining the form of the result. For space of two dimensions the Bipotential does not appear readily to yield to summation in finite

[† See below, p. 51.]

if the distances from the centre of a spherical surface of two points *in the interior* be r, t, and the angle which the line joining them subtends at the centre be ω, then [for a sphere of radius c] the value of the bipotential of the surface at this point-pair is the elliptic integral

$$\int_0^{\frac{\sqrt{(rt)}}{c}} \frac{4\pi\, dx}{\sqrt{(1-2x^2\cos\omega+x^4)}},$$

which I take leave to call the Cardinal Theorem of Spherical Harmonics; for it is the theorem from which spring all the properties relating to the "surface-integral" of the product of any two rational forms of Laplace's coefficients.

Since every spherical harmonic of integral degree is a linear function of the differential derivatives of $(x^2+y^2+z^2)^{-\frac{1}{2}}$, the whole theory of the diplo-spherical-harmonic-surface integral is contained in the annexed equation,

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terms. Thus at one blow the theory of spherical harmonics has been extended to "globoidal" harmonics in general; and the chief cases of statical distribution of electricity heretofore solved may be regarded as virtually solved *mutatis mutandis* for space of any number of dimensions, of course with the proviso that the law of attraction (in consonance with the hypothetical principle of force-emanation to which the English school of physicists seem to be returning) is always to be supposed to vary as the $(i-1)$th power of the distance in space of i dimensions.

The actual expression for $\phi(m, n)$ when n is 3 we know is $\frac{4\pi}{2m+1}$. In general when n is any other odd number, I find that its value is

$$\frac{2\,(2\pi)^{\frac{n-1}{2}}}{(2m+n-2)\,(m+n-3)\,(m+n-4)\ldots\left(m+\frac{n-1}{2}\right)}.$$

As this expression may be split up into partial fractions, it is obvious that the value of the Bipotential may be expressed by means of the sum of integrals of the form

$$\int_\infty^1 \frac{u^j\,du}{\{\sqrt{(u^2+Au+B)}\}^{n-2}},$$

and one of the form

$$\int_\infty^1 \frac{du}{\{\sqrt{(u^4+Au^2+B)}\}^{n-2}};$$

so that it involves no transcendents of a higher order than an ordinary elliptic function. I think also that it follows from the limits to the value of j that the other integrals are mere algebraical functions. The less interesting case when n is an even number (being very much pressed for time and within twenty-four hours of steaming back to Baltimore) I have not taken the trouble to work out in detail.

The determination of the Bipotential constitutes in itself a vast accession to the theory of definite integrals, and promises to be fruitful in yielding whole new families of such when subjected to the usual processes performed under the sign of integration. But does the theory stop here? The success of my method for the Bipotential depends solely upon the discovery that, as regards internal points of *prise*, it may be regarded as a function of only two variables, rr' and $\cos\omega$. Now a Tripotential will obviously at first sight be a function of not more than six variables, viz. the three quantities r, r', r'' and the cosines of the angles between them; but it becomes a question whether this number also may not be reduced to be less than six, themselves simple functions of the six parts of a tetrahedron; and so for a multipotential of any order the question arises, Is it a function of $\frac{1}{2}m(m+1)$ quantities or of a smaller number? and if so, of what number of what variables?

which springs immediately from the expression found above for the bi-potential of a spherical surface at two internal points (slightly modified by taking $-h, -k, -l; -h', -k', -l'$ for the coordinates of the points) by means of the simple and familiar principle that any differential derivative with respect to x, y, z of a function of x, y, z is identical with what the corresponding derivative with respect to h, k, l of the like function of $x+h$, $y+k$, $z+l$ becomes when h, k, l are made to vanish.

Let U stand for $u^4 - 2u^2\Sigma hh' + \Sigma h^2 . \Sigma h'^2$, and let

$$V(h, k, l;\ h', k', l') = \int_\infty^1 \Phi\left(\frac{d}{dh}, \frac{d}{dk}, \frac{d}{dl}\right) \Psi\left(\frac{d}{dh'}, \frac{d}{dk'}, \frac{d}{dl'}\right) \frac{du}{\sqrt{U}},$$

where Φ and Ψ are forms of function which denote series, whether finite or infinite, containing only positive integer powers of the variables. Then, if $\rho = (x^2+y^2+z^2)^{-\frac{1}{2}}$ and dS is the element of a spherical surface of unit radius, the complete integral

$$\iint dS \left\{\Phi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\rho\Psi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\rho\right\} = 4\pi V(0, 0, 0;\ 0, 0, 0).$$

When Φ and Ψ are homogeneous forms of function, each of the degree i, if we write

$$T = 1 - 2\Sigma hh' + \Sigma h^2 . \Sigma h'^2,$$

and make

$$\Omega(h, k, l;\ h', k', l') = \Phi\left(\frac{d}{dh}, \frac{d}{dk}, \frac{d}{dl}\right)\Psi\left(\frac{d}{dh'}, \frac{d}{dk'}, \frac{d}{dl'}\right)\frac{1}{\sqrt{T}},$$

the value of the corresponding harmonic surface integral becomes

$$\frac{4\pi}{2m+1}\Omega(0, 0, 0;\ 0, 0, 0).$$

I am not aware that a rule for finding such integral so simple in form and of such absolute generality in operation as the one above has been given before; the interesting rule furnished by Professor Clerk Maxwell, *Electricity and Magnetism* (vol. I. p. 170), assumes that Φ and Ψ have been each reduced to the form of the product of linear functions of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$—a reduction which cannot practically be effected, as it involves the solution of systems of equations of a high order—not, however, so high as might at first sight be inferred from Professor Maxwell's statement that, for the case of i factors, it depends on the solution of a system of $2i$ equations of the ith degree, as the equations referred to (evidently those obtained by the use of the method of indeterminate coefficients in its crude form) would be of a special character: thus, for example, when $i=2$, the order of the system of the four quadratic equations sinks down from 4.2^3 or 32 (its value in the general case) to be only 3, as will presently be seen.

The method of poles for representing spherico-harmonics, devised or developed by Professor Maxwell, really amounts to neither more nor less than the choice of an apt canonical form for a ternary quantic, subject to the condition that the sum of the squares of its variables (here differential operators) is zero; and I am quite at a loss to understand how it can at all assist "in making the conception of the general spherical harmonic of an integral degree perfectly definite," or what want of definiteness apart from the use of this canonical form can be said to exist in the subject.

Since $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2$ retains its form when any orthogonal linear substitutions are impressed on x, y, z, we recognize *à priori* that a harmonic distribution on the surface of a sphere is invariantive in the sense that it bears no intrinsic relation to the particular set of axes which may happen to be used to express the value of the harmonic at each point of the surface; and the great merit, it seems to me, of Professor Maxwell's beautiful conception of harmonic poles is that it puts this fact in evidence: for it is easy to see at a glance, from the use of successive linear operators, that the harmonic at any variable point on the surface for any given degree (n) will depend in an absolutely determinate manner (save as to an arbitrary constant factor) on the cosines of the arcs joining it with n arbitrarily assumed fixed points on the sphere, and of the arcs joining those n points with one another (being in fact a symmetrical function of each of the two sets of cosines), so that intrinsic poles are substituted for extrinsic Cartesian axes. I am a little surprised that this distinguished writer should not have noticed that there is always one, and only one, *real* system of poles appertaining to any given harmonic, and that to find this system it is not necessary, as he has stated, to employ a system of n equations each of the order $2n$, but one single equation of that order. For calling $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ by the names ξ, η, ζ, then any given harmonic of the nth degree may be reduced by the use of mere linear equations to the form $(\xi, \eta, \zeta)^n \frac{1}{r}$, and the problem to be solved in order to find its poles is the purely algebraical one of converting the quantic

$$(\xi, \eta, \zeta)^n + \Lambda(\xi^2 + \eta^2 + \zeta^2),$$

where Λ is a quantic of the order $(n-2)$, into a product of linear factors. Now this again is merely the problem of finding a pencil of rays that shall pass through the intersections of the curve $(\xi, \eta, \zeta)^n$ with the curve $(\xi^2 + \eta^2 + \zeta^2)$; that is to say, any dispersal of the $2n$ intersections into n sets of two each will give a system of n polar factors in Professor Maxwell's problem. We have therefore only to find the values of $\xi : \eta : \zeta$ in the two simultaneous equations $(\xi, \eta, \zeta)^n = 0$, $\xi^2 + \eta^2 + \zeta^2 = 0$, and this leads to a resolving equation

of the $2n$th order. From the form of the second equation we see that the values $x:y:z$ are *all imaginary*; consequently there will be one, and but one, system of real rays, that is, real polars corresponding to the distribution of the $2n$ roots of the resolving equation into n *conjugate pairs.* The remaining systems (there are in all $1.3.5\ldots(2n-1)$ of them) will each contain imaginary elements, so that all or some of the poles become imaginary.

In the case of $n=2$, the problem becomes the familiar one of finding the principal axes of a cone of the second order; and instead of employing a biquadratic resolvent we make the discriminant of $(\xi, \eta, \zeta)^2+(\xi^2+\eta^2+\zeta^2)$ vanish, which of course only requires the solution of a cubic equation; but as subsequently (when the pair is to be divided into its elements) a new quadratic surd is introduced, we are virtually solving a biquadratic, in accordance with the general rule that, to find the poles of a spherical harmonic of the degree n, it is necessary to solve an equation of the degree $2n$.

To put the coping-stone to Professor Clerk Maxwell's method of poles, I think it would be desirable to find an intrinsic definition of spherical harmonics to correspond with their representation referred to intrinsic axes: I mean we ought to be able to dispense with the Laplacian operator altogether, and to define a Harmonic with sole reference to some algebraical or geometrical (but certainly not physical) condition which it satisfies in regard to its poles. With all possible respect for Professor Maxwell's great ability, I must own that to deduce purely analytical properties of spherical harmonics, as he has done, from "Green's theorem" and the "principle of potential energy" (*Electricity and Magnetism,* vol. I. p. 168), seems to me a proceeding at variance with sound method, and of the same kind and as reasonable as if one should set about to deduce the binomial theorem from the law of virtual velocities, or make the rule for the extraction of the square root flow as a consequence from Archimedes' law of floating bodies.

POSTSCRIPT. NOTE ON SPHERICAL HARMONICS.

The value of $\phi(m, n)$ is stated inaccurately in the long footnote at pp. [46, 47]. If

$$\Omega_i = \frac{(2\pi)^i}{1.3.5\ldots(2i-1)}$$

and $$R = \sqrt{(1 - 2\Sigma hh'.t^2 + \Sigma h^2.\Sigma h'^2.t^4)}$$

then I find $$\phi(m, 2i+1) = \frac{(2i-1)\,\Omega_i}{2m+2i-1};$$

and accordingly the Bipotential in space of $2i+1$ dimensions is

$$\int_1^0 \frac{\Omega_i dt^{2i-1}}{R^{2i-1}}.$$

Also I find that in space of $2i+2$ dimensions the prospherical Bipotential is

$$\frac{2\pi^i}{1.2.3\ldots i}\int_1^0 \frac{dt^i}{(1 - 2\Sigma hh'.t + \Sigma h^2.\Sigma h'^2.t^2)^i}.$$

The above results may be extended to general quadric surfaces and prosurfaces. Thus, for example, if an indefinitely thin ellipsoidal shell be contained between two concentric surfaces, the equation to one of which is $G(x, y, z) = 1$, where G is a general quadric, and if the squared density at x, y, z is the reciprocal of

$$G(x-h,\ y-k,\ z-l).G(x-h',\ y-k',\ z-l'),$$

then the mass of the shell divided by its volume is

$$\int_1^0 \frac{dt}{\sqrt{(1 - At^2 + Bt^4)}},$$

where $$A = \Sigma\left(h\frac{d}{dx}\right).\Sigma\left(h'\frac{d}{dx}\right)G(x, y, z),$$

and $$B = G(h, k, l).G(h', k', l').$$

It is further noticeable that if F and G are contravariantive forms, each numerator of the fractions expressing the differential derivatives of

$\dfrac{1}{\sqrt{\{G(x, y, z)\}}}$ is nullified by the operator

$$F\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right);$$

and conversely, every rational integer function of x, y, z so nullifiable is a linear function of such numerators. And so in general the Theory of Spherical and Prospherical Harmonics merges in a theory of Conicoidal and Proconicoidal Harmonics.

6.

SUR LES INVARIANTS FONDAMENTAUX DE LA FORME BINAIRE DU HUITIÈME DEGRÉ.

[*Comptes Rendus*, LXXXIV. (1877), pp. 240—244, 532—534.]

On sait que le nombre des invariants linéairement indépendants de l'ordre j, appartenant à une forme binaire du degré i, est égal à la différence de deux nombres dont l'un, le plus grand, est le nombre de manières de représenter $\frac{ij}{2}$ comme la somme de j nombres (avec des répétitions à volonté) choisis entre les nombres 0, 1, 2,..., i, et l'autre est le nombre de manières de former $\frac{ij}{2}-1$ selon la même loi.

Ainsi le nombre d'invariants de l'ordre n, appartenant à la forme binaire du degré 8, est la différence entre deux *dénumérants*, l'un du système

$$\begin{aligned} y + 2z + 3t + 4u + 5v + 6w + 7\rho + 8\sigma &= 4n, \\ x + y + z + t + u + v + w + \rho + \sigma &= n; \end{aligned}$$

l'autre du système

$$\begin{aligned} y + 2z + 3t + 4u + 5v + 6w + 7\rho + 8\sigma &= 4n - 1, \\ x + y + z + t + u + v + w + \rho + \sigma &= n. \end{aligned}$$

On comprendra que le dénumérant d'une équation ou d'un système d'équations simultanées en nombres entiers veut dire le nombre de solutions que cette équation ou ce système admet en nombres entiers.

Or j'ai démontré ailleurs que le dénumérant d'un système quelconque d'équations simultanées peut toujours s'exprimer au moyen de dénumérants simples, c'est-à-dire appartenant chacun à une seule équation, et l'on trouvera sans difficulté que la différence entre les deux dénumérants dont il est ici question sera le coefficient de t^n dans la fonction génératrice

$$G = \frac{1 + t^8 + t^9 + t^{10} + t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)}. \tag{1}$$

Ce résultat est parfaitement d'accord avec l'expression donnée par M. Cayley dans son *Second Memoir on Quantics*, c'est-à-dire

$$\frac{(1-x)(1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{12}-x^{13}+x^{15}+x^{16})}{(1-x^2)^2(1-x^3)^2(1-x^4)(1-x^5)(1-x^7)};$$

car on trouvera, par un calcul algébrique des plus simples, que ces deux fonctions génératrices sont identiques en valeur.

En vertu de la forme donnée à G dans l'équation (1), on peut immédiatement déduire les conséquences suivantes, que je nommerai désormais, si j'ai occasion de les citer, *principes:*

(1) Il existe des invariants, appartenant à la forme binaire octavique des ordres 2, 3, 4, 5, 6, 7, que je nomme les *invariants primaires.*

(2) Il existe* quatre invariants des ordres 8, 9, 10, 18, disons x, y, z, θ, tels, que chaque autre invariant peut s'exprimer comme une fonction linéaire de x, y, z, θ, les coefficients et le terme constant de telle fonction étant des fonctions rationnelles et entières des invariants primaires.

(3) Il sera impossible de former aucune équation linéaire de la nature exprimée plus haut entre x, y, z, θ.

(4) x, y, z seront indépendants entre eux. Quant à θ, il y aura deux hypothèses à faire: ou il est indépendant de x, y, z, ou l'on peut prendre pour sa valeur une fonction linéaire quelconque de xz et y^2.

Je démontrerai que la dernière hypothèse doit être rejetée, c'est-à-dire qu'il existe en effet un invariant fondamental de l'ordre 18, de sorte que le système complet des invariants se composera de six, que je nomme *primaires*, dont les ordres sont 2, 3, 4, 5, 6, 7, et cinq dont les ordres sont 0, 8, 9, 10, 18, que je nommerai *secondaires,* car on ne doit jamais oublier que la constante 1 est un invariant du degré zéro.

Traitons désormais les invariants primaires comme des constantes, cela facilitera beaucoup la parole dans cette dissertation.

Supposons que y^2, zx ne puissent pas s'exprimer séparément comme fonctions linéaires de x, y, z, 1; puisque, pour une valeur quelconque de x, y, z, on peut substituer $ax+b$, $cy+d$, $ez+fx+g$, on verra facilement qu'on peut instituer les équations suivantes entre x, y, z:

$$xz-y^2=T, \tag{2}$$

$$x^2=Ax+By+Cz+D, \tag{3}$$

$$xy=A'x+B'y+C'z+D', \tag{4}$$

$$yz=L'x+M'y+N'z+P', \tag{5}$$

$$z^2=Lx+My+Nz+P, \tag{6}$$

[* Cf. p. 62 below.]

où l'on remarquera que l'on a fait disparaître le terme xz ou y^2 dans l'équation pour z^2 par le moyen du multiplicateur arbitraire λ, qu'on peut ajouter à z. Multiplions

(2) par x et z,
(3) y,
(4) x et z,
(5) x et z,
(6) y.

Il est facile de voir qu'en faisant les éliminations dialytiques convenables, on obtiendra cinq équations linéaires entre x, y, z, xz, y^2, 1. De plus, dans chacune de ces équations, le coefficient de xz doit être égal à celui de y^2; car, si c'est combiné avec (2), on trouvera xz et y^2 comme fonctions linéaires contraires à l'hypothèse de x, y, z. Ainsi chacune de ces cinq équations doit être une identité et fournira ainsi cinq liaisons entre les coefficients, de sorte qu'on pourrait attendre de trouver vingt-cinq de ces liaisons; mais, en faisant le calcul, on trouvera qu'il n'y a plus que onze indépendantes, que j'écris de la manière suivante:

(1) Le groupe $B' = A$, $C' = B$, $L' = M$, $M' = N$, $N' = A'$; de sorte que l'on peut substituer respectivement les lettres

$$\begin{matrix} A, & B, & C, \\ K, & A, & B, \\ M, & N, & K, \\ L, & M, & N, \end{matrix}$$

au lieu de

$$\begin{matrix} A, & B, & C, \\ A', & B', & C', \\ L', & M', & N', \\ L, & M, & N. \end{matrix}$$

Il y aura encore un groupe de cinq équations que voici :

$$D = BK - CN, \tag{12}$$
$$P = MK - AL, \tag{13}$$
$$D' = CM - AK, \tag{14}$$
$$P' = LB - NK, \tag{15}$$
$$T = CL - K^2, \tag{16}$$

et finalement

$$CL - AN = 0. \tag{17}$$

Mais on peut obtenir encore une nouvelle équation identique en multipliant (2) par xz, (3) par (6), et (4) par (5); car on a

$$xzT = x^2 . z^2 - xz . yz.$$

Les cinq liaisons qui en résultent seront indépendantes entre elles-mêmes, mais une d'elles ne sera qu'une répétition de (16). Les quatre qui restent sont toutes nouvelles et peuvent s'écrire

$$2(KBM - CMN - ALB + ANR) = 0, \tag{18}$$

$$LB^2 - 2KNB + CN^2 + AT = 0, \tag{19}$$

$$CM^2 - 2AKM + LA^2 + NT = 0, \tag{20}$$

$$2(BM - AN)T = 0. \tag{21}$$

Ainsi l'on a

$$\text{ou } T = 0, \quad \text{ou } BM = AN.$$

Si $BM = AN$, on obtient, en combinant avec (18) et (19), $AT = 0$, et, en combinant avec (18) et (20), $NT = 0$. Donc

$$\text{ou } T = 0, \quad \text{ou } A = 0, \quad N = 0, \quad \text{et } \quad BM = 0.$$

Mais si

$$N = 0 \quad \text{et} \quad B = 0, \quad D = 0,$$

et si

$$A = 0 \quad \text{et} \quad M = 0, \quad P = 0.$$

Donc

$$\text{ou } T = 0, \quad \text{ou } A = 0, \quad B = 0, \quad D = 0, \quad \text{ou } N = 0, \quad M = 0, \quad P = 0.$$

Donc on a

$$xz = y, \quad \text{ou } x^2 = Cz, \quad \text{ou } z^2 = Lx;$$

mais chacune de ces équations est inadmissible. Donc l'hypothèse que θ n'est pas indépendant est fausse, et nous avons établi que les invariants *secondaires* de la forme binaire octavique sont respectivement de l'ordre

$$0, \ 8, \ 9, \ 10, \ 18.$$

J'ajoute que, pour obtenir les *principes* qui ont conduit à ce résultat, on n'a besoin de s'appuyer sur aucune autre chose que la forme même de la fonction génératrice prise en conjonction avec la vérité intuitive que chaque combinaison d'invariants est elle-même un invariant.

A cause d'une erreur qui s'est glissée dans le *Second Memoir on Quantics* de M. Cayley, dans son explication des conséquences qui découlent de la fonction génératrice pour les covariants appartenant aux formes au-dessus du quatrième et les invariants au-dessus du sixième degré, on a pensé (voir *Théorie des formes binaires,* de M. Faà de Bruno, p. 150) que la théorie elle-même est en défaut et que les équations linéaires qui conduisent à cette fonction, après qu'un certain point est passé, cessent d'être indépendantes. J'ai examiné cette question de près et j'arrive à la *certitude* du contraire.

En effet, l'indépendance de ces équations est une conséquence d'un théorème très-curieux que j'ai découvert, un théorème plutôt de position que d'arithmétique que voici. Prenons trois nombres quelconques i, j, w avec la seule condition que w ne soit pas plus grand que $\frac{ij}{2}$. Formons toutes les combinaisons possibles avec les chiffres $0, 1, 2, \ldots, i$, qui donnent la somme w: que le nombre de ces partitions soit m et qu'elles soient nommées $P_1, P_2, \ldots, P_m$.

De même formons toutes les partitions semblables avec la somme $w-1$, que leur nombre soit μ et nommons-les $\Pi_1, \Pi_2, \ldots, \Pi_\mu$.

On doit observer que le nombre m ne peut jamais devenir plus petit que μ, à cause de la condition que w n'est pas plus grand que $\frac{ij}{2}$.

Quand un Π quelconque, disons Π_λ, peut être déduit d'un P quelconque, disons P_l, par moyen de diminuer un des chiffres qui y entrent par l'unité, je nomme Π_λ une dérivée de P_l et dans le cas contraire une non-dérivée.

Formons un rectangle de m sur μ et à la tête des colonnes écrivons les sommes P et à côté de chaque ligne les sommes Π. De cette manière on peut dire que chaque place dans le rectangle aura une certaine longitude désignée par un P et une latitude désignée par un Π. Dans chaque place dont la latitude est une *dérivée* de la longitude, écrivons un signe quelconque, par exemple une croix, et dans toutes les autres places insérons des zéros. Par une diagonale d'une *matrice* carrée, comprenons une combinaison quelconque des positions occupées par ces éléments qui entrent dans la valeur du déterminant qui y appartient. Ces diagonales se diviseront, selon la règle élémentaire pour le calcul des déterminants, en deux espèces positives et négatives. De plus on peut sous-entendre par une diagonale *effective* une diagonale dans laquelle il n'entre nul *zéro*.

Or, avec le rectangle dont j'ai parlé, formons toutes les matrices carrées *complètes* possibles, c'est-à-dire des carrés de μ^2 plans. Il peut arriver que, pour un certain nombre d'entre elles, il n'y aura nulle diagonale effective, mais on peut démontrer qu'il en existe toujours une au moins qui possède une ou plusieurs diagonales. S'il n'y a qu'une seule diagonale effective, évidemment le déterminant ne peut pas s'évanouir; mais s'il y en a plusieurs, alors je dis que toutes ces diagonales effectives pour un déterminant donné porteront le même signe, de sorte que, si l'on donne des valeurs positives quelconques aux éléments désignés par des croix, la somme des produits qui correspondent à ces diagonales ne peut pas devenir égale à zéro. Cette proposition, fort remarquable, suffit pour démontrer la suffisance de la règle mise en doute par M. de Bruno. Pour trouver le nombre total de covariants appartenant à une forme donnée du degré i, d'un ordre donné j dans les

coefficients, et d'un degré donné k dans les variables, on n'aura qu'à prendre la différence de deux dénumérants de deux systèmes de deux équations simultanées dans l'une desquelles les termes constants seront $\frac{ij-k}{2}$ et dans l'autre $\frac{ij-k}{2}-i$. Comme conséquence de ce théorème, il est facile de démontrer que le nombre total des covariants de l'ordre j n'est autre chose que le nombre de manières de former la somme $\frac{ij}{2}$ ou $\frac{ij-1}{2}$ avec j des chiffres $0, 1, 2, 3, \ldots, i$.

Toutes ces conclusions se trouvent peut-être étendues à des *systèmes* de formes binaires.

Par exemple, si l'on considère le cas de deux formes binaires seulement, disons des degrés i et i', et si, pour plus de simplicité, on traite le problème du nombre *total* de covariants de l'ordre j dans ces coefficients par rapport à une des deux formes, et j' par rapport à l'autre, ce nombre sera le dénumérant d'un système ternaire d'équations simultanées en nombres entiers, que voici:

$$y+2z+3t+\ldots+it+\eta+2\zeta+\ldots+i'\tau=\frac{ij+i'j'+\epsilon}{2}, \qquad (1)$$

$$x+y+z+\ldots+t=j, \qquad (2)$$

$$\zeta+\eta+\quad\ldots+\tau=j', \qquad (3)$$

ϵ étant égal à zéro si $ij+i'j'$ est pair, et à -1 dans le cas contraire, et ainsi, en général, pour un système contenant un nombre de formes quelconques.

Le théorème qui porte à la démonstration de l'indépendance des équations linéaires dont il a été fait mention plus haut peut être mis sous une forme plus générale, que voici:

Soit Q une quantité quelconque d'un ou plusieurs systèmes de variables $x, y, z, \ldots, x', y', \ldots, x'', y'', \ldots$.

Prenons l'émanant de cette quantité par rapport à $\xi, \eta, \ldots, \xi', \eta', \ldots, \xi'', \eta'', \ldots$.

Substituons, pour $\xi, \eta, \ldots$, des fonctions linéaires *omnipositives* quelconques de $x, y, z \ldots$, pour $\xi', \eta', \ldots$ des fonctions linéaires *omnipositives* de $x', y', \ldots$, de sorte qu'on obtiendra une nouvelle quantité tout à fait semblable, dans sa constitution, à Q, mais dont les coefficients seront fonctions linéaires des coefficients de Q. Alors je dis que ces fonctions linéaires seront nécessairement indépendantes entre elles. Par une fonction linéaire omnipositive on comprendra facilement que je désigne une fonction linéaire dont tous les coefficients sont des quantités qui ne sont ni négatives ni nulles.

7.

SUR UNE MÉTHODE ALGÉBRIQUE POUR OBTENIR L'ENSEMBLE DES INVARIANTS ET DES COVARIANTS FONDAMENTAUX D'UNE FORME BINAIRE ET D'UNE COMBINAISON QUELCONQUE DE FORMES BINAIRES.

[*Comptes Rendus*, LXXXIV. (1877), pp. 1113—1116, 1211—1213.]

J'AI complètement résolu ce grand problème de trouver le système complet des invariants et covariants fondamentaux, que j'appellerai désormais les radicaux (*grundformen*) d'une forme binaire ou d'une combinaison quelconque de formes binaires, par une méthode purement algébrique tirée de l'équation partielle différentielle, à laquelle chaque *différentiant* binaire est assujetti. Par le mot *différentiant*, je désigne une fonction rationnelle quelconque des différences des racines d'une forme binaire donnée ou de chacune de telles formes, s'il y en a plus d'une, données. A l'aide de cette équation, j'obtiens une fonction, dite *génératrice* pour le système, sous la forme d'une fraction rationnelle contenant une variable, en raison du nombre des formes dans le système donné, laquelle fraction étant développée d'une telle façon que, dans la série qui en résulte, toutes les puissances des variables portent des indices positifs; le coefficient de chacune de ces puissances répondra au nombre des covariants ou invariants, linéairement indépendants, dont le degré et les ordres sont égaux respectivement aux indices de la puissance. Pour obtenir les radicaux (*grundformen*) du système, cette fonction doit être présentée, non sous sa forme réduite, mais d'une telle façon, que les indices des facteurs dont le dénominateur sera composé répondront chacun au degré et aux ordres d'un invariant ou covariant actuellement existant, comme il est toujours possible de le faire. Alors les indices du dénominateur répondront aux indices, pour ainsi dire, d'un radical appartenant à ce que j'appelle la *classe des primaires.*

Les radicaux secondaires seront obtenus au moyen du numérateur de la génératrice, en soumettant à une règle très-simple de *tamisement* l'ensemble des termes portant des coefficients positifs qui s'y présentent. J'ajouterai, pour plus de clarté, la génératrice dans quatre cas où j'aurai

le moyen de comparer mes résultats avec ceux de M. le professeur Gordan. Pour les trois premiers cas, l'accord entre les deux méthodes est parfait; pour le quatrième cas, sur trente radicaux donnés par M. Gordan, vingt-huit se présentent dans mon résultat, les deux qui manquent ayant disparu dans le procédé dit de *tamisement*. J'ai démontré catégoriquement que M. Gordan s'est trompé sur ces deux formes en les supposant fondamentales; elles doivent être et sont, en effet, décomposables, c'est-à-dire peuvent être exprimées comme sommes de combinaisons des radicaux inférieurs, de sorte que, pour un système de deux formes du quatrième degré, le nombre des covariants fondamentaux biquadratiques est 7 et des covariants du sixième degré 5, et non pas 8 et 6, comme M. Gordan l'avait pensé. J'ai même déterminé les coefficients numériques qui entrent dans ces deux sommes, de sorte qu'il ne reste pas la moindre ombre de doute sur la justesse de cette rectification. C'est le grand avantage que possède cette nouvelle méthode sur l'ancienne. De l'aveu même de M. Gordan, on ne peut jamais, en se servant de cette méthode (la méthode des hyperdéterminants), s'assurer d'une manière absolue que les formes réputées fondamentales sont telles en effet. Dans ma méthode, qui distingue les radicaux en deux classes, les primaires se présentent immédiatement à première vue, et les secondaires s'obtiennent en *tamisant* (selon une règle numérique des plus simples) un ensemble de formes qui se présentent simultanément avec les primaires.

(1) Soit donnée une seule forme binaire du cinquième degré.

La génératrice, sous sa forme canonique, sera la fraction dont le dénominateur est

$$(1-t^4)(1-t^8)(1-t^{12})(1-tu^5)(1-t^2u^6)(1-t^2u^2)$$

et le numérateur

$$\begin{aligned}1 &+ t^{18} + (t^5+t^7+t^{11}+t^{13})\,v + (t^6+t^8+t^{10}+t^{12}+t^{16}-t^{20})\,v^2\\ &+ (t^3+t^5+t^9+t^{11})\,v^3 + (t^4+t^6+t^8+t^{10}+t^{14}-t^{18})\,v^4\\ &+ (t^3+t^7+t^9-t^{19})\,v^5 + (t^4-t^{14}-t^{15}-t^{20})\,v^6\\ &+ (t^5-t^9-t^{13}-t^{15}-t^{17}-t^{19})\,v^7 - (t^{12}+t^{14}+t^{18}+t^{20})\,v^8\\ &+ (t^3-t^7-t^{11}-t^{13}-t^{15}-t^{17})\,v^9 - (t^{10}+t^{12}+t^{16}+t^{18})\,v^{10}\\ &- (t^5+t^{23})\,v^{11}.\end{aligned}$$

On déduit immédiatement du dénominateur 4, 0; 8, 0; 12, 0, trois invariants, 1, 5; 2, 6; 2, 2, trois covariants (dont le premier est la forme donnée elle-même); ces sept formes sont les radicaux primaires de la forme donnée.

Pour trouver les radicaux secondaires, on soumet au procédé de *tamisement* les formes ayant pour indices

18, 0; 5, 1; 7, 1; 11, 1; 13, 1; 6, 2; 8, 2; 10, 2; 12, 2; 16, 2; 3, 3; 5, 3; 9, 3; 11, 3; 4, 4; 6, 4; 8, 4; 10, 4; 14, 4; 3, 5; 7, 5; 9, 5; 1, 6; 1, 7.

La règle de tamisement enseigne à négliger les couples 10, 2; 12, 2; 16, 2; 11, 3; 10, 4; 14, 4; 9, 5, parce que ces couples se forment en additionnant des couples inférieurs (l'addition des couples $f.g$, $h.k$ signifie le couple $(f+k).(g+k)$). Il reste dix-sept couples qui répondent aux ordres et aux degrés des radicaux secondaires.

Voici la règle générale pour le tamisement:

Supposons que par le tamisement on ait déjà obtenu un certain nombre de couples irréductibles et qu'on trouve un nouveau couple $i.j$ avec le coefficient μ. On détermine le nombre M de manière à former ce dernier couple en additionnant les couples inférieurs avec eux-mêmes ou les uns avec les autres. Alors, si M est inférieur à μ, on aura $(\mu - M)$ radicaux secondaires avec les indices $i.j$, on comptera $\mu - M$ fois ce couple et l'on continuera le procédé de tamisement comme auparavant. Si $\mu - M$ est zéro ou négatif, il n'y aura aucun secondaire du type $i.j$. Dans le dernier cas, la valeur numérique de la différence $\mu - M$ indiquera l'existence de ce nombre de rapports syzygétiques entre les radicaux des deux espèces des degrés i pour les coefficients et j pour les variables.

Dans le cas traité ci-dessus, toutes les valeurs de μ sont l'unité. Il résulte de ce qui a été fait que l'ensemble du système radical contient vingt-trois formes que l'on trouvera identiques avec celles données par Clebsch dans son *Traité sur les formes binaires*, p. 277.

(2) Prenons la forme binaire du sixième degré. On trouve pour génératrice la fraction dont le dénominateur est

$$(1-t^2)(1-t^4)(1-t^6)(1-t^{10})(1-t^2v^4)(1-t^2v^8)(1-tv^6)$$

et le numérateur

$$\begin{aligned}(1+t^{15}) &+ (t^3+t^5+t^7+t^8+t^{10}+t^{12})\,v^2\\ &+ (t^4+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{13}-t^{17})\,v^4\\ &+ (t^3+t^4+2t^6+t^8+t^9+t^{11}-t^{16})\,v^6\\ &+ (t^3+t^5+t^7-t^{13}-t^{15}-t^{17})\,v^8\\ &+ (t^4-t^9-t^{11}-t^{12}-2t^{14}-t^{16}-t^{17})\,v^{10}\\ &+ (t^3-t^7-t^9-t^{10}-t^{11}-t^{12}-t^{13}-t^{14}-t^{15}-t^{16})\,v^{12}\\ &- (t^5+t^{20})\,v^{16} - (t^8+t^{10}+t^{12}+t^{13}+t^{15}+t^{17})\,v^{14}.\end{aligned}$$

Le procédé de tamisement fera disparaître

6, 4; 8, 4; 10, 4; 11, 4; 13, 4; 8, 6; 9, 6; 11, 6; 7, 8.

Il y aura donc sept radicaux primaires et dix-neuf secondaires, en tout les vingt-six *bildungen* posés par Clebsch (*Formen binären*, p. 296).

(3) Prenons le système comprenant deux formes binaires, l'une biquadratique, l'autre quadratique. En faisant rapporter la variable T à la quadratique et t à la biquadratique, je trouve que la génératrice, sous sa forme canonique, aura pour dénominateur

$$(1-t^2)(1-t^3)(1-T^2)(1-T^2t)(1-T^2t^2)(1-Tv)(1-tv^2)(1-t^2v^2)$$

et pour numérateur

$$\begin{aligned}(1+T^3t^3)&+[(T+T^2)t+(T+T^2)t^2+(T^2-T^4)t^3]v^2\\&+[Tt+Tt^2+(T-T^3)t^3-T^3t^4-T^3t^5]v^4\\&+[(1-T^2)t^3-(T^2+T^3)t^4-(T^2+T^3)t^5]v^6-(Tt^3+T^4t^6)v^8.\end{aligned}$$

Ici aucun des termes du numérateur ne disparaît par l'opération du tamisement, et il y aura 8 primaires, 10 secondaires, 18 *grundformen* en tout, ce qui est d'accord avec les résultats déjà obtenus. (Voir *Salmon's Lessons*, 3e édition, p. 200.)

Finalement, je considérerai le cas *crucial*, où M. Gordan et moi nous sommes en désaccord, de deux formes biquadratiques. Pour plus de brièveté, je ne donnerai que la première moitié des termes du numérateur; on peut obtenir le reste de ces termes (qui n'influe nullement sur le résultat, tous les coefficients positifs dans cette partie, 25 en nombre, s'évanouissant dans le procédé de *tamisement*) par la règle suivante: *A chaque terme, dans la première partie, correspondra un terme dans la seconde partie du numérateur, tel que le produit des deux termes sera* $T^7 . t^7 . v^{14}$.

Or je dis que le dénominateur de la génératrice sera

$$(1-T^2)(1-T^3)(1-t^2)(1-t^3)(1-Tt)(1-Tt^2)(1-tT^2)(1-Tv^4)$$
$$(1-T^2v^4)(1-tv^4)(1-t^2v^{14}),$$

et la première partie du numérateur (la seule effective) sera

$$\begin{aligned}&(1+T^2t^2+T^4t^4)\\&+[(Tt)+(T^2t+Tt^2)+(Tt^3+T^2t^2+T^3t)+(T^2t^3+T^3t^2)+T^3t^3]v^2\\&+(Tt+Tt^2+T^2t+T^2t^2+T^3t^3+T^4t^3+T^3t^4-T^5t^4-T^4t^5-T^6t^4-T^4t^6)v^4\\&+[(Tt+T^3+T^2t+Tt^2+t^3+T^2t^2-Tt^4-T^2t^3-T^3t^2-T^4t\\&\qquad -Tt^5-2T^2t^4-3T^3t^3-2T^4t^2-T^5t-T^5t^2-2T^4t^3\\&\qquad\qquad -2T^3t^4-T^2t^5-T^3t^5-T^5t^3)]v^6.\end{aligned}$$

Par l'opération de *tamisement* opérée sur les termes du numérateur, il ne restera que les triplets

2.2.0, 1.1.2, 2.1.2, 1.2.2, 1.3.2, 2.2.2, 3.1.2, 2.3.2, 3.2.2,
1.1.4, 1.2.4, 2.1.4,
1.1.6, 3.0.6, 0.3.6, 2.1.6, 1.2.6.

Observez que les triplets 2.2.4, 2.2.6 disparaissent, comme étant respectivement les sommes de triplets inférieurs. Ainsi il y aura 17 *grundformen* secondaires et 11 primaires, faisant ensemble le nombre 28.

J'ai calculé aussi la génératrice pour la forme du huitième degré; mais elle est trop longue pour être reproduite ici. La partie de cette fonction appartenant aux invariants a été déjà donnée par moi, dans sa forme canonique, dans la première de mes deux Communications récentes à l'Académie.

Le dénominateur est

$$(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7);$$

le numérateur est

$$1+t^8+t^9+t^{10}+t^{18}.$$

Je profite de cette occasion pour corriger une erreur dans la Communication que j'avais envoyée par dépêche télégraphique. Les radicaux primaires invariants seront, comme je l'avais remarqué, 6 en nombre et des degrés 2, 3, 4, 5, 6, 7 par rapport aux coefficients; mais les secondaires seront 3 et non pas 4 en nombre et des degrés 8, 9, 10 respectivement. Le tamisement fera disparaître l'indice 18 tout à fait, et, comme dans le cas de deux formes biquadratiques, cette opération de tamisement fait disparaître l'invariant double correspondant au terme T^4t^4 dans le numérateur.

J'ai obtenu la génératrice pour les invariants appartenant à la forme du septième et à la forme du dixième degré; dans cette dernière, c'est fort remarquable, un invariant de degré impair 9 figure parmi les radicaux secondaires. Je crois aussi être sur la voie pour faire l'extension de cette méthode aux formes et systèmes de formes de *dimensions* supérieures à la seconde, c'est-à-dire de formes ternaires, quaternaires, &c.; mais il faut réserver pour quelque autre occasion ce que j'ai à dire sur ce sujet et sur la méthode dont je me suis servi pour former la fonction génératrice des formes binaires. Je dois ajouter que l'erreur que j'ai commise dans ma démonstration prétendue de l'existence d'un radical du degré 18, pour les formes octaviques binaires, paraît consister dans l'hypothèse, mal fondée, de l'impossibilité de l'existence d'une équation syzygétique, dans laquelle les x, y, z figurent seulement au premier degré.

8.

SUR LE VRAI NOMBRE DES COVARIANTS ÉLÉMENTAIRES D'UN SYSTÈME DE DEUX FORMES BIQUADRATIQUES BINAIRES.

[*Comptes Rendus*, LXXXIV. (1877), pp. 1285—1289.]

DANS une récente Communication que j'ai eu l'honneur d'adresser à l'Académie, j'ai remarqué que ma méthode pour obtenir les *grundformen* d'un système de deux formes biquadratiques ne donne raison qu'à supposer l'existence de 28 invariants et covariants élémentaires, tandis que M. le professeur Gordan en a fourni une Table de 30. J'ai appris qu'outre M. Salmon, qui a adopté les conclusions de M. Gordan sans examen, M. le professeur Bertini pense aussi, de son côté, en avoir confirmé la justesse. Il importe donc dans le plus haut degré au progrès de l'Algèbre que ce point ne puisse rester douteux; c'est pourquoi j'ai pris la liberté d'exposer dans les *Comptes rendus* la preuve concluante que deux des formes données par M. Gordan sont superflues, c'est-à-dire qu'elles ne sont en effet que des combinaisons algébriques d'autres formes contenues dans sa Table.

On me présente deux corps qu'on affirme être des corps simples: sans me donner la peine de démontrer (comme il sera facile dans le cas actuel) l'impossibilité qu'il en existe de tels possédant les caractères qu'on leur attribue; je vais démontrer l'erreur de cette affirmation, en effectuant pour ainsi dire leur décomposition sous les yeux mêmes du lecteur.

Le travail de cette décomposition sera beaucoup abrégé par la considération suivante. Quand le premier terme d'un covariant quelconque est donné, le covariant lui-même est donné; car, en vertu de l'équation différentielle partielle à laquelle chaque covariant satisfait, de ce premier terme découlent tous les autres au moyen d'opérations explicites de différentiation et d'addition exclusivement. Ainsi, pour prouver qu'un covariant donné est la somme d'autres covariants, il suffit de démontrer que le coefficient du premier terme de l'un est la somme des coefficients des premiers termes des autres.

Or servons-nous en général du symbole ijk pour désigner le coefficient de x^k dans un covariant élémentaire dont l'ordre, par rapport aux coefficients

d'une forme biquadratique binaire donnée, est i, par rapport aux coefficients d'une autre forme semblable j, et dont le degré, relatif aux variables x, y, est k.

Posons

$$a,\ 4b,\ 6c,\ 4d,\ e, \qquad \alpha,\ 4\beta,\ 6\gamma,\ 4\delta,\ \epsilon$$

pour les coefficients des deux formes biquadratiques données.

Alors, en suivant les prescriptions données par M. Gordan lui-même, on trouvera facilement les valeurs suivantes pour les invariants et covariants fondamentaux dont l'existence n'est pas douteuse, c'est-à-dire

$$1.1.2 = a\delta - 3b\gamma + 3c\beta - d\alpha, \qquad 1.1.4 = a\gamma - 2b\beta + c\alpha,$$

$$1.1.0 = a\epsilon - 4b\delta + bc\gamma - 4d\beta + e\alpha, \qquad 1.1.6 = a\beta - b\alpha,$$

$$1.0.4 = a,$$

$$1.2.2 = a(\gamma\delta - \beta\epsilon) + b(\alpha\epsilon + 2\beta\delta - 3\gamma^2) + 3c(\beta\gamma - \alpha\delta) + 2d(\alpha\gamma - \beta^2),$$

$$0.1.4 = \alpha,$$

$$2.1.2 = \alpha(be - cd) - \beta(ae + 2bd - 3c^2) + 3\gamma(ad - bc) - 2\delta(ac - b^2).$$

On comprend que dans ce qui précède j'aie réduit chaque expression à sa forme numérique la plus simple. Le signe algébrique est disponible à volonté, et j'ai attribué à chacune le signe le plus commode pour mettre en évidence le rapport numérique qui lie le produit de chaque couple à la forme 2 . 2 . 6 dite *élémentaire* par M. Gordan, dont la valeur (voir *Salmon's Lessons on higher Algebra*, 3^e^ édition, p. 206) est obtenue de la manière suivante. Dans la *hessienne* d'une des formes données pour x, y, écrivez x_1, y_1, dans l'autre x_2, y_2. Multipliez ces deux hessiennes ainsi modifiées ensemble et opérez sur ce produit avec le symbole $\left(\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}\right)^2$, et, dans le résultat, remplacez x_1, x_2 par x; y_1, y_2 par y; c'est la méthode de M. Gordan pour obtenir sa forme 2 . 2 . 6, traduite dans le langage des hyperdéterminants. Moins 6 fois ce résultat pris dans sa forme arithmétique réduite (et affectée d'un signe algébrique convenable) sera la somme des quatre produits précédents, comme on le verra par la Table ci-jointe, où l'on remarquera que la somme des chiffres de chaque colonne sera égale à zéro.

La manière de comprendre cette Table s'explique d'elle-même. Par exemple, la seconde ligne enseigne que le produit 1 . 1 . 0 par 1 . 1 . 6 sera égal à

$$[a^2\beta\epsilon - ab\alpha\epsilon - 4ab\beta\delta + 6ac\beta\gamma \ldots],$$

et de même pour les autres lignes. La dernière ligne montre que la forme de l'ordre 2 dans chaque système de coefficients et du degré 6 en x et y, citée par M. Gordan comme un covariant fondamental calculé selon la règle donnée par lui, aura le coefficient de x^6 égal à

$$ac\alpha\delta - ac\beta\gamma - b^2\alpha\delta + b^2\beta\gamma - ad\alpha\gamma + ad\beta^2 + bc\alpha\gamma - bc\beta^2.$$

A.—Table *pour effectuer la décomposition de la forme dite élémentaire du type* 2 . 2 . 6 *de M. Gordan*

	$aa\beta\epsilon$	$aa\gamma\delta$	$aba\epsilon$	$ab\beta\delta$	$ab\gamma\gamma$	$aca\delta$	$ac\beta\gamma$	$bba\delta$	$bb\beta\gamma$	$ada\gamma$	$ad\beta\beta$	$bca\gamma$	$bc\beta\beta$	$aea\beta$	$bda\beta$	$cca\beta$	$beaa$	$cdaa$
1 . 1 . 2 × 1 . 1 . 4		−1		+2	+3	−1	−3		−6	+1		+3	+6		−2	−3		+1
1 . 1 . 0 × 1 . 1 . 6	1		−1	−4			6	4			−4	−6		1	4		−1	
1 . 0 . 4 × 1 . 2 . 0	−1	1	1	2	−3	−3	3			2	−2							
0 . 1 . 4 × 2 . 1 . 2						−2		2		3		−3		−1	−2	3	1	−1
−[2 . 2 . 6] de Gordan sextuplé						6	−6	−6	6	−6	6	6	−6					

B.—Table *pour effectuer la décomposition de la forme dite élémentaire du type* 2 . 2 . 4 *de M. Gordan*

	$aa\gamma\epsilon$	$aa\delta\delta$	$ab\beta\epsilon$	$ab\gamma\delta$	$aca\epsilon$	$ac\beta\delta$	$ac\gamma\gamma$	$bba\epsilon$	$bb\beta\delta$	$bb\gamma\gamma$	$ada\delta$	$ad\beta\gamma$	$bca\delta$	$bc\beta\gamma$	$aea\gamma$	$ae\beta\beta$	$bda\gamma$	$bd\beta\beta$	$cca\gamma$	$cc\beta\beta$	$bea\beta$	$cda\beta$	$ceaa$	$ddaa$
1 . 1 . 2 × 1 . 1 . 2		1		−6		6				9	−2			−18			6			9		−6		1
1 . 1 . 0 × 1 . 1 . 4	−1		2	4	−1		−6		−8			4	4	12	−1			−8	−6		2	4	−1	
0 . 2 . 0 × 2 . 0 . 4					1	−4	3	−1	4	−3														
2 . 0 . 0 × 0 . 2 . 4															1	−1	−4	4	3	−3				
1 . 0 . 4 × 1 . 2 . 0	1	−1	−2	2	1	2	−3				−2	2			1	−1								
0 . 1 . 4 × 2 . 1 . 0					1			−1			−2		2		1		2		−3		−2	2	1	−1
−[2 . 2 . 4] de Gordan doublé					−2	−4	6	2	4	−6	6	−6	−6	6	−2	2	−4	4	+6	−6				

Je passe à la considération de la forme 2.2.4, et, comme dans le cas précédent, je me sers du symbole ijk pour représenter le coefficient principal dans le covariant dont les ordres et le degré sont i, j, k.

On trouvera

$$1.1.2 = \alpha\delta - 3b\gamma + 3c\beta - d\alpha,$$

$$1.1.0 = -a\epsilon + 4b\delta - 6c\gamma + 4d\beta - e\alpha, \qquad 1.1.4 = a\gamma - 2b\beta + c\alpha,$$

$$0.2.0 = \alpha\epsilon - 4\beta\delta + 3\gamma^2, \qquad 2.0.4 = ac - b^2,$$

$$2.0.0 = ae - 4bd + 3c^2, \qquad 0.2.4 = \alpha\gamma - \beta^2,$$

$$1.0.4 = a,$$

$$1.2.0 = e(\alpha\gamma - \beta^2) - 2d(\alpha\delta - \beta\gamma) + c(\alpha\epsilon + 2\beta\delta - 3\gamma^2) - 2b(\beta\epsilon - \gamma\delta) + a(\gamma\epsilon - \delta^2),$$

$$0.1.4 = \alpha,$$

$$2.1.0 = \epsilon(ac - b^2) - 2\delta(ad - bc) + \gamma(ae + 2bd - 3c^2) - 2\beta(be - cd) + \alpha(ce - d^2).$$

Finalement la forme de M. Gordan, dont le coefficient principal est représenté par [2.2.4], s'obtient tout à fait comme la forme correspondant à [2.2.6] dans le cas précédent, avec la seule exception que l'opérateur différentiel sur le produit des hessiennes sera la puissance quatrième au lieu de la puissance deuxième du symbole $\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}$. Cette opération donnera

$$(b^2 - ac)(\alpha\epsilon + 2\beta\delta - 3\gamma^2) + (\beta^2 - \alpha\gamma)(ae + 2bd - 3c^2) + 3(ad - bc)(\alpha\delta - \beta\gamma)$$

pour valeur de [2.2.4], ou plutôt je préférerai considérer cette fonction comme la valeur de $-$[2.2.4]. Alors on trouvera que 2 fois [2.2.4] sera la somme des six produits précédents, comme on le voit par la Table ci-contre, où l'on remarquera que la somme de chaque colonne donne la somme zéro, comme dans le cas précédent.

J'ajouterai seulement que cette preuve éclatante de l'insuffisance de la méthode de M. Gordan et de son école, pour séparer les formes véritablement élémentaires des formes superflues qui s'y rattachent (insuffisance reconnue par M. Gordan lui-même de la manière la plus loyale dans son discours inaugural prononcé à Erlangen), n'ôte rien à la valeur immense du service qu'il a rendu à l'Algèbre, en ayant le premier démontré l'existence d'une limite au nombre de ces formes.

(9)

THÉORIE POUR TROUVER LE NOMBRE DES COVARIANTS ET DES CONTREVARIANTS D'ORDRE ET DE DEGRÉ DONNÉS LINÉAIREMENT INDÉPENDANTS D'UN SYSTÈME QUELCONQUE DE FORMES SIMULTANÉES CONTENANT UN NOMBRE QUELCONQUE DE VARIABLES.

[*Comptes Rendus*, LXXXIV. (1877), pp. 1359—1361, 1427—1430.]

POUR plus de clarté, je commencerai par le cas d'une seule forme du degré n à k variables. On se propose de trouver le nombre: (1) des covariants, (2) des contrevariants linéairement indépendants de degré i par rapport aux coefficients et d'ordre j par rapport aux variables.

(1) *Cas des covariants.*—Écrivons

$$\sigma = \frac{in + (k-1)i}{k}, \quad \sigma' = \sigma + 1,$$

et trouvons toutes les solutions en nombres positifs et entiers des équations

$$a_0 + a_1 + a_2 + \dots + a_n = i, \tag{1}$$

$$a_1 + 2a_2 + \dots + na_n = \sigma. \tag{2}$$

Pour une solution quelconque de ces équations, soit S le nombre des *invariants* indépendants appartenant à un système de formes des degrés n, $n-1$, $n-2, \dots$, contenant chacun $k-1$ variables, les ordres de ces invariants quant aux coefficients de ces formes étant respectivement

$$a_0, \ a_1, \ a_2, \ \dots, \ a_{n-1},$$

nous obtiendrons ainsi une somme de nombres que je nommerai ΣS.

Formons le même système d'équations en a', comme plus haut avec des a, avec la différence d'écrire σ' au lieu de σ, et soit S' le nombre des *contrevariants linéaires* appartenant au même système de formes qu'auparavant, les ordres de ces contrevariants par rapport aux coefficients étant respectivement

$$a'_0, \ a'_1, \ a'_2, \dots, \ a'_{n-1};$$

nous obtiendrons ainsi une seconde somme $\Sigma S'$; la différence $\Sigma S - \Sigma S'$ sera le nombre de covariants du degré i et de l'ordre j pour la forme du $n^{\text{ième}}$ degré à k variables.

(2) *Cas des contrevariants.* Écrivons

$$\sigma = \frac{in - (k-1)j}{k}, \quad \sigma' = \sigma - 1,$$

et, avec la nouvelle valeur de σ, trouvons, comme dans le cas précédent, la valeur de ΣS. De même trouvons $\Sigma S'$, comme auparavant, en nous servant de la nouvelle valeur de σ', mais avec cette différence que, pour trouver un S' quelconque, il faut calculer le nombre non pas des contrevariants, mais des covariants linéaires des formes correspondantes. $\Sigma S - \Sigma S'$ sera le nombre des contrevariants du degré i et de l'ordre j, linéairement indépendants, appartenant à une forme du degré n à k variables.

Pour les invariants, on met $j = 0$, et l'on se sert indifféremment de l'une ou de l'autre méthode, c'est-à-dire on écrit

$$\sigma' = \sigma + 1 \quad \text{ou} \quad \sigma' = \sigma - 1$$

à volonté.

Quand $k = 3$, c'est-à-dire pour les formes ternaires, on comprend, en formant S', que la distinction entre les covariants et les contrevariants binaires devient superflue, puisque à chaque covariant d'une forme binaire correspond un contrevariant, et *vice versâ*.

Quand $k = 2$, en se rappelant que pour un système de formes *unitaires* simultanées

$$a_0 x^n, \ a_1 x^{n-1}, \ a_2 x^{n-2}, \ldots, \ a_{n-1} x$$

chaque combinaison des coefficients est un invariant, et multipliée par x un covariant ou contrevariant unitaire, la règle pour trouver le nombre des covariants et des contrevariants binaires revient à la règle connue.

Je passe à présent au cas plus général d'un système de formes $n_1, n_2, \ldots, n_q$ à k variables. On cherche le nombre des covariants et des contrevariants du degré j et des ordres $i_1, i_2, \ldots, i_q$ quant aux coefficients des formes données.

On écrit dans les deux cas respectivement

$$\sigma = \frac{\pm (k-1) j + \Sigma in}{k};$$

le rapport de σ' à σ reste le même, comme auparavant. Au lieu de l'équation (1), on écrit q équations de la forme

$$a_{0,q} + a_{1,q} + a_{2,q} + \ldots + a_{nq,q} = i_q \, [q = 1, \ 2, \ 3, \ldots, \ q],$$

et, au lieu de l'équation (2), on écrit la seule équation

$$\Sigma_{q=q}^{q=1} (a_{1,q} + 2a_{2,q} + \ldots, \ + n_q a_{nq,q}) = \sigma.$$

Alors, pour trouver S, on prend un système de formes à $k-1$ variables, une de chaque degré de 1 jusqu'à n_1, encore une de chaque degré de 1 jusqu'à $n_2, \ldots$, et finalement une de chaque degré de 1 jusqu'à n_q, et l'on trouve pour S le nombre des invariants à $k-1$ variables, dont les ordres respectifs, par rapport à ces formes, sont les valeurs des a données pour une solution quelconque des équations écrites plus haut: ainsi l'on obtient ΣS; de même, en substituant σ' pour σ et des contrevariants linéaires (si l'on s'occupe des covariants) ou des covariants linéaires (si l'on s'occupe des contrevariants), on trouve la valeur de $\Sigma S'$, et la différence $\Sigma S - \Sigma S'$ sera le nombre cherché.

Ainsi l'on voit que le problème pour des systèmes à k variables se réduit au même problème pour $k-1$ variables, de sorte que, par déductions successives, le problème est complétement résolu par une méthode arithmétique pour un nombre quelconque de variables.

Avec l'aide de ce principe, on peut construire, simplifier et réduire à la forme canonique une fonction génératrice ayant par rapport aux formes ternaires, quaternaires, etc., le même genre de rapport que la fonction génératrice dont, sous la forme canonique, j'ai déjà donné des exemples pour les formes binaires: c'est de cela que je m'occupe en ce moment; mais ce travail algébrique, quoique d'une nature très-élémentaire, devient, même pour les formes ternaires, extrêmement laborieux.

Je terminerai cette Note par un seul exemple numérique du calcul indiqué par mon théorème: qu'il soit demandé de trouver le nombre de contrevariants asyzygétiques du douzième ordre et du neuvième degré appartenant à la forme cubique ternaire.

Nous avons ici

$$i = 12, \quad \sigma = \frac{3.12 - 2.9}{3} = 6, \quad \sigma' = 5.$$

Je forme les deux Tables

0	1	2	3	S
10	0	0	2	0
9	1	1	1	1
9	0	3	0	1
8	2	2	0	3
7	4	1	0	5
6	6	0	0	2
3e	2e	1re		12

0	1	2	3	S'
10	0	1	1	0
9	2	0	1	1
9	1	2	0	3
8	3	1	0	7
7	5	0	0	0
3e	2e	1re		11

Dans la Table à gauche, en prenant une ligne horizontale quelconque, la somme du produit de chaque chiffre par le chiffre correspondant à la

tête de la colonne où il se trouve est égale à 6; dans la Table à droite, cette somme de produits est 5; pour l'une et l'autre, la somme des chiffres de chaque ligne est 12. Les chiffres de ces dernières colonnes ne figurent pas dans les calculs; ces chiffres sont les valeurs des S et des S'; le nombre d'invariants ou de covariants linéaires appartenant à chaque partition, par exemple à un système composé d'une cubique quadratique et d'une forme linéaire à deux variables, le nombre des invariants des ordres 8, 2, 2 respectivement pour ces formes est 3, et, appartenant au même système de formes, le nombre des covariants linéaires des ordres 8, 3, 1 quant aux coefficients est 7. La somme des S étant 12 et des S' 11, la différence 1 sera le nombre des contrevariants à la forme cubique ternaire du type donné, et ainsi en général.

Comme second exemple, cherchons s'il y a des invariants cubiques pour les courbes du quatrième degré.

Ici $$i = 3, \quad n = 4, \quad j = 0:$$

donc $$\sigma = \frac{4 \,.\, 3}{3} = 4, \quad \sigma' = \sigma \pm 1.$$

Prenons $\sigma' = 3$. On forme les deux Tables

0	1	2	3	4	S
2	0	0	0	1	1
1	1	0	1	0	1
1	0	2	0	0	1
0	2	1	0	0	0
4^e	3^e	2^e	1^re	r	3

0	1	2	3	4	S'
2	0	0	1	0	0
1	1	1	0	0	2
0	3	0	0	0	0
4^e	3^e	2^e	1^re	r	2

Tous les chiffres, dans les colonnes S et S', correspondent à des résultats ou évidents d'eux-mêmes ou donnés déjà par MM. Clebsch, Gordan et Gundelfinger, sauf la valeur 2 de S', qui représente le nombre des contrevariants linéaires, non pas seulement par rapport à leurs degrés, mais aussi par rapport à chaque système des coefficients appartenant à un système de trois formes des degrés 4, 3, 2 respectivement. Pour trouver ce nombre, on en forme un nouveau

$$\sigma = \frac{1 \,.\, 4 + 1 \,.\, 3 + 1 \,.\, 2 - 1}{2} = 4$$

et un autre $$\sigma' = (\sigma - 1) = 3,$$

et l'on prend la différence de deux *dénumérants :* l'un le nombre de solutions en nombres positifs et entiers du système d'équations

$$\lambda' + 2\lambda'' + 3\lambda''' + 4\lambda^{\text{IV}} + \mu' + 2\mu'' + 3\mu''' + \nu' + 2\nu'' = 4,$$
$$\lambda + \lambda' + \lambda'' + \lambda''' + \lambda^{\text{IV}} = 1, \quad \mu + \mu' + \mu'' = 1, \quad \nu + \nu' + \nu'' = 1;$$

l'autre le dénumérant du même système d'équations quand on remplace

4 par 3. On voit facilement que le premier dénumérant est le nombre des combinaisons

4	0	0
3	1	0
3	0	1
2	2	0
2	0	2
2	1	1
1	3	0
1	2	1
1	1	2
0	2	1
0	2	2

et que le second est le nombre des combinaisons

3	0	0
2	1	0
2	0	1
1	2	0
1	0	2
1	1	1
0	3	0
0	2	1
0	1	2

c'est-à-dire le nombre des contrevariants linéaires qu'on cherche est $11 - 9$ ou 2. La somme des S moins la somme des S' est donc $3 - 2$, et conséquemment il n'y a qu'un et un seul invariant cubique appartenant aux courbes du quatrième degré.

10.

ADDRESS ON COMMEMORATION DAY AT JOHNS HOPKINS UNIVERSITY*
22 FEBRUARY, 1877.

Sir! Ladies and Gentlemen!

It is the custom of this country (which will take no denial) that, through the voice of our truly estimable President, calls upon me to appear before you and render an account of my experiences in connection with this great institution, which, so recently inaugurated, is steadily and solidly rising from its foundations, like the stately pile standing almost at its gates—the magnificent bequest of George Peabody to his fellow-citizens—where day by day, quietly but persistently, to the ring of the hammer and the merry click of the chisel, without haste as without pause, we may witness stone after stone lifted into its position, and each pillar set upright and securely on its base. Had I consulted only my own inclinations in the matter, I would have much preferred to remain silent, and let my work in the future tell its own tale.

It is with unaffected feelings of diffidence that I present myself before you, for, save on rare and exceptional occasions, it has not been my wont to make my voice heard in public assemblies. I know, indeed, and can conceive of no pursuit so antagonistic to the cultivation of the oratorical faculty—that faculty so prevalent in this country that the possession of it is not regarded as a gift, but the want of it as a defect—as the study of Mathematics. An eloquent mathematician must, from the nature of things, ever remain as rare a phenomenon as a talking fish, and it is certain that the more anyone gives himself up to the study of oratorical effect the less will he find himself in a fit state of mind to mathematicize. It is the constant

* The address was written on a rather sudden call, within a few hours, and many marks will be apparent to the practised eye of the haste with which it was composed. Two or three paragraphs have been inserted that were not contained in the address as delivered, and the writer is alone responsible for the opinions or sentiments which it expresses. Some copies of it will be forwarded to England, which he hopes to revisit in June next.

aim of the mathematician to reduce all his expressions to their lowest terms, to retrench every superfluous word and phrase, and to condense the Maximum of meaning into the Minimum of language. He has to turn his eye ever inwards, to see everything in its dryest light, to train and inure himself to a habit of internal and impersonal reflection and elaboration of abstract thought, which makes it most difficult for him to touch or enlarge upon any of those themes which appeal to the emotional nature of his fellow-men. When called upon to speak in public he feels as a man might do who has passed all his life in peering through a microscope, and is suddenly called upon to take charge of an astronomical observatory. He has to go out of himself, as it were, and change the habitual focus of his vision.

Yet it is not without a considerable admixture of feelings of a more agreeable nature that I have acquiesced in taking the part allotted to me in this day's proceedings.

It is always a satisfaction to meet those from whom we have received marks of regard, and whom we know to be favorably disposed towards us; and I should be heartless, indeed, and more callous than the oyster, who, twin-soul to the mathematician, working in silence and seclusion between the folding-doors of his mansion, elaborates the pearl that may, hereafter, deck an empress's brow, could I be insensible to the many proofs of kind and generous feeling which, both within and without the walls of this University, have been so widely and unequivocally accorded to me.

I scruple not to say (for it is strictly the truth) that I have experienced from the authorities of the University a degree of delicate consideration and forbearance from all claims that might be supposed to interfere, in any respect, with my comfort or ease of mind, that, as long as I live, will endear to me the name of the Johns Hopkins University.

But that pleasure of coming as a friend among friends is enhanced by the fact that I stand here a harbinger of glad tidings—that I can honestly congratulate all of you who are interested in the success of the University on the good seed that has been sown, and on the promise which it affords of a rich harvest in no distant future.

One of our great English judges observed on some occasion, when he was outvoted by his brethren on the bench (or, perchance, it may have been the twelfth outstanding juryman, who protested that never before in his life had he been shut up with eleven other such obstinate men) that "opinions ought to count by weight rather than by number," and so I would say that the good done by a university is to be estimated not so much by the mere number of its members as by the spirit which actuates and the work that is done by them. When I hear, as I have heard, of members of this University, only hoping to be enabled to keep body and soul together in order that

they may continue to enjoy the advantages which it affords, it may be for a decade of years to come; when I find classes diligently attending lectures on the most abstruse branches of scholarship and science, remote from all the avenues which lead to fortune or public recognition; when I observe the earnestness with which our younger members address themselves to the studies of the place, and the absence of all manifestations of disorder or levity, without the necessity for the exercise of any external restraint, it seems to me that this establishment, even in its cradle, better responds to what its name should import, more fully embodies the true idea of a university, than if its halls and lecture-rooms swarmed with hundreds of idle and indifferent students, or with students, diligent, indeed, but working not from a pure love of knowledge, not even for the chaplet of olive, or the laurel crown, but for high places in examinations, for marks, as we say in England, the counters or vouchers to enable their fortunate possessor to draw large stakes out of the pool of sinecure fellowships or lucrative civil appointments.

But I look not only to our students, but to the means of instruction at our command, to our chemical and physical and biological laboratories, unsurpassed anywhere in the world for completeness in all essential particulars, furnished and replenished whenever called for without question and without stint, to our libraries and rooms for research and study, where any earnest student can work in comfort and seclusion, with all the materials and aids that he may require to assist him in his investigations, close at hand, and to the multifarious subjects in which (with a necessarily limited staff), even in our inchoate state, all who wish can receive instruction. In the course of time and as opportunities present themselves no doubt our staff of professors and lecturers will receive, as they need, considerable augmentation; but as I have heard it pithily and tellingly expressed, the object of our trustees is to found not chairs but professors.

I have had the pleasure myself of listening to a course of lectures on a very abstruse and important subject, allied to my own, which I am sure could not be surpassed for lucidity of arrangement, strictness of concatenation, aptness, fulness and variety of illustration and application, by lectures given in any university in the world, with which I am acquainted. These were the lectures of our colleague, unfortunately absent on this occasion, owing to ill health, Professor Rowland, on Thermodynamics, in which all the principal conclusions of this wonderful mathematical theory, perhaps the most wonderful since the discovery of universal gravitation, were deduced with geometrical rigor from the two great laws capable of being contained within a few words, the seven last words of the expiring Caloric theory, "*Heat is motion,*" "*Temperature seeks its level.*" *

* That is to say, temperature in regard to the categories of greater and less. Its *measure* Professor Rowland identifies with the integrating factor of a partial differential equation.

I have alluded, as a subject of congratulation, to the absence of all vexatious restraints upon the free action of our students which their conduct justifies. With equal reason may I congratulate myself and the professors and teaching staff with whom I have the happiness to be associated, on the confidence that is reposed in us, and on the free scope that is given to each to carry out in the manner that may seem to him most likely to be conducive to a useful result, the combined objects which this University has been founded to promote, under its two-fold aspect as a teaching body and as a corporation for the advancement and propagation of science and learning.

It has happened to myself, when in a state of despondency and embarrassment as to how I could best divide my energies between the contending claims of the teacher and the investigator, to be released from my difficulty by the cheering words graven lastingly on my memory, "The University desires from you your best and highest work."

And let me take this opportunity of making my profession of faith on a subject much mooted at the present day, as to whether the highest grade of university appointments should be conferred with or without the condition of teaching annexed.

I hesitate not to say that, in my opinion, the two functions of teaching and working in science should never be divorced. I believe that none are so well fitted to impart knowledge (if they will but recognize as existing, and take the necessary pains to acquire, the art of presentation) as those who are engaged in reviewing its methods and extending its boundaries—and I am sure that there is no stimulus so advantageous to the original investigator as that which springs from contact with other minds and the necessity for going afresh to the foundations of his knowledge, which the work of teaching imposes upon him. I look forward to the courses of lectures that I hope to deliver in succession within the walls of this University as marking the inauguration of a new era of productivity in my own scientific existence; nor need I consider any subject too low (as it is sometimes foolishly termed) for me to teach, when I remember to have seen the minutes of the conversation held between the delegates of the Convention, at the time of the French Revolution, and the illustrious Lagrange, the son of the pastry-cook of Turin, possibly the progenitor of the Marquis Lagrange, of turf celebrity (Citoyen Lagrange, as he is styled in the record), who, when asked what subject he would be willing to profess for the benefit of the community, answered meekly, "I will lecture on Arithmetic."

At this moment I happen to be engaged in a research of fascinating interest to myself, and which, if the day only responds to the promise of its dawn, will meet, I believe, a sympathetic response from the Professors of our divine Algebraical art wherever scattered through the world.

There are things called Algebraical Forms. Professor Cayley calls them Quantics. These are not, properly speaking, Geometrical Forms, although capable, to some extent, of being embodied in them, but rather schemes of processes, or of operations for forming, for calling into existence, as it were, Algebraic quantities.

To every such Quantic is associated an infinite variety of other forms that may be regarded as engendered from and floating, like an atmosphere, around it—but infinite in number as are these derived existences, these emanations from the parent form, it is found that they admit of being obtained by composition, by mixture, so to say, of a certain limited number of fundamental forms, standard rays, as they might be termed in the Algebraic Spectrum of the Quantic to which they belong. And, as it is a leading pursuit of the Physicists of the present day to ascertain the fixed lines in the spectrum of every chemical substance, so it is the aim and object of a great school of mathematicians to make out the fundamental derived forms, the Covariants and Invariants, as they are called, of these Quantics.

This is the kind of investigation in which I have for the last month or two been immersed, and which I entertain great hopes of bringing to a successful issue. Why do I mention it here? It is to illustrate my opinion as to the invaluable aid of teaching to the teacher, in throwing him back upon his own thoughts and leading him to evolve new results from ideas that would have otherwise remained passive or dormant in his mind.

But for the persistence of a student of this University in urging upon me his desire to study with me the modern Algebra I should never have been led into this investigation; and the new facts and principles which I have discovered in regard to it (important facts, I believe), would, so far as I am concerned, have remained still hidden in the womb of time*. In vain I represented to this inquisitive student that he would do better to take up some other subject lying less off the beaten track of study, such as the higher parts of the Calculus or Elliptic Functions, or the theory of Substitutions, or I wot not what besides. He stuck with perfect respectfulness, but with invincible pertinacity, to his point. He would have the New Algebra (Heaven knows where he had heard about it, for it is almost unknown in this continent), that or nothing. I was obliged to yield, and what was the consequence? In trying to throw light upon an obscure explanation in our text-book, my brain took fire, I plunged with re-quickened zeal into a subject which I had for years abandoned, and found food for thoughts which have engaged my attention for a considerable time past, and will probably occupy all my powers of contemplation advantageously for several months to come.

I remember, too, how, in like manner, when a very young professor, fresh from the University of Cambridge, in the act of teaching a private pupil the

* See Appendix [p. 85 below].

simpler parts of Algebra, I discovered the principle now generally adopted into the higher text books, which goes by the name of the "Dialytic Method of Elimination." So much for the reaction of the student on the teacher*. May the time never come when the two offices of teaching and researching shall be sundered in this University! So long as man remains a gregarious and sociable being, he cannot cut himself off from the gratification of the instinct of imparting what he is learning, of propagating through others the ideas and impressions seething in his own brain, without stunting and atrophying his moral nature and drying up the surest sources of his future intellectual replenishment.

I should be sorry to suppose that I was to be left for long in sole possession of so vast a field as is occupied by modern mathematics. Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack; it is not a mine, whose treasures may take long to reduce into possession, but which fill only a limited number of veins and lodes; it is not a soil, whose fertility can be

* Not to speak of professor on professor. Thus it was in order to be able to meet the threatened interrogatories of my valued colleague, the irrepressible Mr Rowland, that I was led, on my return passage to England last summer, to look into Prof. Clerk Maxwell's extremely valuable, but ill-digested and somewhat unduly pretentious treatise on Electricity and Magnetism, which led to my theory of the Bipotential, and to my writing the paper published in the *Philosophical Magazine* for October last, which ought to have the effect of causing the author to rewrite one of his leading chapters on Statical Electricity.

I have at present a class of from eight to ten students attending my lectures on the Modern Higher Algebra. One of them, a young engineer, engaged from eight in the morning to six at night in the duties of his office, with an interval of an hour and a half for his dinner or lectures, has furnished me with the best proof, and the best expressed, I have ever seen of what I call the Law of Concomitant Interchange, applicable to permutation systems, *i.e.* the law which affirms that every complete set of permuted elements may be separated into two parts, or if we like to say so, be presented in the form of a diptych with two precisely similar Alæ, such that a single interchange between any two elements is accompanied with a total interchange between the two Alæ. This is the theorem which lies at the basis of the great theory of simple equations, which every school-boy is supposed to understand, but which was not really made out until a bevy of great Mathematicians, including Leibnitz, Laplace and Lagrange, had turned their attention to the subject. Jacobi, I have read somewhere, used to say that if he at all excelled other mathematicians, it was chiefly due to his greater facility in manipulating simple equations that he owed it. The same Jacobi, who, I remember, visited our English Cambridge, and so much relished the Trinity audit ale which he drank there, and who once being asked whether he was brother to the eminent physicist, Professor Jacobi, of St Petersburg, replied: "Quite the contrary—he is my brother." And *apropos* of the zeal of the student in question, let me mention for the benefit of my English friends, I have been agreeably surprised to find how widely diffused a spirit there exists in this country of disinterested love of learning. Out of Italy, especially Tuscany, where my friend Enrico Betti, as I had the opportunity of observing, and in his own country too, where no man is supposed to be a prophet, the neighbourhood of Pistoja, as a Professor is more influential, more honored and courted than he could be if he were a rich Marquis, I believe there is no nation in the world where ability with character counts for so much, and the mere possession of wealth (in spite of all that we hear about the Almighty dollar), for so little as in America, with exception it may be of certain of the Trans-Atlantic cities, which are really only colonies and emporiums for the trading classes of Europe.

exhausted by the yield of successive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined: it is limitless as that space which it finds too narrow for its aspirations; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze; it is as incapable of being restricted within assigned boundaries or being reduced to definitions of permanent validity, as the consciousness, the life, which seems to slumber in each monad, in every atom of matter, in each leaf and bud and cell, and is forever ready to burst forth into new forms of vegetable and animal existence.

I think that I am not claiming too much for my own special pursuit when I affirm that every science becomes more perfect, approaches more closely to its own ideal, in proportion as it imitates or imbibes the mathematical form and spirit. It is, therefore, I think, a just cause of congratulation to us that as shown by our official returns and the evidence of those best acquainted with the aims and pursuits of our students, their interest and their proficiency in mathematics is, to say the least, unsurpassed by that which is evinced by them in any other department of instruction carried on within our walls. Many gentlemen who have graduated years ago in other colleges have come up to us with the sole or principal object of continuing and extending their mathematical studies.

I have reason to think that the taste for mathematical science, even in its most abstract form, is much more widely diffused than is generally supposed over this great continent, and that there is really a demand for the higher instruction which we are, or hope to be, prepared to give.

I know that in response to a circular letter inviting opinions as to the expediency of founding a mathematical journal of a high character under the auspices of the authorities of this University, we have received some scores of replies expressing deep interest in the proposed undertaking, with hopes for its speedy realization, coupled with distinct pledges of support and co-operation.

Such a journal, I venture to vaticinate, would not fail to receive contributions from mathematicians of the highest eminence in Europe, and would form a new chain of connection, of which this University would hold the leading link in its hand, between America and the other nations of the world which lead the van of science.

In contributing my share to the matter and superintendence of this journal I should feel that I was discharging one important duty of my office. Another branch of my duty will consist, as now, in being open to communication, at a stated hour of the day, with all who wish to confer with me in relation to their studies.

A third branch of my duty will be to deliver a succession of lectures on subjects either of special interest in themselves, or in which I may happen to

possess what may seem to me to be new views, or in which I may have succeeded in making discoveries of any general interest.

I ought not to omit to mention here the invaluable aid which I derive from the concurrence of the gentlemen associated with me in the work of mathematical instruction carried on under my general direction.

My associate, Dr Story, has had the advantage of studying for a long course of years in more than one German university, and I can speak, from personal attendance on one of his courses of lectures, from which I have derived both pleasure and instruction, of his thorough mastery over many of the most important and difficult branches of mathematical science.

Our students have thus the advantage of being put in direct communication with, and made participants of, all that has been done and is doing in that classical land of learning, in the way of mathematical research. In thoroughness of exposition, whatever may be the case as regards lucidity of presentation or spontaneity of initiative, I need hardly add my testimony to the general verdict of the world that our Teutonic brethren occupy the foremost rank. Many important *lacunæ*, which I should find otherwise a difficulty in filling up out of my own intellectual resources, are thus completely and efficiently supplied. Added to this, one of the most promising of our Fellows has lent his co-operation in bringing up to the standard of our University instruction such of our junior members as have come here insufficiently prepared, either from a too-short course of study or the lack of competent instruction in the schools or colleges in which they have received their preliminary education, and I am happy to be able to state that our trustees, with wise liberality, have recognized his services by raising him at once to the rank of a stipendiary lecturer.

Any one who will look through the syllabus of the lectures, not merely announced, but *bona fide* delivered and followed by attentive audiences within our walls, will see how respectable a range our courses of mathematical instruction comprehend: Analytical Geometry, Determinants, the Theory of Equations, the Differential and Integral Calculus, Definite Integrals, Rational Mechanics, Thermodynamics, the Theory of Elasticity and Modern Higher Algebra, are the subjects which have been actually taught here to smaller or larger classes of diligent students within the last few months.

Various other courses have been announced and will form part of our programme of instruction in this or future years.

The mention of Germany brings to my mind the importance of universities to the maintenance or development of a national spirit in the countries in which they are fostered and carried on with an animus free from local or sectarian prejudices.

I think that there can be little doubt that the greatest fact in modern history, the resuscitation of the German Empire, the resurrection of the

German people, is mainly to be attributed to the feeling of brotherhood and the spirit of nationality kept alive in those ganglions of thought, those centres of intellectual activity, the German universities.

It is the university professors who have made German unity a possibility, and I cannot but deplore the unpatriotic short-sightedness of those in my own country who, until so late a period, have struggled, and still covertly struggle, to make our universities in England not the representatives of the universal English mind, but the monopoly of a party and the appanage of a sect.

Their work it is that a separation deeper and a chasm more difficult to fill up has been created between the two most free and powerful nations in the world, England and America, than any due to political causes present or past.

Not the strained prerogative of a well-meaning but obstinate and narrow-minded monarch, nor the subservience of his ministers, nor the echoing voice of a misguided people it is, which has set up a permanent wall of separation between these two countries, a separation not founded on any opposition of material interests, but striking to the very groundwork of our mental constitution. Why is it that the flower of American youth resort not where the ties of a common language and of a common kindred would naturally have attracted them, to our English universities, to receive their mental impulse and their higher education, not to Oxford or Cambridge, but to Berlin, Leipzig, Göttingen, Jena or Heidelberg?

It is because there they were welcomed to whatever religious communion they were attached or unattached, without question and without distinction. It is because there they could rest on the bosom of a common mother, who shows kindness to all and favor to none.

If German professors have made Germany what it is, England may thank the narrow-minded class, or section of a class, of its university professors and chiefs (for there are numerous and nobleminded examples of English university leaders who combine the highest genius with the most liberal views; think of the Herschels, the Peacocks, the Sedgwicks, the De la Prymes, the Babbages, the Henslows and Lubbocks of the past, the Sidgwicks, the Stanleys, the Jowetts, the Liddells, the Brodies, the Mark Pattisons, the Prices, the Henry Smiths, His Grace of York, and many other illustrious men, leaders of thought, children of light, of the present generation,)—England, I say, may thank the obscurantist class of her university professors and heads, if the right arm of her spiritual power is shortened—if she is now, and it is to be feared will long remain, so much inferior in intellectual weight and influence in the world to what she ought to, and but for them would have been. They it is who, surrendering to party what was meant for

mankind, and laboring to cut out an English university upon the pattern of the University of Salamanca, have made a rent in the garment that should have been without seam, and alienated from us the intellectual sympathy of a mighty and kindred race. Driven to bay, like Rizzio at Holyrood, cowering behind *their* chairs and covered with *their* academic gowns, Intolerance found its last refuge and received, or is destined to receive, its last stab. Yet we shall probably live to see, as we have seen on former occasions, on the principle of "setting the cat to watch the cream," those very same men entrusted with the task of carrying out and shaping the promised university reforms who have passed their lives in endeavoring to frustrate or avert all substantial reforms up to this time.

I have been struck, almost from the first hour of my landing on these shores, by the manifestations I have everywhere witnessed of the close scholarly alliance which has sprung up between America and Germany. It is German books that are read, German professors who are quoted, German opinion on all matters of science and learning that is appealed to; and as regards community of work and intellectual ties, I do not think it at all extravagant to assert that Germany and America belong to one hemisphere, and we in England to another. If the English and American minds are ever again to be brought into contact, it will have to be on neutral and German soil.

I am old enough to remember when the great universities of England affixed their corporate seals to petitions to Parliament praying that the Crown would refuse to grant a charter to the University of London, then in the course of being founded, to enable it to give degrees, and that, too, at a time when, within their own walls, in many or most of the colleges, a religious test applied even to the admission of students*, and when no student, not a member of the Anglican communion, could be admitted to take a degree, so that not only would the universities not confer their own degrees, but they labored to prevent all Englishmen unwilling to sign the Thirty-nine Articles, from obtaining degrees elsewhere.

Then followed a struggle against Mr James Heywood's bill to open the degrees of the old universities to members of every faith; and in the third stage of this protracted contest, after the awakened intelligence and conscience of the magnanimous English people had overruled the monkish objections of the professorial and other chiefs of the retrograde party, the official head of

* The tutor on "one of the sides" at Trinity College, Cambridge, acting under the express directions of Dr Whewell, the then Master of the College, made strict inquisition of a gentleman, now occupying a Professor's chair in the University of Oxford, whether he professed the faith in which the founder of Christianity was educated, as in that case he must refuse to admit him as a student of the College. If I am not mistaken, the tutor in question was the present highly estimable and learned Master of the College. This incident was reported to me by the gentleman to whom and at the time when it occurred.

Physical Science in my own Alma Mater (for as such and not as an Injusta Noverca, or as a neglectful nurse who leaves her helpless charge whilst she perambulates with others more dear to her, will I ever continue to regard and cherish her) not merely signed, but was, (as I have been credibly informed, the projector and originator, and to my certain knowledge,) the active and leading canvasser for signatures to a petition* to the two Houses of Parliament to estop all others but members of the Church of England from holding any office of instruction in the university! This happened only a very few years ago.

Such is the blinding and blighting effect of early sectarian influences, one-sided culture, and narrow partisan connections, even on minds of a superior intellectual order, and on dispositions amiable by nature. There is a black drop of gall, a taint of congenital rancour and animosity, which infects all it comes in contact with, more indelible, more difficult to wring out or efface, than that dread smear on Lady Macbeth's hand, which could "the multitudinous sea incarnadine."† I doubt not that those who have taken this part, so prejudicial to their country's welfare, believe themselves to have been actuated by honest motives, just as I should not hesitate to admit that Torquemada was actuated by such and believed that he was doing a work acceptable to God when torturing heretics or presiding at the celebration of one of those *Auto da fé's* more horrid but scarcely more brutalizing than the bull-fights which I have seen supply their place.

It is difficult to estimate the lengths to which human self-delusion can be carried. No one questions that a great English statesman believes that he is prompted by the purest motives of philanthropy and patriotism when

* I ought to have said "to *two* petitions," one to shut out non-Anglicans from offices of emolument in the Colleges, the other to shut them out from Professors' Chairs in the University. I can understand upon what grounds (mistaken as they may appear to me) it may have been thought right to retain the management and endowments of the Colleges in the hands of a single denomination, but am really at a loss to conceive what reasonable plea can be offered for petitioning Parliament to exclude any one from teaching Anatomy, Latin and Greek or Mathematics in the University, who should happen not to say his prayers out of the same prayer-book as the signatories to the petition—a petition more worthy, it seems to me, to have proceeded from the members of some red-hot Irish Orange Lodge than from sober-minded Professors in a great National English University.

† A young gentleman, born in County Antrim, near the Giant's Causeway, was one night returning home from a dinner party with three-cornered hat and frilled shirt, "flushed," as we may suppose, not "with the juice of the Tuscan grape," but with run claret or *crooked* potheen, when, in passing over a narrow plank bridge that spanned a mountain torrent, he espied coming towards him an aged priest with lantern in hand, probably on his way to perform some silent deed of charity and mercy. He did *not* throw him over the bridge, but for two years afterwards felt much troubled in mind at the thought of having let slip so favorable an opportunity of doing a good deed. He subsequently emigrated to America, where he often recounted the story, and lived to shudder at the temptation to which he had so nearly succumbed. I have taken this account from the lips of his grandson, one of the most respected and enterprising citizens of Baltimore.

agitating to paralyze a government and overthrow a rival odious to him, and to cast his country at the feet or into the arms of an insidious suitor and foe, or that one who treads in his footsteps believes that he is acting with a single eye to the interests of education when he cries up the University of London, and, like Ham, mocking the nakedness of his parent, with sublime self-abnegation or *matchless* cynicism, derides the venerable university where he was nurtured, where he taught, and which gave him his start in life.*

I think that you in this favored land are so far educated out of such pseudo-religious and antisocial views (survivals of a bygone age), that you will feel almost prompted to doubt the veracity of my statements, or the faithfulness of my recollections on the subject, and I am certain that not a score of signatures could be gathered to any document of such a nature in this country, were the continent canvassed from Maine to Florida, or from Chesapeake Bay to the shores of the Pacific. If I speak with some warmth on this subject, it is because it is one that comes home to me—because I feel what irreparable loss of facilities for domestic and foreign study, for full mental development and the growth of productive power, I have suffered, what opportunities for usefulness been cut off from, under the effect of this oppressive monopoly, this baneful system of protection of such old standing and inveterate tenacity of existence. I cannot easily express myself at any length in cold blood, but require to be warmed by a sense of personal interest in my subject, when I venture to address a public audience; with me *facit indignatio versus,* nor can I sit down to compose except in conformity with the dictates of the Muse to the impassioned Sidney, "Fool! look in thy heart," she said—"there learn to write."

Happy the young men gathered under our wing, who, unfettered and untrammeled by any other test than that of diligence and attainments, have here afforded to them an opportunity of filling up a complete scheme of education, such as a Milton or a Locke would have deemed adequate to their ideal.

How rejoiced should I be, were I of less ripe years and under less peremptory obligations as to the disposal of my time, branching out from mathematics as my natural mental centre of gravity, to diverge into the physical and chemical studies which lie so near to it, and which there are here such ample means accorded of studying under the most competent instructors, and with all the aids that modern ingenuity and the improvements in mechanical science can devise for putting direct questions to

* "Viewed her own feather on the fatal dart,
"And winged the shaft that quivered in her heart;
"Keen were her pangs, but keener far to feel
"She nursed the pinion that impelled the steel."
—*Byron's lines on the death of Henry Kirke White.*

Nature, and complementing and substantiating theory by visible and palpable Experience. For Experimental Physics, like the Practice of Gunnery, thanks in a great measure to the close alliance which the go-between Telegraphy has brought about between Science and Commerce, has in these days almost become a refined branch of Mechanical Engineering, very changed from those when a James Watt worked at his bench, when Priestley may have used a washhand basin for a Pneumatic Trough, or when a Woollaston could point to his cupboard as his Laboratory, and to a saucer holding a watch-glass, a lens and a blowpipe as his Philosophical Apparatus.

How delightful it were to be brought into contact with the treasures of antiquity and the music of the most perfect instrument of language, interpreted with Hellenic taste and wit and subtlest intellectual sympathy by my gifted colleague, whom you have just had the pleasure of listening to*, on whose lips all the bees of Hymettus seem to have settled and left their sweetest honey there; or, under the guidance of our enthusiastic and accomplished junior associate, penetrate to the foundations of our Indo-Germanic tongue; or, if that were not a dream too bright to be realized, to be led to the pure well of English undefiled, by one whose stay among us is, alas! only too short and transitory, who has effected among us what the rewards offered by an Eastern potentate were incompetent to bring forth—the Invention of a New pleasure—the eminent Chaucerian scholar, to whom I shall ever feel I owe a heavy debt of gratitude in supplying an unfailing source of delight, a pillow of repose for my declining years, in bringing to my knowledge and teaching me how to read and enter into the charm of another, a fresher and earlier Shakspeare!

> *'Αλλ' οὐ γὰρ ωὑτὸς πάντ' ἐπίστασθαι βροτῶν*
> *Πέφυκεν.*

Even as the case stands, could our trustees but see their way to the institution of a certain number—I must not say of new mathematical chairs, but of additional mathematical professors, through whom I might supplement my deficiencies, and with them interchange ideas and carry on joint studies—I know not where in the wide world, out of my own country, I could feel more content to abide, or where I could find more conveniently within my reach all the materials for a complete mental equipment than within these walls, in this Temple of the Muses, in this free and law-abiding and

* Mr Gildersleeve, late Professor of Greek in the University of Virginia, who, like all of my other American colleagues, has drunk deep out of and been baptized in the perennial springs, in which whoever has been dipped comes out twice the man he was before he went in, which bubble up in those sacred precincts of science, the German Universities, and whose grammatical and other works are familiar to all scholars. I am informed that he is at present engaged in completing a Magnum Opus on Greek Syntax, which is likely to give him constant occupation for the next four or five years to come.

hospitable land—a land, to borrow the words of a fantastic but, to me, sympathetic rhymester,

> Where tost bark a haven may find,
> And new earth its roots to bind,
> Drawing sap with instinct blind,
> Willow stooping to each wind,
> Oak, the monarch of its kind.

I thank you, ladies and gentlemen, for the kindness with which you have listened to my feeble utterances and, bird of night, given up to moping and brooding on my solitary perch, gladly make way for the lark, the herald of the morn, the star conspicuous amidst the effulgence of those Northern Lights*, which, "shooting madly from their spheres" for the last month past, have wandered into *our* latitudes and glowed in *our* sky, the poet whose name is honored wherever the English language is spoken or read, the author of the "Biglow Papers" and the "Ode to Washington."

APPENDIX.

There are three methods of treating the question of the Scale of Fundamental Invariants and Covariants—the *realistic,* the *symbolic* and the *fatalistic* or *peprotic.* In the first of these methods (the explicit or realistic) the derived Quantics, set out in full or abridged, through the intervention of canonical forms, are dealt with. It was thus that I established the scale for Ternary Cubics and for Binary Quartics and Quintics, in my early papers in the *Philosophical Magazine,* the *Cambridge and Dublin Mathematical Journal,* and in my Trilogy, published in the *Philosophical Transactions.* In the second, (the symbolic, schematic or embryonic method,) the derivations are not regarded as actually deduced, but are studied through the medium of the symbolic processes which gave the key to their existence (this is the method

* During the last month, Professors Childs and James Russell Lowell, both of Harvard University, have been giving lectures on Chaucer and Dante at Johns Hopkins University, and Mr Norton, Professor of the Fine Arts, and Mr Fiske, Assistant Librarian at Harvard, on English Cathedrals and the Aryan language and myths at the Peabody Institute. Professor Whitney is at present holding under the spell of his eloquence an audience of some hundreds of people, of both sexes, at the University, with lectures on the History of the Inflexional Structure of the Indo-European languages. There are many ladies in Baltimore who know Greek, and some who are about to enter upon a course of Sanskrit; others whose skill in singing and playing would command attention in any European concert room; and I have heard Professor Childs say that he never was in any city in the world where there was so pronounced a dramatic instinct as in Baltimore. Not to speak or read French and German is rather the exception than the rule. I mention these facts in order that my friends in Europe may well understand that my lot has not been cast among a barbarous or uncultivated race, and that the University has been planted in a congenial soil. Professor Lowell's recitation of his poem on Washington, at the Johns Hopkins Commemoration, "moved many of the audience, men as well as women, even to tears."

pursued with so large a measure of success by Prof. Gordan). The third, (the Deontological or Peprotic,) which precedes the one last named in the order of time, is the method indicated by Professor Cayley, in his memorable Second Memoir on Quantics, published in the *Philosophical Transactions*, which, owing to an error in its application, committed by its illustrious author, has fallen into neglect, and even the validity of whose substratum has been called into question. In this method the qualities of the derived forms, and the modes in which they can be brought into existence, are equally ignored: they are treated as mere Arithmetical existences, and, through the medium of that subtlest of all instruments for putting Nature and Reason to the question—a Partial Differential Equation—the numerical laws to which they are subject are made to depend on a problem in the Partition of Numbers.

This is the method followed in my researches, the validity of which I have established on an irrefragable basis, and which I have extended and modified so as to recover, by its aid alone, all the results obtained by Professors Clebsch and Gordan, and to go beyond them in showing how, with very great probability, (for at present I have not completed a strict apodictic proof of my Cardinal Principle,) an Algebraical Limit may be set to the degree and order of the Fundamental Derivations.

In order that I may not be supposed to be making a gratuitous assertion, I subjoin the Complete Generating Function, by means of which I can obtain the fundamental Invariants and Covariants given in Clebsch's Binären Formen for the Binary Sextic, demonstrate (with the aid of my Cardinal Principle) that there are none others, and establish all the fundamental Syzygants by which they are connected; for it ought to be noticed that alongside of the problem of determining the fundamental scale of Invariants or Covariants, there is the correlative problem, equally deserving of a solution, of determining the fundamental Syzygants, *i.e.* those rational integral functions of the Invariants or Covariants which, expressed in the terms of the coefficients of the Primary Quantic, are identically zero. I find that the total number of Covariants of the order m in the coefficients, and of n in the facients, (of course when n is zero the Covariants become Invariants,) is the coefficient of $t^m . v^n$ in the fraction whose Denominator is

$$(1-t^2)(1-t^4)(1-t^6)(1-t^{10})(1-t^2v^4)(1-t^2v^8)(1-tv^6)$$

and whose Numerator is

$$\begin{aligned}(1+t^{15}) &+ (t^3+t^5+t^7+t^8+t^{10}+t^{12})v^2\\ &+ (t^4+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{13}-t^{17})v^4\\ &+ (t^3+t^4+2t^6+t^8+t^9+t^{11}-t^{16})v^6\\ &+ (t^3+t^5+t^7-t^{13}-t^{15}-t^{17})v^8\\ &+ (t^4-t^9-t^{11}-t^{12}-2t^{14}-t^{16}-t^{17})v^{10}\\ &+ (t^3-t^7-t^9-t^{10}-t^{11}-t^{12}-t^{13}-t^{14}-t^{15}-t^{16})v^{12}\\ -(t^5+t^{20})v^{16} &- (t^8+t^{10}+t^{12}+t^{13}+t^{15}+t^{17})v^{14}.\end{aligned}$$

I am thus enabled to show that the fundamental Covariants and Invariants are composed of two classes—Primaries, which are got from the Denominator, and Secondaries, from the Numerator—a distinction of the utmost importance, but which does not disclose itself in Professor Gordan's method. And it ought to be noticed that besides the problem of forming the fundamental scale, there is the not less important one in all cases of determining the *total* number of Invariants and Covariants of any given degree and order to which Gordan's method gives no general clue, but which is absolutely and completely resolved by my extension of the Peprotic method, above indicated. I repeat emphatically that no table of fundamental Invariants and Covariants will serve to calculate the *total number* of a given order and degree in the absence of a correlative table of the fundamental syzygies.

So completely had the Peprotic or partial-differential-equation method fallen into discredit, that I believe no allusion is made to it in Clebsch's work; only a slight reference is made to it in a note in Dr Salmon's new edition of his Modern Higher Algebra, and a condemnation is passed upon it by Professor Faà de Bruno, in his Treatise on Binary Forms, in so far as regards its application to Covariants of Quantics exceeding the fourth degree.

11.

ON A GENERALIZATION OF TAYLOR'S THEOREM.

[*Philosophical Magazine*, IV. (1877), pp. 136—140.]

CONNECTED with the study of the Theory of the symmetrical functions of the differences of the roots of an Algebraical Equation, a theorem presents itself in Dr Salmon's *Lessons on Higher Algebra*, 3rd edition, p. 59, art. 63, only partially indicated and insufficiently demonstrated there, which on a closer inspection will be found to be well deserving of notice as containing a true generalization of Taylor's theorem, leading to a development of the same form, subject to a like law of convergence, and easily demonstrable by the same method as that theorem.

Let f be *any function whatever* of $a, b, c, \ldots$, and f_1 the same function of $a_1, b_1, c_1, \ldots$, where

$$a_1 = a, \quad b_1 = b + ah, \quad c_1 = c + 2bh + ah^2,$$
$$d_1 = d + 3ch + 3bh^2 + ah^3, \ldots$$

and let Ω represent the operator

$$a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots;$$

then the theorem in question affirms that

$$f_1 = f + \Omega\,.\,fh + (\Omega\,.)^2 f\frac{h^2}{1\,.\,2} + (\Omega\,.)^3 f\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

On making $a = 1,\ b = x,\ c = 0,\ d = 0 \ldots$, the theorem becomes Taylor's. To prove it in its general form, let

$$\phi x = ax^n + nbx^{n-1} + n\frac{n-1}{2}cx^{n-2} + \ldots;$$

then, on substituting $x+h$ for x, ϕx becomes

$$= a_1 x^n + n b_1 x^{n-1} + n\frac{n-1}{2} c_1 x^{n-2} + \dots.$$

Let h become $h+\delta h$, then obviously

$$\delta f_1 = \frac{d}{dh} f_1 \delta h.$$

But we may obtain the new values of a_1, b_1, $c_1, \dots$ corresponding to the change of h into $h+\delta h$, by substituting in ϕx first $x+\delta h$ and then $x+h$ for x.

The effect of the first substitution is to change $a, b, c, \dots$ into $a+\delta a$, $b+\delta b$, $c+\delta c, \dots$, where

$$\delta a = 0, \quad \delta b = a\delta h, \quad \delta c = 2b\delta h, \quad \delta d = 3c\delta h, \dots.$$

Hence the increment

$$\delta f_1 = \left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d}\dots\right) f_1 \,.\, \delta h;$$

consequently

$$\frac{d}{dh} f_1 = \Omega\,.\,f_1.^*$$

Hence, if we write

$$f_1 = f + Bh + Ch^2 + Dh^3 + \dots,$$

we shall have

$$B + 2Ch + 3Dh^2 + \dots$$
$$= \Omega f + \Omega Bh + 2\Omega Ch^2 + \dots.$$

Hence $\qquad B = \Omega\,.f, \; C = \tfrac{1}{2}(\Omega\,.)^2 f, \; D = \dfrac{1}{2\,.\,3}(\Omega\,.)^3 f \dots;$

and consequently

$$f_1 = f + \Omega\,.f + (\Omega\,.)^2 f\frac{h^2}{1\,.\,2} + (\Omega\,.)^3 f\frac{h^3}{1\,.\,2\,.\,3} + \dots,\dagger$$

and the first part of the theorem is demonstrated. It will of course be understood that $(\Omega\,.)^i$ means not $(\Omega^i).$, but $\Omega\,.\,\Omega\,.\,\Omega\,.$ (to i factors).

* Or without introducing ϕx, the equations between a_1, b_1, $c_1 \dots$ and $a, b, c, \dots$ show by direct inspection that the effect upon the former is the same, whether we augment h by δh or $b, c, d \dots$ respectively and simultaneously by $a\delta h$, $2b\delta h$, $3c\delta h, \dots$ so that $\frac{d}{dh} f_1 = \Omega\,.f_1$, as in the text.

† Consequently, if Ωf vanishes, since also $(\Omega\,.)^i f$ will also vanish for all values of i, we shall have $f_1 = f$. It is this fact of $(\Omega f = 0)$ being the complete solution of $(f_1 = f)$ which constitutes the importance of the theorem in the Calculus of Invariants.

Lagrange's or any other rule for the Remainder in the old Taylor's theorem may be extended to this generalization of it; that is to say, if in the development of f_1 we stop at the nth term, the *remainder* will be equal to

$$\frac{h^n}{\Pi n}(\Omega\,.)^n f(\alpha, \beta, \gamma \ldots),$$

where α, β, $\gamma \ldots$ are what a_1, b_1, $c_1 \ldots$ become when we write θh for h, θ being some proper positive fraction. The demonstration proceeds *pari passu* for the generalized form and for Taylor's case of it. Thus, consider Bertrand's proof as given in Williamson's *Calculus*, second edition, p. 64.

The *lemma* upon which the proof depends takes the form, that if f_1 (supposed continuous between two values of h) has the same value (zero, as it happens in the matter in hand) for two values of h, Ωf must vanish for some intermediate value of h; which is obviously true, since $\delta f = \Omega f \delta h$. The rest of the demonstration remains essentially the same, *mutatis mutandis*, at each point as for Taylor's theorem properly so called.

The theorem above established easily admits of extension to the case of a_1, b_1, $c_1 \ldots$ being the values assumed by a, b, $c \ldots$, when in the quantic $(a, b, c \ldots \between x, y, z)^n$ we substitute $x + hy + kz + \ldots$ for x. We may thus obtain a theorem which will bear to Taylor's theorem for any number of variables the same relation as the theorem given in the text to Taylor's theorem for a single variable.

Since the effect of changing x into $x + h + \delta h$ may be obtained either by first substituting $x + h$ for x and then $x + \delta h$ for x in ϕx, or by a reversal of the order of these two processes, we obtain the interesting consequence that the two operators

$$a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots$$

and

$$a_1\frac{d}{db_1} + 2b_1\frac{d}{dc_1} + 3c_1\frac{d}{d\,.\,d_1} + \ldots$$

are absolutely identical,—a theorem which of course admits, but not without a somewhat complicated process, of an *à posteriori* direct proof; so that the operator Ω is to all intents and purposes what Professor Cayley calls a semi-invariant or pene-invariant, but to which I am accustomed to give the name of a *differentiant* to ϕx.

Finally, it may be observed that a development for f_1 may be obtained by the use of the ordinary Taylor's theorem for *several* variables. If we make use of this method, and write in addition to

$$\Omega = a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots,$$

$$\Omega_1 = a\frac{d}{dc} + 3b\frac{d}{d\,.\,d} + 6c\frac{d}{d\,.\,e} + \ldots,$$

$$\Omega_2 = a\frac{d}{d\,.\,d} + 4b\frac{d}{d\,.\,e} + 10c\frac{d}{d\,.\,f} + \ldots,$$

&c. = &c.,

we shall obtain the noteworthy symbolical and absolute identity

$$e^{h\Omega\,.} = e^{h\Omega + h^2\Omega_1 + h^3\Omega_2 + \ldots}\,*,$$

which may be verified, but not without some little trouble, by direct expansion.

If we use $\Omega\,!$ to signify that Ω is to be used as a pure operator on the matter coming after it (operating that is to say solely on the symbols of quantity a, b, c, ... and not on the operators $\frac{d}{da}$, $\frac{d}{db}$, $\frac{d}{dc}$ ...), we shall have

$$\Omega_1 = \frac{(\Omega\,!\,\Omega)}{1\,.\,2}, \quad \Omega_2 = \frac{(\Omega\,!\,\Omega\,!\,\Omega)}{1\,.\,2\,.\,3},$$

and so on. Hence the "noteworthy" symbolical equation above written may be put under the hypersymbolical form

$$e^{h\Omega\,.} = e^{(e^{h\Omega\,!}-1)\frac{\Omega}{\Omega\,!}}$$

a suggestive identity that may possibly call forth a sneer from the mathematical cynic, but not from the thoughtful mathematician, who, aware that algebra is in its essence a language which it is the proper business of his art to fathom and develop, is prompt to recognize every step in expression as a gain in power.

* If we write

$$\Delta = (-\Omega\,.\,+\Omega)\,h + \Omega_1 h^2 + \Omega_2 h^3 + \ldots,$$

we ought to have $e^{\Delta} - 1 = 0$, and the coefficients in the expansion of $e^{\Delta} - 1$ according to ascending powers of h ought all to vanish identically; and so they will be found to do, provided that in each such coefficient expressed as the sum of the product of powers of Ω_0, Ω_1, Ω_2, ... and of $\Omega\,.$ the power of the *dotted* Ω be taken last in order. As soon as that expansion is made (but of course not before) we may write $\Omega\,.-\Omega = 0$, and we may readily calculate *à priori* the value of each power of $(\Omega\,.-\Omega)$; thus we shall obtain

$$(\Omega\,.-\Omega)^2 = \Omega\,.^2 - 2\Omega\Omega\,. + \Omega^2 = \Omega\,.^2 - 2\Omega^2 + \Omega^2 = 2\Omega_1\,;$$

and so by a similar calculation, having first determined $\Omega\,.^2$, $\Omega\,.^3$, $\Omega\,.^4$, &c., we shall obtain

$$(\Omega\,.-\Omega)^3 = 6\Omega_2, \quad (\Omega\,.-\Omega)^4 = 24\Omega_3 + 12\Omega_1^2, \text{ \&c.};$$

on substituting these values in $\Delta + \frac{\Delta^2}{1\,.\,2} + \frac{\Delta^3}{1\,.\,2\,.\,3} + \ldots$ the coefficients of the several powers of h will be found to vanish.

The appearance in the above process of a zero whose powers are not zero is a phenomenon which will not shock those who are acquainted with Professor Peirce's discussions of possible algebras; but it is new to find it occur in working out a symbolical identity.

The theorem $f_1 = e^{h\Omega} \cdot f$ having, as far as I am aware, been first given by Dr Salmon in a form, if not quite complete, still sufficient for the immediate purpose to which it was to be applied, ought, I think, in justice to bear his name; and I see no reason why Salmon's Theorem in its totality should not be expected in the future to bear new fruit in algebraical expansions and other uses as important as have flowed from the one familiar and simplest case of it, known as Taylor's Theorem. Thus, *ex. gr.*, for the special case where f_1 becomes a function of one only of the quantities b_1, c_1, ... the Salmonian theorem reproduces Arbogast's celebrated one for expanding a rational integral function by the method of derivations, but under a greatly improved form of notation, and with the advantage of a test of convergency supplied by the *limit to the remainder* given in the text above. Who on a first casual reading could have imagined that Arbogast's problem in the differential calculus was virtually solved in an improved form in an article treating "on the symmetrical functions of the differences of the roots of an equation"? "*Que diable allait-il faire dans cette galère là?*" may rise to the lips of many a reader on being made acquainted with the fact*.

* Using Q to denote any rational integral function of x, Salmon's theorem is a theorem for expanding any function of Q, $\frac{dQ}{dx}$, $\frac{d^2Q}{dx^2}$, ... in terms of ascending powers of x.

12.

SUR LES INVARIANTS.

[*Comptes Rendus*, LXXXV. (1877), pp. 992—995, 1035—1039, 1091—1092.]

LA théorie que j'ai exposée dans mes dernières Communications à l'Académie repose sur le théorème suivant. Commençons par le cas d'une seule quantique du degré i, fonction des variables x et y, soit $(a, b, c, \ldots, l)(x, y)^i$. Je nomme différentiant de cette quantique une fonction rationnelle et entière quelconque, qui retient sa valeur quand on substitue pour les coefficients de la quantique donnée les coefficients de la quantique qu'on obtient en substituant $x+hy$ pour x. Alors le nombre de ces différentiants de l'ordre j dans les coefficients et du poids w par rapport à x sera égal à la différence entre deux nombres dont l'un est le nombre de combinaisons de j quelconques des chiffres $0.1.2\ldots.i$ (répétées autant de fois qu'on veut) dont la somme est w, moins le nombre de combinaisons pareilles pour lesquelles la somme est $(w-1)$. Nommons l'opérateur $a\dfrac{d}{db}+2b\dfrac{d}{dc}+3c\dfrac{d}{dd}+\ldots=\Omega$. La condition nécessaire et suffisante pour que D soit un différentiant est que ΩD soit identiquement zéro. De là on déduit facilement que le nombre des D linéairement indépendants, dont le poids est w et l'ordre s, soit $D\,(w:i, j)$, ne peut pas être moins que la différence dont j'ai parlé plus haut, soit la différence $(w:i, j)-\{(w-1);\ i,\ j\}$. Si les équations contenues dans l'identité $\Omega D=0$ sont indépendantes, la valeur de $D(w:i,\ j)$ sera égale à

$$(w:i,\ j)-\{(w-1)\,;\ i,\ j\}\,;$$

si elles ne sont pas indépendantes, ce nombre sera *plus grand* que

$$(w:i,\ j)-\{(w-1)\,;\ i,\ j\}.$$

Dans une Communication que je viens d'envoyer au *Journal de M. Borchardt*, j'ai réussi à donner une démonstration rigoureuse de l'égalité de $D(w:i,\ j)$ à la différence citée qu'on peut nommer $\Delta(w:i,\ j)$; car, si cette égalité n'était pas vraie pour toutes les valeurs de w, en commençant par la plus grande possible, c'est-à-dire $\dfrac{ij}{2}$ ou $\dfrac{ij-1}{2}$, alors on aurait pour cette valeur *maxima* de w

$$D\,(w:i, j)+D\,\{(w-1):i, j\}+D\,\{(w-2):i, j\}+\ldots+D\,(0:i, j)>(wi, j),$$

laquelle inégalité ne peut pas avoir lieu, comme je le démontre par une méthode très-belle et très-facile. C'est à M. Cayley qu'on doit l'énoncé de la proposition $D(w:i,j)=\Delta(w:i,j)$; mais ce grand géomètre n'avait réussi qu'à démontrer rigoureusement l'inégalité $D(w:i,j) =$ ou $> \Delta(w:i,j)$.

On avait même exprimé des doutes sur la vérité de la proposition, désormais mise à l'abri de toute objection, $D(w:i,j)=\Delta(w:i,j)$. Passons au cas de plusieurs quantiques $(a, b, c\ldots)(x, y)^i$, $(a, b, c\ldots)(x, y)^{i'}, \ldots$. J'ai étendu la méthode de M. Cayley à ce cas plus général. Par un procédé analogue au sien pour le cas d'une seule quantique, j'établis la proposition

$$D(w:i,j:i',j':\ldots) = \text{ou} > (w:i,\, j:i',j':\ldots) - \{(w-1):i,j:i',j':\ldots\},$$

où le premier membre de l'équation signifie le nombre de différentiants, linéairement indépendants, appartenant au système de quantiques donné de l'ordre j, $j', \ldots$, dans les quantiques successives et du poids w par rapport à x: $(n:i,j:i',j':\ldots)$, signifiant, pour une valeur quelconque de n, le nombre des combinaisons de j des chiffres $(0, 1, 2, 3, \ldots, i)$, de j' des chiffres $(0, 1, 2, \ldots, i'), \ldots$, dont la somme réunie est égale à n. Alors, par une méthode précisément identique avec celle que j'applique au cas d'une seule quantique, je démontre que l'inégalité

$$\begin{aligned} D(w:i,j:i',j':\ldots) + D\{(w-1):i,j:i',j':\ldots\} + \ldots \\ + D(0:i,j:i',j':\ldots) > (w:i,j:i',j':\ldots), \end{aligned}$$

où w représente la valeur maxima du poids w, ne peut pas avoir lieu et que conséquemment, pour *toutes* les valeurs de w,

$$D(w:i,j:i',j':\ldots) = \Delta(w:i,j:i',j':\ldots).$$

Donc la théorie de la construction de la fonction génératrice dont je me suis servi reste aujourd'hui sur une base inattaquable. Mais, même en l'absence de cette démonstration nouvellement trouvée, l'évidence de sa vérité, fondée sur l'improbabilité *a priori* d'aucune dépendance sur les autres équations de condition données par la formule $\Omega D=0$, conjointe avec l'accord parfait des résultats obtenus, en les supposant indépendants, avec les résultats qu'on obtient par d'autres méthodes pour tous les cas où l'on pouvait faire la comparaison, suffisait provisoirement comme démonstration *morale* de la vérité supposée. Or, chose bien remarquable, une difficulté de même nature revient quand on se sert de la fonction génératrice non pas en l'appliquant au calcul du nombre des dérivées invariantes linéairement indépendantes d'un type donné, mais en déduisant par son moyen l'échelle des dérivées élémentaires (*grundformen*). En un mot, la difficulté qui, aujourd'hui, a disparu quant à la formation de la fraction génératrice subsiste encore quand on passe à l'interprétation de cette fraction qui conduit à l'échelle de *grundformen*, mais avec une certaine différence. Quant à la proposition qui vient d'être nouvellement démontrée, la difficulté autrefois

consistait à démontrer l'absence de rapports syzygétiques quelconques. Mais, dans l'application dont je parle, on admet par nécessité l'existence de certains de ces rapports, qui *se révèlent* comme conséquence de la loi élémentaire de toute combinaison algébrique d'invariants. L'hypothèse que l'on fait, c'est qu'il n'existe pas de tels rapports (pour ainsi dire *cachés*) en dehors de ceux dont l'existence est apparente.

Si l'on voulait nier l'exactitude de cette hypothèse, voici ce qui arriverait: les formes élémentaires (*grundformen*) obtenues en l'admettant ne cesseraient pas de subsister comme telles; seulement il y aurait la possibilité (pour ainsi dire métaphysique) de l'existence d'autres en plus. Prenons, par exemple, le cas de deux biquadratiques. M. Gordan en a donné 30, dont j'ai démontré que 2 sont superflues: il en reste donc 28. La méthode de M. Gordan ne suffit pas pour démontrer que ce nombre n'est pas encore assujetti à une réduction au-dessous de 28; mais ma méthode, au contraire, quoique laissant provisoirement peser un doute métaphysique sur l'existence de plus de 28, n'en laisse aucun sur la certitude qu'au moins ces 28 subsistent. Donc on est assuré que les 28 en question forment l'échelle fondamentale. La méthode de M. Gordan assure qu'il n'y a pas plus que 28, la méthode anglaise qu'il n'y a pas moins que 28 invariants et covariants élémentaires; donc le nombre est 28, ni plus ni moins. On comprend que l'incertitude dont je parle dans l'application de la méthode anglaise n'est que provisoire et, pour ainsi dire, métaphysique; l'évidence, à dire vrai, est accablante et ne peut laisser subsister aucun doute moral que les rapports syzygétiques cachés ou latents, dont j'ai parlé, n'ont aucun lieu dans la sphère de réalité. Cependant il semble bon de confirmer ce *postulatum*, en donnant encore des exemples, comme je vais le faire, de la conformité des résultats auxquels il conduit avec ceux qu'on obtient par d'autres méthodes. De plus, on doit se rappeler que chacune de mes fractions génératrices donne encore des résultats en dehors de la formation de l'échelle fondamentale, qu'on ne sait pas obtenir par la méthode de M. Gordan ni par aucune autre méthode connue. Elle donne absolument, et sans suggestion à aucun doute métaphysique, le nombre total des invariants, covariants, etc., les mouvements indépendants de degrés et d'ordres donnés, et, une fois la vérité absolue de la conclusion quant à l'échelle fondamentale pour un cas donné étant ou admise ou prouvée par l'évidence, elle donne en même temps et immédiatement tous les rapports syzygétiques qui peuvent lier ensemble les formes qui entrent dans l'échelle fondamentale. Bien plus, non-seulement les *grundformen* ne sont pas indépendantes, mais les équations qui les lient, en général, ne seront pas non plus indépendantes. Voici la vraie idée de ces rapports successifs:

On commence avec les *grundformen*. Alors il y aura des fonctions algébriques, qu'on peut nommer des *syzygants* du premier rang et qui auront la propriété de s'évanouir quand on substituera aux *grundformen*

leurs valeurs comme fonctions des coefficients des quantités données. De même il y aura des fonctions algébriques de ces syzygants qu'on peut nommer des syzygants du second rang, qui auront la propriété de s'évanouir quand on substituera pour les syzygants du premier rang leurs valeurs comme fonctions des *grundformen*, et ainsi de suite, de sorte qu'il y aura une succession de syzygants de rangs de plus en plus élevés, et pour les syzygants de chaque rang il y aura une échelle fondamentale finie. Je crois que l'indice des rangs ascendants ne va jamais à l'infini. Sous ce point de vue, on voit que les formes fondamentales (*grundformen*) elles-mêmes peuvent être regardées comme des syzygants du rang zéro. Or ma fraction génératrice donne le moyen d'obtenir l'échelle fondamentale pour les syzygants d'un rang quelconque. Le procédé pour l'obtenir dans les cas du rang zéro et du rang unité est aussi simple pour l'un que pour l'autre. Quant aux syzygants de rang supérieur, le calcul peut être un peu plus compliqué, et je ne me suis pas permis jusqu'à présent d'entrer dans ce calcul. Il est singulier de remarquer l'inversion de rôles qui a lieu entre les deux problèmes, l'un de trouver les formes élémentaires et les syzygants successifs qui en découlent, l'autre de trouver le nombre total de formes dérivées d'un type donné. On aurait pensé *a priori* que la solution du premier problème serait nécessaire pour arriver à la solution du second. Mais, en réalité, la marche de l'investigation est toute contraire. Grâce à l'initiative admirable pour tout jamais de M. Cayley, dans son second Mémoire sur les *Quantics*, on sait comment résoudre d'un seul coup le second problème et de la forme même de cette solution on fait découler pas à pas la solution du premier.

Je vais donner les fractions génératrices pour trois nouveaux cas pour lesquels on peut comparer les résultats quant à l'échelle fondamentale avec des résultats déjà connus. Ces trois cas seront: (1) celui d'un système contenant une forme linéaire et une forme cubique; (2) d'un système contenant une forme quadratique et une forme cubique; (3) d'un système de deux cubiques. Dans une Communication prochaine, je donnerai la théorie qui s'applique aux cas d'un nombre indéfini de formes linéaires et d'un nombre indéfini de formes quadratiques. Entre ces cas il existe un lien vraiment surprenant. Je n'ai pas besoin de dire que, par rapport aux considérations qui limitent l'horizon des recherches de l'école allemande en matière de formes algébriques, ces deux cas n'offrent à peine aucune prise pour construire une théorie, ou pour mieux dire la théorie qu'on construit s'épuise en quelques mots; au contraire, selon les idées constituantes de la méthode anglaise, ces deux cas mènent à une théorie très-étendue et à des recherches du plus haut intérêt. En effet, le premier cas est celui de la théorie des rapports syzygétiques de fonctions des différences d'un nombre

quelconque donné de quantités, théorie qui doit réagir puissamment sur celles de formes de degrés quelconques; de plus, dans le traitement de l'un et l'autre cas, j'aurai occasion de donner une solution de certains problèmes de l'Algèbre ordinaire de la plus grande beauté, en faisant appel à des principes algébriques que je crois être d'un genre tout à fait nouveau.

Commençons par le cas d'un système composé d'une forme linéaire et d'une cubique. Le dénominateur de la fraction génératrice sous la forme canonique sera

$$(1-b^4)(1-b^2a^2)(1-ba^3)(1-ax)(1-b^2x^2)(1-bx^3),$$

où a est le symbole pour la fonction linéaire, et b pour la cubique. Ainsi il y aura six formes fondamentales primaires: L'invariant et la hessienne de la cubique, les deux formes données, leur résultant (typifié par ba^3) et le résultant de la hessienne et la forme linéaire typifiée par b^2a^2.

Le numérateur est

$$\begin{array}{ll} 1+a^3b^3 & +(-ab^3-a^4b^6)\,x^4 \\ +(a^2b+ab^2+a^2b^3-a^4b^3)\,x & +(b^3-a^2b^3-a^3b^4-a^4b^5)\,x^3 \\ +(ab+ab^3-a^3b^3-a^3b^5)\,x^2. & \end{array}$$

Les termes positifs ne perdent rien en étant assujettis au tamisage. Il reste donc sept formes fondamentales secondaires:

1 invariant typifié par	3 . 3 . 0		
3 covariants linéaires	2 . 1 . 1	1 . 2 . 1	2 . 3 . 1
2 covariants quadratiques	1 . 1 . 2	1 . 3 . 2	
1 covariant cubique	0 . 3 . 3		

ce dernier appartenant à la cubique prise séparément.

Prenons, en deuxième lieu, le système composé d'une quadratique et d'une cubique. Le symbole a appartiendra à la première, b à la seconde.

La fraction génératrice, sous sa forme canonique, aura pour dénominateur

$$(1-a^2)(1-b^4)(1-ab^2)(1-a^3b^2)(1-ax^2)(1-bx^3)(1-b^2x^2)$$

et pour numérateur

$$\begin{array}{l} (1+a^3b^4) \\ +(ab+a^2b+ab^3+a^2b^3)\,x \\ +(ab^2+a^2b^2+a^3b^2+a^2b^4-a^4b^4-a^3b^6)\,x^2 \\ +(ab+b^3-a^2b^3-ab^4-a^2b^5-a^3b^5)\,x^3 \\ +(-a^2b^4-a^3b^4-a^2b^6-a^3b^6)\,x^4 \\ +(-ab^3-a^4b^7)\,x^5. \end{array}$$

Le produit constant de chaque couple conjugué est, comme on voit, $-a^4b^7x^5$, et le rapport, qui est toujours constant entre les termes conjugués qui figurent dans la partie sans x, et la partie qui multiplie la plus haute puissance de x de ces fractions génératrices, est $-ab^3$.

Ainsi on a sept formes fondamentales primaires: les deux invariants des formes données, prises séparément; deux autres invariants dont l'ordre, dans les coefficients de la quadratique et de la cubique, respectivement, est pour l'un (1, 2) et pour l'autre (3, 2), les deux formes données elles-mêmes et la hessienne de la cubique.

Quant au numérateur, on voit que les seuls coefficients positifs qui disparaissent sous le tamisage sont: $a^2b^2x^2$, $a^3b^2x^2$, $a^2b^4x^2$. Il reste les sept formes fondamentales secondaires, figurées par ces nombres:

1 invariant		3 . 4 . 0		
4 covariants linéaires........	1 . 1 . 1	2 . 1 . 1	1 . 3 . 1	2 . 3 . 1
1 covariant cubique		1 . 2 . 2		
2 covariants cubiques.........		1 . 1 . 3	0 . 3 . 3	

ce dernier appartenant à la cubique donnée, prise séparément.

Comme dernier cas prenons le système composé de deux cubiques binaires ayant a et b pour leurs symboles.

Le dénominateur de la fraction génératrice canonique sera

$$(1-a^4)(1-b^4)(1-ab)(1-ab^3)$$
$$\times(1-a^3b)(1-ax^3)(1-a^2x^2)(1-bx^3)(1-b^2x^2)$$

donnant neuf formes fondamentales primaires dont les invariants et les hessiennes des cubiques données constituent 6 et en outre les trois invariants ayant pour symboles $ab : ab^3 : a^3b$. Son numérateur sera

$$\begin{aligned}
&1+a^2b^2+a^3b^3+a^5b^5\\
&\quad+(ab^2+a^2b+ab^4+a^4b+a^3b^2+a^2b^3+a^3b^4+a^4b^3)\,x\\
&\quad+(ab+ab^3+a^2b^2+a^3b+a^4b^4-a^5b^7-a^7b^5)\,x^2\\
&\quad+(a^3+a^2b+ab^2+b^3-a^4b-ab^4-ab^6-2a^3b^4-2a^4b^3-a^6b-2a^5b^6-2a^6b^5)\,x^3\\
&\quad+(ab-ab^5-a^2b^4-a^3b^3-a^4b^2-a^5b-a^2b^6-2a^3b^5-2a^4b^4\\
&\qquad-2a^5b^3-a^6b^2-a^3b^7-a^4b^6-a^5b^5-a^6b^4-a^7b^3+a^7b^7)\,x^4\\
&\quad+(-2a^2b^3-2a^3b^2-a^2b^7-2a^4b^5-2a^5b^4\\
&\qquad-a^7b^2-a^4b^7-a^7b^4+a^5b^8+a^6b^7+a^7b^6+a^8b^5)\,x^5\\
&\quad+(-ab^3-a^3b+a^4b^4+a^5b^7+a^6b^6+a^7b^5+a^7b^7)\,x^6\\
&\quad+(a^4b^5+a^5b^4+a^5b^6+a^6b^5+a^4b^7+a^7b^4+a^6b^7+a^7b^6)\,x^7\\
&\quad+(a^3b^3+a^5b^5+a^6b^6+a^8b^8)\,x^8.
\end{aligned}$$

On remarquera que le produit constant général pour les termes conjugués est ici $+a^8b^8x^8$; bien entendu que chaque terme précédé par un coefficient, disons k, doit être compté comme k termes avec le coefficient unité dont chacun aura été conjugué. On remarquera aussi le rapport constant de $1 : a^3b^3$ entre les quatre termes au commencement et les coefficients des

quatre à la fin, et de plus le produit constant partiel pour ces deux groupes, c'est-à-dire a^5b^5 pour l'un, et conséquemment $a^{11}b^{11}$ pour l'autre. Ces trois théorèmes, le produit constant général, le produit constant pour la partie qui symbolise ces invariants et le rapport constant entre les termes de cette partie et les coefficients en nombre égal à la fin, sont des caractères permanents pour toutes les fractions génératrices dont on se sert dans le calcul des invariants, et qu'on peut démontrer *a priori*.

En soumettant les termes positifs au tamisage, on trouvera sans peine que les seuls qui restent seront les suivants:

$$a^2b^2,\ a^3b^3.$$
$$ab^2x,\ a^2bx,\ ab^4x,\ a^4bx,\ a^3b^2x,\ a^2b^3x,\ a^3b^4x,\ a^4b^3x,$$
$$abx^2,\ ab^3x^2,\ a^2b^2x^2,\ a^3bx^2.$$
$$a^3x^3,\ a^2bx^3,\ ab^2x^3,\ b^3x^3.$$
$$abx^4.$$

Donc il y a 19 formes fondamentales secondaires, *savoir*:

2 invariants	typifiés par	2.2.0 3.3.0
8 covariants linéaires...	„	1.2.1 2.1.1 1.4.1 4.1.1 3.2.1 2.3.1 3.4.1 4.3.1
4 covariants quadratiques	„	1.1.2 1.3.2 2.2.2 3.1.2
4 covariants cubiques...	„	3.0.3 2.1.3 1.2.3 0.3.3
1 covariant biquadratique	„	1.1.4.

Les résultats sont en parfait accord avec le résumé de M. Salmon, fondé sur les travaux de MM. Clebsch et Gordan: *Lessons on Higher Algebra*, 3e édition, p. 186, qui se trouvent ainsi pleinement confirmés, de sorte qu'on sait *apodictiquement* que rien de superflu ne peut être contenu dans leur Table des *Grundformen* pour ce cas-ci.

Ici, il est nécessaire de faire une remarque très-importante sur une omission d'un certain procédé, qui, dans ma méthode, doit précéder celui de tamisage. Cette omission n'a aucune importance pour les cas que nous avons considérés, car les circonstances qui rendent nécessaire l'application de ce procédé additionnel n'existent pas pour ces cas-là, et il semble souvent ne pas arriver que dans le cas où il y a un très-grand nombre de formes comprises dans le système donné, lequel nombre, apparemment, croît avec les degrés de ces formes. C'est dans l'étude des systèmes de formes linéaires ou quadratiques que ce phénomène, dont je vais parler, s'était présenté pour la première fois, et seulement quand ce système ne comprend pas moins de quatre formes. Dans toutes les huit fractions génératrices que j'ai données dans cette Note et dans mes Communications précédentes, on trouvera facilement que, si l'on développe ces fractions en séries, les coefficients positifs ne subiront pas une diminution quelconque. Mais, quand cela arrive, c'est-à-dire quand un tel coefficient ou disparaît, ou subit une diminution, alors il faut substituer, au lieu du coefficient

dans le numérateur, le chiffre diminué (qui peut être zéro, mais, comme je l'ai démontré dans l'article* cité, destiné au *Journal de Crelle,* jamais négatif). Donc, comme règle générale (quoique presque jamais nécessaire dans la pratique), il faut soumettre chaque coefficient à cet examen, auquel je donne le nom de *triage.* Voici donc le tableau complet de mes procédés pour arriver à l'échelle des formes invariantes des dérivées fondamentales:

(1) Formation de la fraction génératrice dans sa forme crue dont le développement donnerait une série allant vers l'infini dans deux directions qu'on pourrait nommer série *bivergente*;

(2) Retranchement de la partie contenant des indices négatifs et substitution d'une fraction génératrice réduite, dont le développement en série sera *univergent*;

(3) Multiplication du numérateur et du dénominateur de la fraction réduite par un facteur commun propre à mettre le dénominateur sous une forme telle, que chaque facteur, comme $1 - a^\alpha b^\beta c^\gamma \ldots x^\lambda$, qu'il contient, correspondra à un covariant ou invariant, dont le type est $\alpha, \beta, \gamma, \ldots, \lambda$, laquelle condition sera satisfaite si, en faisant le développement en série, le terme $a^\alpha b^\beta c^\gamma \ldots x^\lambda$ ne se trouve pas aboli. La fraction est alors canonique;

(4) Triage appliqué à la diminution ou suppression des coefficients positifs du numérateur, quand cela est nécessaire;

(5) Tamisage appliqué aux coefficients ainsi triés.

Il est bon aussi de remarquer que, sans former la fonction génératrice, on peut appliquer ma méthode à la solution complète par des méthodes purement arithmétiques du problème suivant, qui, en effet, est la partie laissée incomplète dans la théorie de M. Gordan:

Étant donnés les types d'une assemblée de formes entre lesquelles sont composées toutes les GRUNDFORMEN *d'un système de formes données, on désire éliminer toutes celles qui sont superflues.*

C'est ainsi que j'ai mis à l'épreuve les résultats donnés par M. Gundelfinger, pour le cas d'un système composé d'une forme cubique et une forme biquadratique, car j'ai reculé, pour le moment, devant le travail énorme qui serait nécessaire pour former la fraction génératrice applicable à ce cas, et, comme résultat de cet examen (sauf la possibilité d'erreurs d'Arithmétique), je crois pouvoir affirmer que, sur les soixante-quatre *grundformen* prétendues, deux sont superflues, mais que les autres soixante-deux restent bonnes. Je compte revenir sur ce cas spécial dans une autre Communication que j'espère avoir l'honneur de faire à l'Académie sur ce sujet.

[* p. 218 below.]

13.

ON THE LIMITS TO THE ORDER AND DEGREE OF THE FUNDAMENTAL INVARIANTS OF BINARY QUANTICS.

[*Proceedings of the Royal Society of London,* XXVII. (1878), pp. 11, 12.]

THE developments which I have recently given to Professor Cayley's second method of dealing with invariants (the first method being that which has been exclusively used by Professor Gordan), has led me through the theory of the Canonical Generating Fraction to the following results, showing that the degree and order of the fundamental invariants and covariants to a quantic or system of quantics are subject to algebraical limits of a very simple kind, and I think it right that these results should not be withheld from the knowledge of those who are pursuing another and, as it seems to me, much more arduous and less promising direction of inquiry into the same subject.

By order I mean the dimensions of a derived form in the coefficients of its primitive (Clebsch and Gordan's *grad*), and by degree the dimensions in the variables (Clebsch and Gordan's *ordnung*).

First as to degree.

If there be a system of $n, n', n''\,..$ odd degreed quantics and $\nu, \nu', ..$ &c., even ones, then (with the exception of the case when the system reduces to a single linear function or a single quadratic) the degree of any irreducible covariant to the system has for a superior limit $\Sigma\left(\frac{n^2+1}{2}\right)+\Sigma\left(\frac{\nu^2}{2}\right)-2$.

Thus, for example, where there is but one quantic, the limit is $\frac{n^2-3}{2}$ or $\frac{\nu^2-4}{2}$, according as the degree is n odd or ν even.

Secondly, as to order.

As the expressions become somewhat complicated when there are several quantics, I shall confine myself to a statement applicable to a single quantic,

distinguishing between the three cases when n (its degree) is evenly even, oddly even, and odd.

A. When n contains 4, the superior limits for the order of the invariants and covariants respectively are for the former $\frac{(n+1)(n-4)}{2}$, and for the latter $\frac{(n+2)(n-3)}{2}$.

B. When n is even, but not divisible by 4, and is greater than 2, the limits for the two species are $\frac{3n^2-6n-12}{4}$ and $\frac{(n+2)(3n-8)}{4}$ respectively.

C. When n is any odd number greater than 3, the order of the invariants has for its limit $\frac{3}{2}(n+1)(n-3)$, and when it is any odd number greater than unity, the order of the covariants has for its limit $\frac{3n^2-4n-9}{2}$.

Further investigations will, I have good reason to believe, lead to considerably lower limits than those given for cases *B* and *C*.

Although morally certain, the three formulæ *A*, *B*, *C* cannot be considered at present apodictically established; the formula respecting the limit to *degree* may, I believe, be regarded as admitting of a complete demonstration. There exists, however, a superior limit to the orders of the fundamental invariants or covariants, which may be regarded as subject to direct demonstration even in our present state of knowledge; this when n is even is n^2-2n-3 for invariants, and n^2-n-4 for covariants; and when n is odd, the corresponding limits are $2n^2-3n-5$ for invariants, and $2n^2-2n-5$ for covariants. But I have no moral doubt whatever of the validity of the formulæ *B* and *C* as they stand, and next to none of the validity of formula *A*.

14.

CHEMISTRY AND ALGEBRA.

[*Nature*, XVII. (1877—1878), pp. 284, 309.]

IT may not be wholly without interest to some of the readers of *Nature* to be made acquainted with an analogy that has recently forcibly impressed me between branches of human knowledge apparently so dissimilar as modern chemistry and modern algebra. I have found it of great utility in explaining to non-mathematicians the nature of the investigations which algebraists are at present busily at work upon to make out the so-called *Grundformen* or irreducible forms appurtenant to binary quantics taken singly or in systems, and I have also found that it may be used as an instrument of investigation in purely algebraical inquiries. So much is this the case that I hardly ever take up Dr Frankland's exceedingly valuable *Notes for Chemical Students,* which are drawn up exclusively on the basis of Kekulé's exquisite conception of *valence,* without deriving suggestions for new researches in the theory of algebraical forms. I will confine myself to a statement of the grounds of the analogy, referring those who may feel an interest in the subject and are desirous for further information about it to a memoir which I have written upon it for the new *American Journal of Pure and Applied Mathematics,* the first number of which will appear early in February.

The analogy is between atoms and *binary* quantics exclusively.

I compare every binary quantic with a chemical atom. The number of factors (or rays, as they may be regarded by an obvious geometrical interpretation) in a binary quantic is the analogue of the number of *bonds,* or the *valence,* as it is termed, of a chemical atom.

Thus a linear form may be regarded as a monad atom, a quadratic form as a duad, a cubic form as a triad, and so on.

An invariant of a system of binary quantics of various degrees is the analogue of a chemical substance composed of atoms of corresponding *valences.*

The order of such invariant in each set of coefficients is the same as the number of atoms of the corresponding *valence* in the chemical compound.

A co-variant is the analogue of an (organic or inorganic) compound radical. The orders in the several sets of coefficients corresponding, as for invariants, to the respective valences of the atoms, the free valence of the compound radical then becomes identical with the degree of the co-variant in the variables.

The weight of an invariant is identical with the number of the bonds in the chemicograph of the analogous chemical substance, and the weight of the leading term (or basic differentiant) of a co-variant is the same as the number of bonds in the chemicograph of the analogous compound radical. Every invariant and covariant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph. But not every chemicograph is an algebraical one. I show that by an application of the algebraical law of reciprocity every algebraical graph of a given invariant will represent the constitution in terms of the roots of a quantic of a type reciprocal to that of the given invariant of an invariant belonging to that reciprocal type. I give a rule for the geometrical multiplication of graphs, that is, for constructing a *graph* to the product of in- or co-variants whose separate graphs are given. I have also ventured upon a hypothesis which, whilst in nowise interfering with existing chemicographical constructions, accounts for the seeming anomaly of the isolated existence as "monad molecules" of mercury, zinc, and arsenic—and gives a rational explanation of the "mutual saturation of bonds."

I have thus been led to see more clearly than ever I did before the existence of a common ground to the new mechanism, the new chemistry, and the new algebra. Underlying all these is the theory of pure colligation, which applies undistinguishably to the three great theories, all initiated within the last third of a century or thereabouts by Eisenstein, Kekulé, and Peaucellier.

15.

SUR LA LOI DE RÉCIPROCITÉ POUR LES INVARIANTS ET COVARIANTS DES QUANTICS BINAIRES.

[*Comptes Rendus*, LXXXVI. (1878), 446—448.]

A UN invariant ou covariant donné d'un quantic binaire du degré i de l'ordre j dans les coefficients, M. Hermite a montré qu'il répond toujours un invariant ou covariant (du même degré) de l'ordre i dans les coefficients, mais appartenant à un quantic du degré j, et il a fourni un procédé pour passer de l'un à l'autre.

Je vais donner une généralisation de ce théorème en l'étendant à un système de quantics binaires, et une méthode plus facile pour faire la transformation pour le cas ou d'un seul quantic ou d'un système. Soit D un invariant ou covariant du degré δ appartenant à un système de quantics binaires des degrés respectifs $i, i', i'', \ldots$, dont l'ordre dans les coefficients des quantics est respectivement $j, j', j'', \ldots$. Je dis qu'il y répond un invariant Δ ou covariant du degré δ appartenant à un système de quantics binaires des degrés respectifs $j, i', i'', \ldots$, dont l'ordre dans ces coefficients des quantics est respectivement $i, j', j'', \ldots$, de sorte qu'à une forme comprise dans le type $i, j\,;\ i', j'\,;\ i'', j'', \ldots,\ :\ \delta$ il en répond une autre comprise dans le type $j, i\,;\ i', j'\,;\ i'', j'', \ldots,\ :\ \delta$.

Cela étant vrai pour le couple d'indices i, j sera nécessairement vrai pour tous les couples ou pour une combinaison quelconque des couples $i, j, \ldots;\ i', j', \ldots;\ i'', j'', \ldots$; mais il suffit évidemment de donner les règles de transformation pour l'échange entre eux d'un seul couple d'indices conjugués i et j.

Pour l'effectuer, voici tout ce qui est nécessaire:

Regardons le coefficient de x^i dans le quantic du degré i comme égal à un; alors tous les coefficients de ce quantic deviennent fonctions symétriques des racines $e_1, e_2, \ldots, e_i$. Qu'ils soient exprimés ainsi, alors chaque terme de D sera de la forme $Me_1^{\alpha} e_2^{\beta} e_3^{\gamma} \ldots e_i^{\lambda}$; bien entendu qu'un ou plusieurs des chiffres $\alpha, \beta, \gamma, \ldots, \lambda$ peuvent devenir zéro.

Au lieu de ce terme, écrivons

$$M\eta_\alpha \eta_\beta \eta_\gamma \ldots \eta_\lambda, \quad \text{ou} \quad \eta_\alpha = (-)^\alpha \epsilon_\alpha,$$

$\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_\alpha$, étant les éléments du quantic général $(\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_j)(x, y)^j$.

L'expression ainsi obtenue sera évidemment de l'ordre i, quant aux coefficients ϵ, et de plus elle sera un invariant ou un covariant (du même degré que le primitif).

La preuve en est facile, ne dépendant que de l'application de l'équation partielle différentielle, qui sert pour définir un invariant ou différentiant : elle est donnée avec des exemples de son application dans un Mémoire qui doit paraître prochainement dans l'*American Journal of Mathematics*, publié à Baltimore.

Je me borne ici à ajouter quelques mots sur l'usage du terme *réciprocité,* sans lesquels on pourrait aisément se tromper sur la véritable portée du théorème; et, pour plus de clarté, je ne sortirai pas du cas le plus simple, celui d'un seul quantic du type $i, j : \delta$, dont le type conjugué sera $j, i : \delta$.

Supposons que de D appartenant au premier type on ait passé à Δ appartenant au type conjugué $j, i : \delta$. Qu'on répète le procédé, on retournera au type donné $i, j : \delta$. Or il importe beaucoup de savoir si ou non on retournera à la forme donnée D en regardant si l'on veut comme identiques les formes qui ne diffèrent l'une de l'autre que par un multiplicateur numérique.

Pour répondre à cette question, il sera bon de se servir d'une nouvelle définition. J'appelle la *multiplicité* d'un type $j, i : \delta$ le nombre de formes linéairement indépendantes qui y sont attachées, ou, ce qui revient au même, le nombre de paramètres numériques arbitraires de la forme la plus générale qui est représentée par ce type. On peut nommer ces formes ou ces types monadelphiques, diadelphiques, etc., selon la valeur de la multiplicité.

Or, pour ces types monadelphiques en retournant au même type, on retourne nécessairement à la même forme, de sorte que la question que j'ai proposée se limite nécessairement aux types polyadelphiques. Or je suis en mesure d'affirmer qu'en général, en transformant deux fois un quantic appartenant à un type de la multiplicité k, il n'y a que k formes particulières qui se reproduisent identiquement. En donnant des valeurs arbitraires aux k paramètres, on retourne au même type, sans retourner à la même forme, de sorte que D ne peut pas se déduire de Δ comme Δ de D; et ainsi la *réciprocité,* tellement nommée, est essentiellement une réciprocité de types et non pas de formes. Quant aux formes spéciales (disons principales) qui se reproduisent et qui possèdent des réciproques dans un sens étroit, il est facile de voir qu'on peut les déterminer avec l'aide d'une équation algébrique du degré k, très-analogue à l'équation pour trouver les axes principaux d'une courbe ou surface, ou hypersurface, etc., du second degré; j'ai expérimenté,

comme on peut voir dans le Mémoire cité, sur des types diadelphiques, et je trouve, dans les cas que j'ai étudiés, que les exercices de l'équation quadratique à résoudre sont rationnels; mais je ne puis affirmer que cela aura toujours lieu. L'équation dont je parle exprime le rapport numérique entre chaque forme principale et, si je puis me servir de l'expression, seconde *image*, c'est-à-dire l'image de Δ comme Δ est l'image de D. Ses racines ou au moins leurs rapports sont indépendants de toute convention, et sont en effet des constantes absolues de la raison humaine; ainsi il me paraît que la constitution de ces équations mérite d'être étudiée à fond. Sans la règle simplifiée que j'ai donnée pour trouver les images, le travail nécessaire dans le cas des types polyadelphiques serait, à cause de sa longueur, presque inexécutable, et même avec cette simplification le travail est assez pénible. Quoique la nouvelle méthode de former l'image d'une dérivée invariantive possède (il me semble) un avantage considérable quant à la facilité du calcul, cependant la route frayée par M. Hermite a une très-grande utilité, car avec son aide on voit instinctivement que chaque invariant ou covariant binaire équivaut à un hyperdéterminant, et l'on peut même calculer par un procédé direct l'hyperdéterminant qui représente un invariant ou covariant binaire donné.

16.

SUR LA THÉORIE DES FORMES ASSOCIÉES DE MM. CLEBSCH ET GORDAN.

[*Comptes Rendus*, LXXXVI. (1878), pp. 448—450.]

DANS le Traité de Clebsch sur les formes binaires, on trouve un théorème très-remarquable sur ce qu'il appelle les *formes associées*, et sur le système le plus simple des formes associées.

Je me bornerai à l'exposition et à la généralisation de cette dernière. Voici le théorème comme on le trouve dans le travail de M. Clebsch : Soient Q un quantic binaire quelconque du degré i, f un invariant ou covariant quelconque de Q. En choisissant convenablement le chiffre μ, $Q^\mu f$ sera une fonction entière et rationnelle de i invariants et covariants, constants et connus de Q, dont le premier sera Q et les autres successivement de l'ordre 2 et 3 dans les coefficients de Q. Si l'on examine de près ce théorème avec l'aide de la conception et des propriétés des différentiants, voici à quoi il équivaut: Prenons la forme $x^i + px^{i-1} + qx^{i-2} + \ldots + l$.

On sait bien qu'une fonction symétrique quelconque de ses racines sera une fonction rationnelle et entière des i coefficients donnés. Mais, si l'on se borne à une fonction symétrique des *différences* des racines, on peut ajouter (et voilà en quoi consiste essentiellement ce théorème de M. Clebsch ou de M. Gordan) qu'elle sera une fonction rationnelle et entière de $i-1$ fonctions alternativement de l'ordre 2 et de l'ordre 3 des coefficients, dont chacune sera elle-même une fonction des différences des racines.

C'est par une analyse assez compliquée que MM. Clebsch et Gordan établissent leur théorème. Je le déduis par un calcul tout à fait élémentaire et presque instantané en me servant seulement de l'équation partielle différentielle qui sert à définir les invariants et les différentiants et avec ce grand avantage que, avec son aide, je passe immédiatement à l'extension du théorème au cas de système de quantics. Voici en effet le résultat auquel j'arrive avec cette méthode.

Soit $Q_1, Q_2, Q_3, \ldots, Q_\lambda$ un système de quantics binaires. Prenons $(\lambda-1)$ jacobiens indépendants quelconques des Q combinés en paires qu'on peut nommer $J_1, J_2, \ldots, J_{\lambda-1}$ et de plus prenons les a formes associées dans leur forme la plus simple qui appartiennent à $Q_1, Q_2, \ldots, Q_\lambda$ prises séparément. Alors, je dis que, f étant un invariant ou covariant quelconque du système des Q, on aura, en choisissant convenablement les chiffres $\mu_1, \mu_2, \ldots, \mu_\lambda$, $Q_1{}^{\mu_1} Q_2{}^{\mu_2} \ldots Q_\lambda{}^{\mu_\lambda} f$ une fonction rationnelle et entière des formes associées propres à $Q_1, Q_2, \ldots, Q_\lambda$ et des quantités $J_1, J_2, \ldots, J_{\lambda-1}$.

J'ajouterai encore un théorème que je crois être nouveau et qui se déduit immédiatement de ce dernier.

Soient $a_1, b_1, \ldots; a_2, b_2, \ldots; a_\lambda, b_\lambda$ les deux premiers coefficients de $Q_1, Q_2, \ldots, Q_\lambda$ et prenons la forme linéaire $a_k x + b_k y$ (k étant choisi arbitrairement), que je nommerai u. Soit un invariant ou un covariant quelconque du système exprimé comme fonction de u et de y, alors tous les coefficients de F seront des différentiants en x, ce que M. Cayley nomme des *semi-invariants*. Ainsi, par exemple, si l'on prend le covariant bien connu

$$(ac-b^2)\,x^2 + (ad-cb)\,xy + (bd-c^2)\,y^2$$

appartenant à un seul quantic $(a, b, c, d)(x, y)^3$, on peut le mettre sous la forme

$$\frac{1}{a^2}\{(ac-b^2)(ax+by)^2 + (a^2d-3abc+2b^3)(ax+by)\,y - (ac-b^2)^2 y^2\},$$

où, en supposant $a=1$, tous les coefficients deviennent des fonctions des différences des racines de $(1, b, c, d)(x, y)^3 = 0$.

La preuve de ces théorèmes sera donnée dans l'*American Journal of Mathematics* publié à Baltimore (États-Unis de l'Amérique), qui doit paraître prochainement.

17.

DÉTERMINATION D'UNE LIMITE SUPÉRIEURE AU NOMBRE TOTAL DES INVARIANTS ET COVARIANTS IRRÉDUCTIBLES DES FORMES BINAIRES.

[*Comptes Rendus*, LXXXVI. (1878), pp. 1437—1441, 1491, 1492, 1519—1522.]

LA méthode que je vais exposer s'applique aux cas de systèmes quelconques des formes binaires; mais, pour plus de concision, je me bornerai au cas d'un seul quantic de degré pair: cela suffira pour donner une idée nette de la méthode, ce qui est tout ce que je me propose de faire dans cette première Communication.

Je démontre facilement que le nombre total des invariants ou covariants appartenant au quantic binaire du degré $2t$, de l'ordre μ, dans les coefficients du quantic, sera le coefficient de t^μ dans le développement de

$$\frac{F(t)}{(1-t)(1-t^2)^2(1-t^3)\ldots(1-t^{2i-i})}$$

en puissances ascendantes de t, où $F(t)$ est une fonction rationnelle et entière de t, qu'on sait comment obtenir.

Je donne le nom de *covariants primaires* aux $2i$ covariants, pour lesquels les coefficients de la plus haute puissance de x [en représentant le quantic par $(a, b, c, d, e, f, \ldots)(x, y)^{2i}$] sont

$$a : ac - b^2 : ae - 4bd + 3c^2 : a^2d - 3abc + 2b^3$$
$$: a(a^2f - \ldots) : a^3(ag - \ldots) : a^3(a^2h - \ldots) : a^5(ak - \ldots),$$

et je nomme *covariants* (invariants compris) *adjoints* ceux qui, pris en conjonction avec les primaires, formeront un système tel, que tout autre covariant sera une fonction rationnelle et entière de ceux qui sont compris dans ce système.

Je regarde la fonction $F(t)$, qui ne contient en effet qu'un nombre fini de termes actuels, comme si elle contenait un nombre infini de puissances positives de t, dont les coefficients qui correspondent aux termes qui manquent sont des zéros.

Prenons un terme quelconque en $F(t)$, disons lt^λ. Le nombre des adjoints linéairement indépendants de l'ordre λ peut être, ou égal à l, ou plus grand, ou plus petit. Quand ce nombre est plus grand, je nomme la différence *l'excès* pour l'indice λ; quand il est plus petit, *le défaut* (en faisant exception du cas $\lambda = 0$, que je regarde comme n'ayant ni manque ni excès).

Quand il y a excès, je distingue arbitrairement les adjoints en deux groupes: l'un contenant le nombre l et l'autre l'excès; et, en mettant de côté pour le moment ces derniers, je regarde tous les autres adjoints comme formant un seul système, que je nomme *système d'auxiliaires.*

Soit σ la somme des coefficients positifs en $F(t)$, Δ la somme de tous les défauts, et conséquemment $\sigma - 1 - \Delta$ le nombre des auxiliaires. Or, supposons qu'il existe au moins n adjoints surnuméraires, c'est-à-dire des adjoints pour lesquels la somme des excès est n; je démontre rigoureusement qu'en nommant τ le nombre des coefficients négatifs (s'il y en a), il existera au moins $n + \tau - \Delta$ équations entre les primaires et les auxiliaires, linéaires par rapport à ces derniers, et linéairement indépendantes les unes des autres. Donc, puisque les primaires évidemment n'admettent pas de liaison quelconque entre elles-mêmes, il s'ensuit que le nombre $n + \tau - \Delta$ ne peut pas excéder $\sigma - \tau - 1$; donc le nombre total des adjoints ne peut pas excéder $2\sigma - \tau - \Delta - 2$ et, à plus forte raison, ne peut pas excéder $2\sigma - \tau - 2$.

Parmi ces adjoints, se trouvera nécessairement la partie indépendante des puissances du quantic de tous les primaires, à l'exception des quatre premiers, qui sont les seuls indécomposables. Donc la limite supérieure totale devient $2\sigma - \tau + 2$, ou bien $S + \sigma + 2$ si l'on prend S égal à la somme algébrique des coefficients, c'est-à-dire à $\sigma - \tau$.

Quant aux valeurs de S et σ, j'ai trouvé par induction, et je ne doute nullement, que $\tau = 0$. Pour prouver cette proposition, on n'a besoin que de l'Algèbre ordinaire; mais, en attendant la preuve, que je n'ai pas encore trouvée, on peut se servir d'une limite supérieure à σ a lieu de sa valeur exacte. Quand on aura démontré que $\tau = 0$, la limite deviendra tout simplement $2S$.

Or on trouve facilement que

$$S = \frac{1}{i}\left[i^{2i-1} - 2i\,(i-1)^{2i-1} + 2i\,\frac{2i-1}{2}\,(i-2)^{2i-1}\right] \dots \pm \frac{\Pi 2i}{\Pi\,(i-1)\,\Pi\,(i+1)}\,1^{2i-1}$$

et
$$\sigma < \frac{1}{i}\left[i^{2i-1} + 2i\,\frac{2i-1}{2}\,(i-2)^{2i-1} + \dots\right],$$

la dernière série ne contenant que les termes positifs de s. $S + \sigma + 2$ est donc la limite supérieure rigoureusement démontrée; mais il n'est pas douteux, sous le point de vue moral, que $2S + 2$ peut être pris pour cette limite.

J'ajouterai que le point de départ, dans cette démonstration nouvelle du théorème de Gordan, est la règle numérique trouvée par M. Cayley, qui exprime le nombre total des covariants linéairement indépendants d'un ordre et de degré donné appartenant à un quantic de degré donné, règle dont la démonstration rigoureuse a été faite, pour la première fois, par moi-même dans le *Philosophical Magazine** (mars 1878) et dans le dernier tome du *Journal de Borchardt*†. C'est ainsi que, dans le cas considéré plus haut, on établit que ce nombre total sera le coefficient de $t^j u^\epsilon$ dans le développement de la fraction génératrice

$$\frac{1}{(1-tu^{2i})(1-tu^{2i-2})\dots(1-tu^{-2i+2})(1-tu^{-2i})},$$

j étant l'ordre et ϵ le degré du covariant donné: cela mène à la représentation de ce nombre, comme le coefficient de t^j, dans la fraction plus simple

$$\frac{Ft}{(1-t)(1-t^2)^2(1-t^3)\dots(1-t^{2i-1})}.$$

De même, pour le cas où le degré du quantic donné est $2i+1$, on établit que le nombre correspondant sera le coefficient de t^j dans le développement en série de puissances ascendantes de t de la fraction

$$\frac{\Phi t}{(1-t)(1-t^2)(1-t^4)\dots(1-t^{4i})}.$$

Dans ce cas, on se sert d'une série connue de covariants dont les ordres successifs seront $1, 2, 4, \dots, 4i$ comme primaires, et, en nommant S la somme algébrique des coefficients de Φt et Σ la somme des coefficients positifs exclusivement, on trouvera, comme auparavant, que $S+\Sigma-2$ sera une limite supérieure au nombre total des adjoints; et, comme la série de primaires que j'adopte, pour ce cas, ne contient que deux covariants irréductibles, la limite totale des formes irréductibles sera $S+\Sigma$. En admettant, ce qui est certainement vrai, mais non encore prouvé, que Φt comme Ft est omnipositif, on aurait pour la limite $2S$, c'est-à-dire le double d'une certaine série de termes exponentiels connus, qui seront successivement positifs et négatifs: en attendant la preuve de cette loi d'omnipositivité, la limite privée sera cette même série avec seulement les termes positifs doublés.

On peut obtenir d'autres limites supérieures en se servant de la forme canonique pour les invariants, pris séparément, et de la forme canonique à *deux* variables pour les invariants et les covariants combinés; mais on introduit ainsi une difficulté de plus, car on aurait besoin de démontrer *a priori* l'existence et le caractère exact du dénominateur de ces formes canoniques: ce qui n'a pas été encore fait. De même, en se servant de la fonction génératrice que j'ai employée ici, pour des valeurs *données* de $2i$ et $2iH$, on peut trouver des dénominateurs plus simples que le dénominateur

[* p. 117 below.] [† p. 232 below.]

général, auxquels répondront aussi des primaires connues : par exemple, pour le cas de $2i = 8$, on trouvera que l'on peut prendre pour le dénominateur

$$(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)(1-t^5)(1-t^7),$$

au lieu de

$$(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7);$$

et le numérateur restera encore omnipositif: ainsi la limite au nombre des adjoints sera réduite de la moitié ; mais mon objet a été de trouver une limite supérieure *universelle*, c'est-à-dire algébrique, et en même temps de ne pas admettre un principe quelconque reposant en aucun degré sur l'induction ou sur la probabilité. M. Camille Jordan a trouvé et publié, dans le *Journal de Liouville*, une méthode pour déterminer une limite supérieure à l'ordre ou degré des *grundformen* en se servant des principes de M. Gordan, mais je ne sais pas si ce grand géomètre ou aucun autre a réussi à déterminer une limite supérieure à leur nombre. La méthode de MM. Gordan et Jordan est le développement de la première de M. Cayley (celles des hyperdéterminants), comme la mienne est le développement de sa seconde méthode, celle qui repose sur l'emploi de l'equation partielle différentielle, nécessaire et suffisante pour déterminer l'existence des invariants et covariants proposés.

Je donne en conclusion les formes actuelles de la fonction génératrice pour les covariants pris sans distinction quant à leur degré (ce qui revient à dire la fonction génératrice pour les *différentiants*) pour les quantics binaires de tous degrés de 2 jusqu'à 8. Soit μ le degré du quantic, G la fonction génératrice qui y répond.

Quand $\mu = 2$,

$$G = \frac{1}{(1-t)(1-t^2)}.$$

Quand $\mu = 3$,

$$G = \frac{1+t^3}{(1-t)(1-t^2)(1-t^4)}.$$

Quand $\mu = 4$,

$$G = \frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}.$$

Quand $\mu = 5$,

$$G = \frac{1+t^2+3t^3+3t^4+4t^5+4t^6+6t^7+6t^8+4t^9+5t^{10}+3t^{11}+3t^{12}+t^{13}+t^{15}}{(1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)}.$$

Quand $\mu = 6$,

$$G = \frac{1+t^2+3t^3+4t^4+4t^5+4t^6+3t^7+3t^8+t^{10}}{(1-t)(1-t^2)(1-t^2)(1-t^3)(1-t^4)(1-t^5)}.$$

Quand $\mu = 7$,

$$G = \frac{\left[\begin{array}{l}1+2t^2+6t^3+10t^4+19t^5+28t^6+44t^7+61t^8+79t^9+102t^{10}+129t^{11}+156t^{12}\\ +173t^{13}+196t^{14}+215t^{15}+230t^{16}+231t^{17}+231t^{18}+230t^{19}+\ldots+2t^{34}+t^{35}\end{array}\right]}{(1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})}.$$

Quand $\mu = 8$,

$$G = \frac{1 + 2t^2 + 6t^3 + 12t^4 + 19t^5 + 25t^6 + 31t^7 + 36t^8 + 38t^9 + 36t^{10} + \ldots + t^{18}}{(1-t)(1-t^2)(1-t^2)(1-t^3)(1-t^3)(1-t^4)(1-t^5)(1-t^7)},$$

où l'on remarquera que, pour $\mu = 8$, G peut être changé dans la forme normale en multipliant son numérateur et son dénominateur par $1 + t^3$.

En commençant par la forme brute

$$\frac{1 - u^{-2}}{(1-tu^8)(1-tu^6)(1-tu^4)(1-tu^2)(1-t)(1-tu^{-2})(1-tu^{-4})(1-tu^{-6})(1-tu^{-8})},$$

on connaît que le nombre des covariants qui appartiennent à la forme binaire du huitième degré, et qui sont de l'ordre j dans les coefficients et du degré ϵ dans les variables, est le coefficient $t^j . u^\epsilon$ dans le développement de cette fraction selon les puissances ascendantes de t. De là on conclut que, j et ϵ étant positifs, on peut substituer à cette fraction la fraction dont

$$(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6) \times (1-t^7)(1-tu^8)(1-t^2u^{12})(1-t^2u^8)(1-t^2u^4)$$

est le dénominateur, et dont le numérateur est

$$\begin{aligned}
&1 + t^8 + t^9 + t^{10} + t^{18} \\
&\quad + u^2 (t^5 + t^6 + 2t^7 + 2t^8 + 3t^9 + 2t^{10} + 2t^{11} + t^{12} + t^{13}) \\
&\quad + u^4 (t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^8 + t^9 + t^{10} + t^{11} + t^{12} + 2t^{13} + 2t^{14} + t^{15} + t^{16} - t^{20}) \\
&\quad + u^6 (t^3 + t^4 + 2t^5 + 3t^6 + 3t^7 + 3t^8 + 3t^9 + 2t^{10} + t^{11} + t^{12}) \\
&\quad + u^8 (t^3 + t^4 + t^5 + 2t^6 + 2t^7 + 3t^8 + 2t^9 + t^{10} + t^{11} + t^{12} - t^{15} - t^{16} - t^{18} - t^{19} - t^{20}) \\
&\quad + u^{10} (t^3 + 2t^4 + 3t^5 + 2t^6 + 2t^7 + t^8 - t^9 - 2t^{10} \\
&\qquad\qquad - 4t^{11} - 4t^{12} - 3t^{13} - 3t^{14} - 2t^{15} - t^{16} - t^{17}) \\
&\quad + u^{12} (t^3 + t^4 - t^8 - t^9 - 2t^{10} - 2t^{11} - 2t^{12} - 2t^{13} - 4t^{14} \\
&\qquad\qquad - 4t^{15} - 4t^{16} - 3t^{17} - 2t^{18} - t^{19} - t^{20} + t^{21} + t^{22}) \\
&\quad + u^{14} (t^3 + t^4 + t^5 + t^6 - t^7 - t^8 - 3t^9 - 5t^{10} - 6t^{11} \\
&\qquad\qquad - 6t^{12} - 6t^{13} - 4t^{14} - 4t^{15} - 2t^{16} - t^{17}) \\
&\quad + u^{16} (- t^8 - 2t^9 - 4t^{10} - 4t^{11} - 6t^{12} - 6t^{13} - 6t^{14} \\
&\qquad\qquad - 5t^{15} - 3t^{16} - t^{17} - t^{18} + t^{19} + t^{20} + t^{21} + t^{22}) \\
&\quad + u^{18} (t^3 + t^4 - t^5 - t^6 - 2t^7 - 3t^8 - 4t^9 - 4t^{10} - 4t^{11} \\
&\qquad\qquad - 2t^{12} - 2t^{13} - 2t^{14} - 2t^{15} - t^{16} - t^{17} + t^{21} + t^{22}) \\
&\quad + u^{20} (- t^8 - t^9 - 2t^{10} - 3t^{11} - 3t^{12} - 4t^{13} - 4t^{14} \\
&\qquad\qquad - 2t^{15} - t^{16} + t^{17} + 2t^{18} + 2t^{19} + 3t^{20} + 2t^{21} + t^{22}) \\
&\quad + u^{22} (- t^5 - t^6 - t^7 - t^9 - t^{10} + t^{13} + t^{14} + t^{15} \\
&\qquad\qquad + 2t^{16} + 3t^{17} + 2t^{18} + 2t^{19} + t^{20} + t^{21} + t^{22}) \\
&\quad + u^{24} (t^{13} + t^{14} + 2t^{15} + 3t^{16} + 3t^{17} + 3t^{18} + 3t^{19} + 2t^{20} + t^{21} + t^{22})
\end{aligned}$$

$$+ u^{26} (- t^5 + t^9 + t^{10} + t^{11} + 2t^{12} + t^{13} + t^{14} + t^{15}$$
$$+ t^{16} + 2t^{17} + 2t^{18} + 2t^{19} + 2t^{20} + 2t^{21} + t^{22})$$
$$+ u^{28} (t^{12} + t^{13} + 2t^{14} + 2t^{15} + 3t^{16} + 2t^{17} + 2t^{18} + t^{19} + t^{20})$$
$$+ u^{30} (t^7 + t^{15} + t^{16} + t^{17} + t^{25}).$$

Cette fraction a été prise sous sa forme canonique au moyen de l'introduction, dans le numérateur et le dénominateur, du facteur commun

$$(1 + tu^6)(1 + tu^4)(1 + tu^2).$$

En opérant sur les termes positifs de ce numérateur par la méthode générale du tamisage et en combinant les résultats avec les *primaires* donnés par le dénominateur, on obtient la table suivante pour le système complet des *grundformen* du quantic du huitième degré :

Ordre dans les coefficients.	Degré dans les variables.									
	0	2	4	6	8	10	12	14	16	18
1.........					1					
2.........	1		1		1		1			
3.........	1		1	1	1	1	1	1		1
4.........	1		2	1	1	2	1	1		1
5.........	1	1	2	2	1	3		1		
6.........	1	1	2	3	1	1				
7.........	1	2	2	3						
8.........	1	2	2	2						
9.........	1	3	1							
10.........	1	2								
11.........		2								
12.........		1								

Dans cette table un chiffre quelconque dans l'intérieur du cadre exprime le nombre des formes dérivées irréductibles de l'ordre qui se trouve au commencement de la ligne et du degré que se trouve à la tête de la colonne dans laquelle le chiffre est situé. Ainsi, par exemple, il y aura trois covariants irréductibles de l'ordre 6 et du degré 6, 2 de l'ordre 8 et du degré 6, et ainsi en général. Le nombre total de ces formes irréductibles est 69, le degré le plus élevé 18, l'ordre le plus élevé 12. La limite supérieure donnée par la méthode expliquée dans ma dernière Communication (qui sort de la considération de la génératrice à une seule variable) est $2(302) + 2 = 606$, qui est beaucoup trop grand. Mais, en se servant de la même méthode appliquée à la fonction génératrice *à deux variables* dans sa forme canonique donnée ci-dessus, au lieu de la fonction génératrice à une seule variable, on obtiendra comme limite supérieure

$$(2\sigma - \tau - 2) + \epsilon + \nu,$$

σ étant la somme des coefficients positifs, τ la somme des coefficients négatifs dans le numérateur, ϵ le nombre des liaisons algébriques entre les *primaires*

qui répondent aux indices des facteurs du dénominateur, et ν le nombre de ces facteurs.

On aura donc

$$\sigma = 70, \quad \tau = 70, \quad \nu = 10, \quad \epsilon = \nu - 8 = 2,$$

et la limite supérieure devient 80, qui n'est pas beaucoup plus grand que le nombre 69 qu'on a trouvé.

De même, pour le cas d'une fonction du sixième degré, la limite supérieure tirée de la fonction génératrice (dans sa forme canonique) à deux variables sera $(2\sigma - \tau - 2) + \epsilon + \nu$, où l'on trouvera

$$\sigma = 29, \quad \tau = 29, \quad \nu = 7, \quad \epsilon = \nu - 6 = 1,$$

et conséquemment la limite devient 35, le vrai nombre étant 27.

La limite inférieure est évidemment dans tous les cas le nombre donné par la règle du tamisage: par conséquent, dans tous les exemples qu'on a précédemment traités, cette limite coïncide avec le nombre actuel des *grundformen.* On peut à peine douter que cette identité, qui est conforme à la *loi de parcimonie,* et soutenue par une induction à peu près irrésistible, ne soit d'application universelle, et il serait fort à désirer que M. Gordan ou quelqu'un de ses élèves fît connaître, s'il ne l'a pas déjà fait, le système des *grundformen* pour le quantic du huitième degré obtenu par sa méthode, afin qu'on pût le comparer avec celui qui se déduit de la mienne.

Pour éviter toute ambiguïté, je dois ajouter que le fonction génératrice à une variable est celle qui sert à donner le nombre total des covariants d'un ordre donné dans les coefficients sans que le degré dans les variables soit spécifié, tandis que la fonction génératrice à deux variables est celle qui sert pour l'énumération des covariants dont l'ordre et le degré sont tous les deux donnés. Les deux fonctions deviennent algébriquement égales quand, dans la dernière, on aura fait $u = 1$; mais le facteur commun au numérateur et au dénominateur ne sera pas en général le même dans les deux expressions.

18.

PROOF OF THE HITHERTO UNDEMONSTRATED FUNDAMENTAL THEOREM OF INVARIANTS.

[*Philosophical Magazine*, v. (1878), pp. 178—188.]

I AM about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this should be done, because the theorem has been supposed to lead to false conclusions, and its correctness has consequently been impugned*. But, of the two suppositions that might be made to account for the observed discrepancy between the supposed consequences of the theorem and ascertained facts—one that the theorem is false and the reasoning applied to it correct, the other that the theorem is true but that an error was committed in drawing certain deductions from it (to which one might add a third, of the theorem and the reasoning upon it being both erroneous)—the wrong alternative was chosen.

* Thus in Professor Faà de Bruno's valuable *Théorie des Formes Binaires*, Turin, 1876, at the foot of page 150 occurs the following passage:—"Cela suppose essentiellement que les équations de condition *soient toutes indépendantes entr'elles, ce qui n'est pas toujours le cas*, ainsi qu'il résulte des recherches du Prof. Gordan sur les nombres des covariants des formes quintique et sextique."

The reader is cautioned against supposing that the consequence alleged above does result from Gordan's researches, which are indubitably correct. This supposed consequence must have arisen from a misapprehension on the part of M. de Bruno of the nature of Professor Cayley's rectification of the error of reasoning contained in his second memoir on Quantics, which had led to results discordant with Gordan's. Thus error breeds error, unless and until the pernicious brood is stamped out for good and all under the iron heel of rigid demonstration. In the early part of this year Mr Halsted, a Fellow of Johns Hopkins University, called my attention to this passage in M. de Bruno's book; and all I could say in reply was that "the extrinsic evidence in support of the independence of the equations which had been impugned rendered it to my mind as certain as any fact in nature could be, but that to reduce it to an exact demonstration transcended, I thought, the powers of the human understanding."

At the moment of completing a memoir, to appear in Borchardt's Journal, demonstrating my quarter-of-a-century-old theorem for enabling Invariants to procreate their species, as well by an act of self-fertilization as by conjugation of arbitrarily paired forms, the unhoped and unsought-for prize fell into my lap, and I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.

An error was committed in reasoning out certain supposed consequences of the theorem; but the theorem itself is perfectly true, as I shall show by an argument so irrefragable that it must be considered for ever hereafter safe from all doubt or cavil. It lies at the basis of the investigations begun by Professor Cayley in his *Second Memoir on Quantics*, which it has fallen to my lot, with no small labour and contention of mind, to lead to a happy issue, and thereby to advance the standards of the Science of Algebraical Forms to the most advanced point that has hitherto been reached. The stone that was rejected by the builders has become the chief corner-stone of the building.

I shall for greater clearness begin with the case of a single binary quantic $(a, b, c, \ldots, l \between x, y)^i$. Any rational integral function of the *elements* $a, b, c, \ldots l$ which remains unchanged in value when for them are substituted the elements of the new quantic obtained by putting $x + hy$ instead of x in the original one, I call a Differentiant in x to the given quantic.

By a differentiant of a given weight w and order j, I mean one in every term of which the combination of the elements is of the jth order and the sum of their weights w, the weights of the successive elements $(a, b, c, \ldots l)$ themselves being reckoned as $0, 1, 2, \ldots i$ respectively.

The proposition to be proved is, that the number of arbitrary constants in the most general expression for such differentiant is the difference between the number of ways in which w can be made up with j of the integers $0, 1, 2, 3, \ldots i$ (repetitions allowable), less the number of ways in which $w - 1$ can be made up with the same integers. We may denote these two numbers by $(w : i, j)$, $\{(w-1) : i, j\}$ respectively, and their difference by $\Delta (w : i, j)$. Then, if we call the number of arbitrary constants in the differentiant of weight w and order j belonging to a binary quantic of the ith order $D (w : i, j)$, the proposition to be established is that $D (w : i, j) = \Delta (w : i, j)$.

Let us use Ω to denote the operator

$$a \frac{d}{db} + 2b \frac{d}{dc} + \ldots + ik \frac{d}{dl},$$

and O to denote the operator

$$ib \frac{d}{da} + (i-1) c \frac{d}{db} + \ldots + l \frac{d}{dk}.$$

Then it is well known that the necessary and sufficient condition for D being a differentiant in x is that the *identity* $\Omega D = 0$ be satisfied.

Let us study the relations of Ω and O in respect to D.

In the first place, let U be any rational integral function of the elements of order j and weight w; then I say that

$$\Omega . O . U - O . \Omega . U = (ij - 2w) U.$$

For if we use $*$ to signify the act of pure differential operation, it is obvious that

$$\Omega \,.\, O \,.\, U = (\Omega \times O)\, U + (\Omega * O)\, U,$$
$$O \,.\, \Omega \,.\, U = (\Omega \times O)\, U + (O * \Omega)\, U;$$

Therefore $\Omega \,.\, O \,.\, U - O \,.\, \Omega \,.\, U = \{(\Omega * O) - (O * \Omega)\}\, U$

$$= ia\frac{d}{da} + 2(i-1)\, b\frac{d}{db} + 3(i-2)\, c\frac{d}{dc} + \ldots + ik\frac{d}{dk}$$
$$- ib\frac{d}{db} - 2(i-1)\, c\frac{d}{dc} - \ldots - 2(i-1)\, k\frac{d}{dk} - il\frac{d}{dl}$$
$$= ia\frac{d}{da} + (i-2)\, b\frac{d}{db} + (i-4)\, c\frac{d}{dc} - \ldots - (i-2)\, k\frac{d}{dk} - 2l\frac{d}{dl}.$$

If now $\rho a^p \,.\, b^q \,.\, c^r \ldots l^t$, where ρ is a number, be any term in U, we have

$$\left.\begin{array}{r} p + q + r + \ldots + t = j \\ q + 2r + \ldots + it = w \end{array}\right\} \text{ by hypothesis;}$$

therefore $\Omega \,.\, O \,.\, U - O \,.\, \Omega \,.\, U,$

that is

$$i\left(a\frac{d}{da} + b\frac{d}{db} + c\frac{d}{dc} \ldots + l\frac{d}{dl}\right) U$$
$$-2\left(\qquad b\frac{d}{db} + 2c\frac{d}{dc} \ldots + il\frac{d}{dl}\right) U$$
$$= \Sigma\rho\, (ij - 2w)\, (a^p \,.\, b^q \,.\, c^r \ldots l^t)$$
$$= (ij - 2w)\, U, \text{ as was to be proved.}$$

If now for U we write D a differentiant in x, we have $\Omega D = 0$, and therefore

$$\Omega \,.\, O \,.\, D = \delta D,$$

where $\delta = ij - 2w$.

Again,

$$\Omega \,.\, O\, (O \,.\, D) - O \,.\, \Omega\, (O \,.\, D) = \{ij - 2(w+1)\}\, O \,.\, D;$$

for $O \,.\, D$ is of the weight $w + 1$;

therefore $\Omega^2 \,.\, O^2 \,.\, D = \Omega \,.\, O\, \delta D + (\delta - 2)\, \Omega \,.\, O \,.\, D$

$$= (2\delta - 2)\, \Omega \,.\, O \,.\, D$$
$$= \delta\, (2\delta - 2)\, D.$$

Similarly it will be seen that

$$\Omega^3 \,.\, O^3 \,.\, D = \delta\, (2\delta - 2)\, (3\delta - 6)\, D,$$

and in general

$$\Omega^q \,.\, O^q \,.\, D = \delta\, (2\delta - 2)\, (3\delta - 6) \ldots \{q\delta - (q^2 - q)\}\, D$$
$$= (1 \,.\, 2 \,.\, 3 \ldots q)\, \{\delta \,.\, (\delta - 1)\, (\delta - 2) \ldots (\delta - q + 1)\}\, D,$$

the successive numbers δ, $2\delta - 2$, $3\delta - 6$, &c. being the successive sums of the arithmetical series δ, $\delta - 2$, $\delta - 4$, $\delta - 6$, &c.

To find the most general differentiant in question, we must take every combination of the elements whose weight is w and order j, of which the number is obviously $(w : i, j)$, and prefix an indeterminate constant to each such combination; then operating upon this form with Ω, we shall reduce its weight by unity, and shall obtain as many combinations of this reduced weight (the order j remaining unchanged) as there are units in $\{(w-1) : i, j\}$. Each of these combinations will have for its coefficient a linear function of the assumed indeterminate coefficients; and in order to satisfy the identity $\Omega D = 0$, each such linear function must be made equal to zero. There are therefore $(w : i, j)$ quantities connected by $\{(w-1) : i, j\}$ homogeneous equations. *Supposing the equations to be independent*, the number of the indeterminate coefficients left arbitrary is obviously the difference between these quantities, namely, $\Delta (w : i, j)$. The difficulty consists in proving this independence—a difficulty so great that I think any one attempting to establish the theorem, as it were by direct assault, in this fashion, would find that he had another Plevna on his hands. But a position that cannot be taken by storm or by sap may be *turned* or starved into surrender; and this is how we shall take our Plevna. Be the equations of condition linearly independent or not, it is obvious that we must have $D (w : i, j)$ equal to or greater than $\Delta (w : i, j)$. I shall show by aid of a construction drawn from the resources of the Imaginative Reason, and founded on the reciprocal properties that have just been exhibited by the famous O and Ω, that this latter supposition, of the first member of the equation being greater than the second, is inadmissible and must be rejected. Observe that $(0 : i, j)$, the number of ways of making up 0 with j combinations of $0, 1, 2, \ldots i$, is 1; also that $D (0 : i, j)$, the number of arbitrary constants in the most general differentiant in x to the quantic $(a, b, c, \ldots \between x, y)^i$ of order j and weight 0, is also 1; for such differentiant is obviously λa^n.

Thus we have for all values of w,

$$D (w : i, j) = \text{ or } > (w : i, j) - \{(w-1) : i, j\},$$

and also

$$D (0 : i, j) = (0 : i, j);$$

therefore

$$D (w : i, j) + D\{(w-1) : i, j\} + D\{(w-2) : i, j\} + \ldots + D (0 : i, j)$$
$$= \text{ or } > (w : i, j).$$

If in the above condition, for any assumed value of w, $>$ is the sign to be employed, then the equation $D (w : i, j) = \Delta (w : i, j)$ cannot be satisfied for all values of w. If, on the other hand, $>$ is not the sign to be employed, then this equation, for *every value of w*, commencing with the assumed one down to 0, must be satisfied. The greatest value of w for given values of i, j, it is well known, is $\frac{ij}{2}$ for ij even, and $\frac{ij-1}{2}$ for ij odd. Let us give to w this

maximum value in the above "greater or equal" relation; for brevity, denote the differentiants whose types are $[w, i, j]$, $[(w-1), i, j] \dots$ by $[w]$, $[w-1]$, $[w-2]$, &c. respectively, i and j being regarded as constants. It will be convenient to substitute for the number of arbitrary constants in any of these differentiants the same number of linearly independent specific values of them; so that we shall have $D(w:i, j)$ of linearly independent $[w]$'s, $D\{(w-1):i, j\}$ of linearly independent $[w-1]$'s, and so on. Now, instead of $D\{(w-q):i, j\}$ differentiants $[w-q]$, let us substitute the same number of the derived forms $O^q[w-q]$. I shall prove that the quantities (*all of the same weight* w) thus obtained are linearly independent of one another.

For suppose that those belonging to any one set $O^q.[w-q]$ are not independent, but are connected by a linear equation. Then, operating upon this equation with $\Omega^q.$, we shall obtain a linear equation between the quantities $[w-q]$, for each quantity $\Omega^q.O^q.[w-q]$ is a numerical multiple of $[w-q]$; which is contrary to the hypothesis. Again, let there be a linear equation between the quantities contained in any number of sets of the form $O^q.[w-q]$ for which m is the greatest value of q. Then, operating upon this with Ω^m, it is clear that all the quantities in the sets for which $q<m$ will introduce quantities of the form $\Omega^{m-q}[w-q]$ where $m-q>0$, and which consequently vanish. There will be left, therefore, only quantities of the form $[w-q]$, between which a linear equation would exist, contrary to hypothesis, as in the preceding case. Therefore all the quantities in all the sets are linearly independent. But these are all of the weight w, that is,

$$\left[\frac{ij}{2} \text{ or } \frac{ij-1}{2}\right],$$

and are therefore linear functions of the number of ways in which the integers $0, 1, 2, 3, \dots i$ can be combined i and j together so as to give the weight w. Therefore being linearly independent, as just proved, their number cannot exceed this last-named number, that is, cannot exceed $(w:i, j)$. That is to say,

$$D(w:i, j)+D\{(w-1):i, j\}+\dots+D(O:i, j)$$

cannot exceed $(w:i, j)$. Therefore every one of the equations

$$D(w:i, j)=\Delta(w:i, j)$$

must be satisfied from the maximum value of w down to the value 0, which proves *the great hitherto undemonstrated fundamental theorem* for a single quantic.

For any number of quantics the demonstration is precisely similar at all points: there will be as many systems of i, j as there are quantics. $(w:i, j:i', j':\&c.)$ will denote the number of ways of making up w with j

of the integers 0, 1, 2, ... i, with j' of the integers 0, 1, 2, ... i', and so on. The theorem to be demonstrated will be

$$D(w:i,j:i',j':\ldots)=\Delta(w:i,j:i',j':\ldots).$$

$$\Omega \text{ will become } \Sigma\left(a\frac{d}{db}+2b\frac{d}{dc}+\ldots\right),$$

$$O \quad ,, \quad ,, \quad \Sigma\left(ib\frac{d}{da}+(i-1)c\frac{d}{db}+\ldots\right).$$

It will still be true that $\Omega^q . O^q . D$—where D is a differentiant in x (that is, a function of the elements in all the given quantics which withstand change when these are transformed by writing $x+hy$ for x)—is a numerical multiple of D; and D will be subject to the identity $\Omega D=0$. We shall still have

$$D(w:i,j:i',j':\ldots)= \text{ or } >\Delta(w:i,j:i',j':\ldots),$$

and $$D(0:i,j:i',j':\ldots)=(0:i,j:i',j':\ldots),$$

and shall be able in precisely the same way as before to demonstrate the impossibility of $\sum\limits_{k=w}^{k=0} D(w-k:i,j:i',j':\ldots)$ being *greater* than $(w:i,j:i',j':\ldots)$, and so shall be able to infer by the same logical scheme

$$\Delta(w:i,j:i',j':\ldots)=D(w:i,j:i',j':\ldots).$$

This is my extension of Professor Cayley's theorem, which leads direct to the Generating Fractions given in my recent papers in the *Comptes Rendus*.

In a series of articles which I hope to publish in the *American Journal of Pure and Applied Mathematics*, I propose to give a systematic development of the Calculus of Invariants, taking a differentiant as the primordial germ or unit. I have spoken of a differentiant in x, and of course might have done so equally of a differentiant in y. If we call the former D_x, it is capable of being shown, from the very natures of the forms O and Ω, that if the quantity $ij-2w$, which may be called the *degree* of D_x, be called δ, then $O^\delta D_x$ becomes a differentiant in y. These may be termed simple differentiants; but the principle of continuity forbids that we should omit to comprise in the same scheme the intermediate forms $O^p D_x$ or $\Omega^q D_y$, through which simple differentiants in x and y pass into each other. These may be termed mixed differentiants; $O^p D_x$ may be termed a differentiant p removed (as we speak of *cousins* once, twice, &c. removed) from x, which will be the same thing as $O^q D_y$ (a differentiant q removed from y) if $p+q$ is equal to the degree, namely, $ij-2w$. Now all these differentiants, whether simple or mixed, possess a wonderful property, which may be deduced by means of Salmon's Theorem, given in the *Philosophical Magazine* for August 1877. They are all, in an enlarged sense of the term, Invariants—in this sense to wit, that if the elements are made to undergo a substitution consequent upon or, as we may say, induced by a general linear substitution impressed on the variables, which for greater simplicity of enunciation may be

supposed to have unity for the determinant of its matrix, then every differentiant, whether single or double (the latter being equivalent to an invariant), and whether simple or mixed, will remain a Constant Function of the Coefficients of the impressed substitution. To wit, if the differentiant be p removes from x and q removes from y (so that its degree is $p+q$), and if the impressed substitution be $lx+\lambda y$ for x, and $mx+\mu y$ for y, where $l\mu-\lambda m=1$, then will the differentiant be a constant bipartite quantic in the two sets of coefficients l, m and λ, μ, of the degree q in the former and p in the latter—a theorem which amounts almost to a revolution in the whole sphere of thought about Invariants.

I have borrowed the term "Imaginative Reason" from a recent paper of Mr Pater on Giorgione, in which, as in many of those of Mr Symonds (I will instance one on Milton in particular), I find a continued echo of my own ideas, and in the latter many of the very formulæ contained in my *Laws of Verse*, where versification in sport has been made æsthetic in earnest. Surely the claim of Mathematics (its "*Andersstreben*") to take a place among the liberal arts must be now admitted as fully made good. Whether we look to the advances made in modern geometry, in modern integral calculus, or in modern algebra, in each of these a free handling of the material employed is now possible, and an almost unlimited scope left to the regulated play of the fancy. It seems to me that the whole of æsthetic (so far as at present revealed) may be regarded as a scheme having four centres, which may be treated as the four apices of a tetrahedron, namely Epic, Music, Plastic, and Mathematic. There will be found to be a *common* plane to every three of these, *outside* of which lies the fourth; and through every two may be drawn a common axis *opposite* to the axis passing through the remaining two. So far is certain and demonstrable. I think it also possible that there is a centre of gravity to each set of three, and that the lines joining each such centre with the outside apex will intersect in a common point the centre of gravity of the whole body of æsthetic; but what that centre is or must be I have not had time to think out.

Postscript.—In the first fervour of a new conception, I fear that in the manuscript which is now on its way to England I may have expressed myself with some want of clearness or precision on the subject of pure and mixed differentiants. I will therefore add a few more explanatory and vaticinatory words on this subject, through the medium of which I catch a glimpse of the possibility of obtaining a simple proof of Gordan's theorem, just as through the medium of pure differentiants taken *per se* I caught a glimpse (almost immediately afterwards to be converted into a certainty) of the proof of Cayley's theorem given in this memoir. I conceive that what the *ensemble* of pure differentiants have done for the one, the larger *ensemble* of all sorts of

differentiants, pure and mixed, taken together, will enable me or some one else to accomplish for the other.

Any function of the coefficients of a quantic which is nullified by the operation upon it of Ω, which we may call the *revector* symbol, or in other words, whose first *revect* is zero, is a pure differentiant in x. So, of course, if nullified by the operation upon it of O, which may be called the *provector* symbol, it is a pure differential in y. We may call $ij-2w$, where i is the degree of the quantic, j the order of a pure differentiant, and w its weight in x, the grade of the differentiant, and denote this grade by δ.

The δth provect of a pure differentiant in x is of course a pure differentiant in y, which is δ *removes* from x, as the pure differentiant in x is δ removes from y. If q be less than δ, the qth provect of a pure differentiant in x is a mixed differentiant q removes from x, or, if we like to say so, $(\delta-q)$ removes from y. The *grade* of a mixed differentiant may be defined to be the same as that of the pure differentiant of which it is a provect or revect.

Then, in the first place, we have this proposition:—If any linear substitution whatever be impressed on the variables of a quantic, the transformed value of any of its differentiants will separate into two factors, of which one will be the determinant of substitution raised to the power w, where w is the weight corresponding to the order and grade of the differentiant and the degree of the quantic. The remaining factor will be a function of the coefficients of substitution, and may be called the outstanding factor. Of this I shall proceed to speak.

Let x be replaced by $hx+ly$,
y „ „ $kx+my$.

Then the outstanding factor for the transformed D (a pure differentiant in x of the grade δ) may be proved by repeated applications of Salmon's theorem to be equal to

$$\left(1+\frac{k}{m}O\,.+\left(\frac{k}{m}\right)^2\frac{O^2.}{1\,.\,2}+\ldots\right)m^\delta D,$$

where of course the series of terms in the development will, after the $(\delta+1)$th term, vanish spontaneously. In other words, the outstanding factor of the transformed D is $m^\delta e^{\frac{kO\,.}{m}}\,.\,D$, where it will be noticed that only the coefficients of substitution due to the change in y make their appearance.

If now we take any mixed differentiant, say the qth provect of D, that is, $O^q.\,D$, its outstanding factor, I find, will be the qth emanent of the outstanding factor for D, that is, will be

$$\left(h\frac{d}{dk}+l\frac{d}{dm}\right)^q\left(m^\delta e^{\frac{kO\,.}{m}}\right)D.$$

And here for the present I end. The subject is, as it was, a vast one; and this conception of mixed differentiants opens out still vaster horizons. Every thing grown on American soil, or born under the influence of its skies, as its lakes, its rivers, its trees, and its political system, seems to have a tendency to rise to colossal proportions.

I will merely add one remark which has occurred to me relating to Sturm's theorem and the process of Algebraical common measure in general. If $f(x, y)$ be a rational integral function of x, y, and $f'(x, y)$ its derivative in respect to x, and we perform the process of common measure between them regarded as functions of x, we know that the irreducible part of the successive remainders taken in ascending order, say $U_0, U_1, U_2, \ldots$, will have for their leading coefficients (say $D_0, D_1, D_2 \ldots$) the discriminants of f and of its successive derivatives in respect to x respectively.

Here D_0 is an invariant of the given form;

D_1 (a differentiant in x) will be the leading coefficient of the covariant

$$D_1x^2 + O \,.\, D_1xy + \frac{O^2\cdot}{1\,.\,2} D_1y^2;$$

D_2 (another differentiant in x) will be the leading coefficient of the covariant

$$D_2x^4 + O \,.\, D_2x^3y + \frac{O^2\cdot}{1\,.\,2} D_2x^2y^2 + \frac{O^3\cdot}{1\,.\,2\,.\,3} D_2xy^3 + \frac{O^4\cdot}{1\,.\,2\,.\,3\,.\,4} D_2y^4,$$

and so on until we come back to the first Sturmian remainder of $(x, y)^i$, the irreducible part of which (or we may call it the Sturmian Auxiliary Proper) is the Hessian differentiated down from being of the degree $2i-4$ to the degree $i-2$, that is, to half of what it was at first; and so in like manner every Sturmian Auxiliary Proper is, so to say, a Covariant differentiated down to half its original dimensions.

The above invariant and the following covariants may be called $V_0, V_1, V_2, \ldots$ respectively. The interesting point in question is that (to numerical factors *près*)

$$U_0 = V_0, \quad U_1 = \frac{d}{dx} V_1, \quad U_2 = \left(\frac{d}{dx}\right)^2 V_2, \quad U_3 = \left(\frac{d}{dx}\right)^3 V_3,$$

and so on.

So more generally for any two functions $f(x, y)$, $\phi(x, y)$, the irreducible part of the remainders obtained by common-measuring them with respect to x will all be derivatives in regard to x of covariants of the two given quantics. If we take for our quantics

$$(a, b, c, \ldots h, k, l \between x, y)^i : (a', b', c', \ldots h', k', l' \between x, y)^i,$$

the covariants in question will all be educts of (that is, functions having for their leading coefficients) the successive resultants of the forms

$$[(a, \ldots h, k, l), \;\; (a', \ldots h', k', l')],$$

of the forms $[(a, \dots h, k), \ (a', \dots h', k')]$,

of the forms $[(a, \dots h), \ (a', \dots h')]$,

and so on, the discriminants of which may be called *partial* resultants of the given forms; in a word, the simplified residues arising in the process of common-measuring in respect to one of their variables two given binary quantics are differential derivatives, in respect to that variable, of the educts of their partial resultants (of course with the understanding that the last simplified residue is the complete resultant itself).

This seems to point to the existence of some generalized statement of Sturm's theorem in which the same Educts as above referred to shall appear, but where, instead of their derivatives in respect to one of the variables being made use of, perfectly general Emanants of them shall be employed as the Criterion functions. For I need hardly add that all Educts (although not written so as to show it in what precedes) are in fact symmetrical in respect to the two sides of the quantic to which they belong.

On various *à priori* grounds I suspect the generalized theorem to be as follows. If $X_{2\mu}$ is the covariant (of degree 2μ) whose μth derivative in respect to x is a Sturmian Auxiliary Proper to $F(x, y)$, we may substitute throughout for all the values of μ, instead of each such derivative, the more general one $\left(f\frac{d}{dx} - g\frac{d}{dy}\right)^{\mu} X_{2\mu}$, where f and g are any assumed positive constants, of course with the understanding that the second criterion also is to be $\left(f\frac{d}{dx} - g\frac{d}{dy}\right)f$ in lieu of $\frac{dF}{dx}$. And the method of Sturm will still be applicable for finding the positions of the real roots of $\frac{x}{y}$ in $f(x, y) = 0$ when we use these more general derivatives as the criteria instead of Sturm's own. When $g = 0$ the theorem is that of Sturm; when $f = 0$ it is an immediate deduction from this theorem applied to finding the positions of the root values of $\frac{y}{x}$, when it is borne in mind that the motions of $\frac{x}{y}$ and of $\frac{y}{x}$, as regards ascent and descent (excluding the moment for which either of these ratios is indefinitely near to zero) are inverse to each other. It is this that accounts for the negative sign which precedes g.

It is difficult to conceive by what theorem other than the assumed one the chasm between those extreme cases can be bridged over; and all analogy and all belief in continuity veto the supposition that no such bridge exists. "Divide *et impera*" is as true in algebra as in statecraft; but no less true and even more fertile is the maxim "auge *et impera*." The more to do or to prove, the easier the doing or the proof.

19.

SUR LES COVARIANTS FONDAMENTAUX D'UN SYSTÈME CUBO-BIQUADRATIQUE BINAIRE.

[*Comptes Rendus*, LXXXVII. (1878), pp. 242—4, 287—9.]

LE seul cas du dénombrement des *grundformen* binaires qui restait à déterminer par ma méthode, hors de ceux qui ont été calculés par la méthode de Gordan, est celui de la combinaison d'une forme biquadratique avec une forme cubique binaire.

Grâce à la coopération intelligente et à la grande habileté, comme calculateur, de M. J. Franklin, un de mes élèves à Baltimore, je suis en état de présenter à l'Académie le tableau des invariants et covariants fondamentaux, donné par la méthode de tamisage.

En partant de la forme primitive

$$\frac{1-u^{-2}}{(1-tu^4)(1-tu^2)(1-t)(1-tu^{-2})(1-tu^{-4})(1-\tau u^3)(1-\tau u)(1-\tau u^{-1})(1-\tau u^{-3})},$$

on parvient à la fraction génératrice canonique, dont le dénominateur est

$$(1-t^2)(1-t^3)(1-t^2u^4)(1-tu^4)(1-\tau^4)(1-\tau^2u^2)(1-\tau u^3)(1-t^2\tau^4)$$
$$(1-t\tau^4)(1-t^3\tau^2)(1-t^3\tau^4),$$

et dont le numérateur contient 338 termes, dont ceux qui portent des coefficients positifs sont égaux en nombre à ceux qui portent le signe négatif. En effet, à chaque terme $kt^\alpha.\tau^\beta.u^\gamma$ correspond un terme

$$-kt^{\alpha'}.\tau^{\beta'}.u^{\gamma'},$$

où $\alpha+\alpha'$, $\beta+\beta'$, $\gamma+\gamma'$ sont des nombres constants, lesquels (si je ne me trompe, car j'ai eu le malheur de perdre le manuscrit) sont respectivement 12, 17, 11.

En représentant un terme $kt^\alpha.\tau^\beta.u^\gamma$ par le symbole $(\alpha.\beta.\gamma)^k$, voici le tableau des termes positifs.

2 . 4 . 0	1 . 1 . 3	7 . 8 . 4	$(10 . 11 . 5)^2$	7 . 13 . 7	$(6 . 7 . 9)^2$
2 . 6 . 0	1 . 3 . 3	7 . 10 . 4	10 . 13 . 5	8 . 7 . 7	$(6 . 9 . 9)^2$
$(3 . 4 . 0)^2$	2 . 1 . 3	$(7 . 12 . 4)^4$	3 . 0 . 6	$(8 . 9 . 7)^5$	7 . 7 . 9
$(3 . 6 . 0)^3$	$(2 . 3 . 3)^3$	$(7 . 14 . 4)^3$	4 . 10 . 6	$(8 . 11 . 7)^4$	$(7 . 9 . 9)^3$
$(4 . 4 . 0)^2$	3 . 1 . 3	$(8 . 8 . 4)^2$	4 . 12 . 6	$(9 . 9 . 7)^3$	$(8 . 9 . 9)^3$
$(4 . 6 . 0)^2$	$(3 . 3 . 3)^5$	8 . 10 . 4	$(5 . 10 . 6)^2$	$(9 . 11 . 7)^4$	$(9 . 9 . 9)^3$
5 . 4 . 0	$(3 . 5 . 3)^2$	$(8 . 12 . 4)^2$	$(5 . 12 . 6)^2$	10 . 9 . 7	10 . 9 . 9
5 . 6 . 0	3 . 11 . 3	$(8 . 14 . 4)^2$	6 . 8 . 6	$(10 . 11 . 7)^2$	4 . 4 . 10
1 . 1 . 1	$(4 . 3 . 3)^3$	9 . 8 . 4	$(6 . 10 . 6)^2$	11 . 11 . 7	$(4 . 6 . 10)^2$
$(1 . 3 . 1)^2$	$(4 . 5 . 3)^3$	9 . 14 . 4	$(6 . 12 . 6)^2$	11 . 13 . 7	4 . 8 . 10
1 . 5 . 1	4 . 11 . 3	10 . 14 . 4	6 . 14 . 6	3 . 4 . 8	5 . 4 . 10
2 . 1 . 1	5 . 3 . 3	11 . 14 . 4	7 . 8 . 6	3 . 6 . 8	$(5 . 6 . 10)^3$
$(2 . 3 . 1)^3$	$(5 . 5 . 3)^2$	1 . 1 . 5	$(7 . 10 . 6)^4$	4 . 6 . 8	$(5 . 8 . 10)^2$
$(2 . 5 . 1)^2$	5 . 11 . 3	2 . 1 . 5	$(7 . 12 . 6)^3$	4 . 8 . 8	$(6 . 6 . 10)^2$
$(3 . 3 . 1)^2$	5 . 13 . 3	4 . 11 . 5	(7 . 14 . 6)	5 . 6 . 8	$(6 . 8 . 10)^3$
$(3 . 5 . 1)^3$	6 . 13 . 3	$(5 . 11 . 5)^2$	8 . 8 . 6	$(5 . 8 . 8)^3$	7 . 6 . 10
(4 . 3 . 1)	$(7 . 13 . 3)^2$	5 . 13 . 5	$(8 . 10 . 6)^5$	5 . 10 . 8	$(7 . 8 . 10)^2$
$(4 . 5 . 1)^2$	7 . 15 . 3	$(6 . 11 . 5)^2$	$(8 . 12 . 6)^4$	$(6 . 8 . 8)^3$	8 . 8 . 10
5 . 5 . 1	8 . 13 . 3	$(6 . 13 . 5)^3$	(8 . 14 . 6)	$(6 . 10 . 8)^3$	9 . 8 . 10
6 . 5 . 1	8 . 15 . 3	6 . 15 . 5	$(9 . 10 . 6)^4$	$(7 . 8 . 8)^4$	3 . 3 . 11
$(1 . 2 . 2)^2$	9 . 15 . 3	7 . 9 . 5	$(9 . 12 . 6)^5$	$(7 . 10 . 8)^3$	5 . 7 . 11
$(1 . 4 . 2)^2$	1 . 2 . 4	$(7 . 11 . 5)^4$	9 . 14 . 6	$(8 . 8 . 8)^2$	5 . 9 . 11
$(2 . 2 . 2)^3$	$(2 . 2 . 4)^2$	$(7 . 13 . 5)^4$	$(10 . 10 . 6)^2$	$(8 . 10 . 8)^3$	$(6 . 7 . 11)^2$
$(2 . 4 . 2)^4$	$(3 . 2 . 4)^3$	7 . 15 . 5	$(10 . 12 . 6)^3$	$(9 . 10 . 8)^4$	$(6 . 9 . 11)^3$
$(3 . 2 . 2)^2$	$(3 . 4 . 4)^2$	$(8 . 9 . 5)^2$	11 . 12 . 6	$(10 . 10 . 8)^2$	$(7 . 7 . 11)^2$
$(3 . 4 . 2)^5$	4 . 2 . 4	$(8 . 11 . 5)^5$	5 . 11 . 7	11 . 10 . 8	$(7 . 9 . 11)^2$
4 . 2 . 2	4 . 4 . 4	$(8 . 13 . 5)^4$	5 . 13 . 7	11 . 12 . 8	8 . 7 . 11
$(4 . 4 . 2)^3$	4 . 12 . 4	8 . 15 . 5	$(6 . 9 . 7)^2$	3 . 5 . 9	8 . 9 . 11
4 . 12 . 2	$(5 . 12 . 4)^3$	$(9 . 9 . 5)^2$	$(6 . 11 . 7)^3$	$(4 . 5 . 9)^2$	
5 . 4 . 2	5 . 14 . 4	$(9 . 11 . 5)^4$	$(6 . 13 . 7)^2$	$(4 . 7 . 9)^2$	
5 . 12 . 2	6 . 10 . 4	$(9 . 13 . 5)^4$	7 . 7 . 7	$(5 . 5 . 9)^2$	
9 . 16 . 2	$(6 . 12 . 4)^4$	(9 . 15 . 5)	$(7 . 9 . 7)^5$	$(5 . 7 . 9)^3$	
0 . 3 . 3	$(6 . 14 . 4)^2$	10 . 9 . 5	$(7 . 11 . 7)^4$	5 . 9 . 9	

En effectuant le tamisage, ces combinaisons se réduisent aux 50 suivantes:

2 . 4 . 0	1 . 1 . 1	$(1 . 2 . 2)^2$	0 . 3 . 3	1 . 2 . 4	1 . 1 . 5	3 . 0 . 6
2 . 6 . 0	$(1 . 3 . 1)^2$	$(1 . 4 . 2)^2$	1 . 1 . 3	2 . 2 . 4	2 . 1 . 5	
$(3 . 4 . 0)^2$	1 . 5 . 1	$(2 . 2 . 2)^2$	1 . 3 . 3	3 . 2 . 4		
$(3 . 6 . 0)^3$	2 . 1 . 1	$(2 . 4 . 2)^2$	2 . 1 . 3			
$(4 . 4 . 0)^2$	$(2 . 3 . 1)^3$	3 . 2 . 2	2 . 3 . 3			
$(4 . 6 . 0)^2$	$(2 . 5 . 1)^2$		3 . 1 . 3			
5 . 4 . 0	$(3 . 3 . 1)^2$		3 . 3 . 3			
5 . 6 . 0	$(3 . 5 . 1)^2$					
	4 . 3 . 1					

En ajoutant à ces 50 *grundformen* secondaires les 11 primaires qui proviennent du dénominateur dont les types sont

2 . 0 . 0	0 . 2 . 2
3 . 0 . 0	0 . 1 . 3
0 . 4 . 0	1 . 0 . 4
1 . 4 . 0	2 . 0 . 4
2 . 4 . 0	
3 . 2 . 0	
3 . 4 . 0	

on retrouve les 64 types calculés par M. Gundelfinger, selon la méthode de M. Gordan, avec l'exception des 3 suivants: 3 . 4 . 2, 3 . 4 . 2, 4 . 5 . 1.

Il reste à considérer les 3 covariants qui y correspondent; pour cela, je n'ai pas besoin de savoir la construction des *grundformen* données par M. Gundelfinger, car on peut procéder par un calcul algébrique direct pour déterminer si, oui ou non, le nombre des covariants linéairement indépendants appartenant à un quelconque de ces types peut être comblé par la combinaison de certains des 61 covariants connus. Ce nombre, on peut toujours le déterminer *a priori* par le théorème fondamental de M. Cayley, et, de plus, étant donné le type d'un covariant, on peut toujours trouver le covariant lui-même.

C'est par cette méthode, abrégée avec l'aide de quelques considérations appartenant à la théorie générale de la fraction génératrice, que je me suis convaincu de l'exactitude des résultats donnés par le tamisage pour le cas de deux biquadratiques, et que les deux formes, dites *irréductibles*, qui se trouvaient dans le tableau de M. Gordan, mais qui ne figuraient pas dans le mien, étaient superflues.

C'est la méthode la plus courte. Cependant, afin d'ôter toute nécessité d'expliquer la base du raisonnement, au lieu de suivre cette méthode dans la Note insérée dans les *Comptes rendus*, je jugeai préférable de prendre les deux formes qu'on obtient par la construction donnée par M. Gordan et d'en effectuer la décomposition, pour ainsi dire, sous les yeux du lecteur. J'espère, dans une prochaine Communication à l'Académie, par l'une ou l'autre de ces méthodes, pouvoir démontrer que les 3 *grundformen* supposées dont il est question sont superflues aussi, et que le véritable nombre des invariants et covariants irréductibles pour le système cubo-biquadratique binaire est effectivement 61 et non pas 64, comme le pensait M. Gundelfinger. En tout cas, je ferai savoir le vrai nombre de ces *grundformen*.

Pour m'assurer de l'exactitude des résultats précédemment donnés, j'ai fait calculer la fraction génératrice (fonction seulement de t et τ) dont le développement ne contient que les puissances positives de ces lettres, et tel

que le coefficient numérique de $t^n . \tau^\nu$ coïncide avec le nombre des covariants (d'un ordre *quelconque* dans les variables) des degrés n, ν dans les coefficients de la biquadratique et la cubique respectivement. Cette fraction se déduit de la génératrice primitive

$$\frac{1}{(1-tu^4)(1-tu^2)(1-t)(1-tu^{-2})(1-tu^{-4})(1-\tau u^3)(1-\tau u)(1-\tau u^{-1})(1-\tau u^{-3})}$$

(qui ne diffère de celle dont je me suis déjà servi que dans le numérateur où se trouve 1 au lieu de $1-u^{-2}$) de la manière suivante. En la traitant comme une fonction de u, et en la décomposant en fractions partielles, on prend la somme des coefficients (fonctions de t et τ) de celles de ces fractions qui ont pour dénominateurs les facteurs de

$$1-tu^4,\ 1-tu\ ;\ 1-\tau u^3,\ 1-\tau u:$$

cette somme sera la fraction génératrice cherchée. Or il est facile de démontrer que, en mettant $u=1$ dans la fraction génératrice canonique déjà obtenue, les deux fractions doivent devenir égales: on a fait ce calcul et, en comparant les deux expressions, on a trouvé entre elles un accord parfait sans qu'il y ait eu occasion d'introduire, dans l'une ou l'autre, un changement numérique quelconque, preuve satisfaisante de l'exactitude des résultats et, en même temps, de l'habileté très-peu commune du calculateur (M. Franklin), qui, par son dévouement consciencieux et opiniâtre à ce long et pénible travail, a rendu un véritable service au progrès de la science algébrique.

Ce qui ajoute considérablement à la difficulté du travail est la circonstance suivante, qui est assez intéressante en elle-même pour que je la cite ici. En faisant la décomposition en fractions partielles de la génératrice primitive, on trouvera contenus, dans les coefficients de celles mêmes qu'on doit conserver, les facteurs

$$\frac{1}{t-\tau^2},\ \frac{1}{t-\tau^4},\ \frac{1}{t^3-\tau^2},\ \frac{1}{t^3-\tau^4},$$

lesquels ne doivent et ne peuvent pas paraître dans la fraction canonique, de sorte qu'on sait d'avance que $t-\tau^2$, $t-\tau^4$, $t^3-\tau^2$, $t^3-\tau^4$ seront diviseurs exacts du numérateur de la fraction qui conduit à la fraction canonique. C'est, en effet, un théorème général que (quel que soit le nombre des *quantics* donnés), le dénominateur de la fraction génératrice canonique ne peut jamais contenir des facteurs où les lettres prises avec des exposants positifs sont distribuées entre deux groupes.

Toujours des facteurs de cette forme se présenteront dans le cours du calcul; mais, à la fin, quand toutes les sommations auront été effectuées, ils doivent nécessairement disparaître par voie de division dans le numérateur. Sans cette propriété, qu'on peut démontrer *a priori*, un théorie de la fonction génératrice pour des systèmes de *quantics* binaires aurait été impossible ou tout à fait inutile.

En ajoutant aux fractions canoniques que j'ai déjà données dans les *Comptes rendus* celle qui appartient à deux quadratiques, c'est-à-dire

$$\frac{1 - t\tau u^2}{(1-t^2)(1-\tau^2)(1-t\tau)(1-tu^2)(1-\tau u^2)},$$

on voit qu'on est à présent en possession des génératrices canoniques pour tous les systèmes binaires qui proviennent des combinaisons deux à deux des ordres 2, 3, 4, c'est-à-dire 2.2, 2.3, 2.4, 3.3, 3.4, 4.4; et en ajoutant les génératrices déjà connues pour les *quantics* linéaires, quadratiques, cubiques et biquadratiques, pris séparément, à celles que j'ai données dans les *Comptes rendus* pour les *quantics* des ordres 5, 6, 8, on aura de même les génératrices appartenant aux *quantics* pris séparément d'un ordre quelconque, compris entre les limites 1 et 8, avec l'exception de 7, lequel cas M. Cayley a entrepris de calculer. De plus, j'ai donné, dans le second numéro du *American Mathematical Journal*, la génératrice pour la partie invariantive du *quantic* de l'ordre 10, et je me propose de la compléter en faisant calculer, en outre, sa partie covariantive.

J'ai aussi obtenu la génératrice générale pour un nombre quelconque donné des formes linéaires, et la même pour les formes quadratiques, entre lesquelles deux génératrices il existe un rapport algébrique vraiment remarquable, de sorte que, par le moyen d'une substitution algébrique des plus simples, on peut passer immédiatement de l'une à l'autre; mais ce travail n'a pas encore été publié.

Si quelqu'un voulait bien entreprendre le calcul de la génératrice des formes fondamentales pour le *quantic* de l'ordre 9, on aurait une collection très-compacte et assez étendue de ces fonctions importantes.

Je saisis cette occasion pour renouveler mes instances auprès des disciples de M. Gordan, si nombreux et si largement disséminés dans l'Allemagne, l'Italie et ailleurs, de vouloir bien faire exécuter entre eux, par sa méthode, les travaux nécessaires pour confirmer ou réfuter le dénombrement, que j'ai récemment publié dans les *Comptes rendus*, des covariants irréductibles appartenant au *quantic* du huitième degré. Ce serait manquer aux devoirs imposés par la science et la grande renommée de M. Gordan que de ne pas répondre à cet appel. Quant aux résultats que j'ai donnés ici pour le cas de la combinaison des ordres 3 et 4, il est bon d'ajouter que l'ordre le plus élevé des covariants irréductibles 6 est d'accord avec la limite supérieure pour le cas d'un nombre quelconque de *quantics* dont l'ordre de chacun n'excède pas 4, selon la formule donnée par M. Camille Jordan dans une séance toute récente de l'Académie. On trouvera, en effet, que, pour le cas supposé, cette limite est le nombre 6 lui-même.

20.

SUR LE VRAI NOMBRE DES FORMES IRRÉDUCTIBLES DU SYSTÈME CUBO-BIQUADRATIQUE.

[*Comptes Rendus*, LXXXVII. (1878), pp. 445—448.]

EN addition aux 61 formes irréductibles que j'ai trouvées dans une Communication précédente faite à l'Académie, M. Gundelfinger affirme l'existence de trois autres: deux du type 3 . 4 . 2 et une du type 4 . 5 . 1, où le premier, le deuxième et le troisième chiffre expriment respectivement le degré ou l'ordre de la forme dans les coefficients de la biquadratique, de la cubique et dans les variables.

Je me bornerai dans cette Note à démontrer qu'il n'existe nul covariant du type 3 . 4 . 2.

Ce que M. Gordan nomme une *Ueberschiebung*, je le nommerai une *alliance*: si f et ϕ représentent

$$(a_0, a_1, a_2, \dots)(x, y)^m; \quad (b_0, b_1, \dots)(x, y)^n,$$

l'alliance $(f, \phi)^i$, i n'étant pas plus grand ni que m ni que n, sera un covariant de l'ordre $m+n-2i$, dont le coefficient de x^{m+n-2i}, que je nommerai *son représentant*, est

$$(1, -1)^i (a_0 b_i, a_1 b_{i-1}, a_2 b_{i-2}, \dots, a_i b_0).$$

Je considérai le système spécial composé de $(a, b, c, d, e)(x, y)^4$ et de $(1, \beta, 0, 1)(x, y)^3$, où β sera traité comme un infinitésimal.

On aura donc

$$\begin{aligned}
0.1.3 &= x^3 + 3\beta x^2 y + y^3, \\
0.2.2 &= 0x^2 + xy + \beta y^3, \\
0.3.3 &= x^3 + 3\beta x^2 y - y^3, \\
0.4.6 &= x^6 - y^6 + 6\beta x^5 y, \\
1.0.4 &= ax^4 + 4bx^3 y + 6cx^2 y^2 + 4dxy^3 + ey^4, \\
2.0.4 &= Ax^4 + 4Bx^3 y + 6Cx^2 y^2 + 4Dxy^3 + Ey^4,
\end{aligned}$$

où $$A = ac - b^2, \quad C = \frac{2ae - bd - c^2}{3}, \quad D = \frac{be - cd}{2};$$

$$0.4.4 = x^2y^2 + \ldots,$$
$$2.0.8 = \ldots + e^2y^8,$$
$$0.3.9 = x^9 + \ldots.$$

Faisons $l = (1.0.4,\ 0.1.3)^3$, dont le type est $1.1.1$,
$m = (2.0.4,\ 0.1.3)^3$, " $2.1.1$,
$n = \ldots$, " $2.0.0$,
$p = \ldots$, " $3.2.0$,
$r_1 = (1.0.4,\ 0.3.3)^3$, " $1.3.1$,
$r_2 = (1.0.4,\ 0.3.5)^4$, " $1.3.1$,
$s_1 = (2.0.4,\ 0.3.3)^3$, " $2.3.1$,
$s_2 = (2.0.4,\ 0.3.5)^4$, " $2.3.1$,
$s_3 = (2.0.8,\ 0.3.9)^8$, " $2.3.1$,
$t_1 = (1.0.4,\ 0.4.6)^4$, " $1.4.2$,
$t_2 = (1.0.4,\ 0.4.4)^3$, " $1.4.2$,
$u = \ldots$ " $0.2.2$.

Alors les huit produits $mr_1, mr_2, ls_1, ls_2, ls_3, nt_1, nt_2, pu$ seront tous du type $3.4.2$.

En se servant de la notation $R\phi$ pour exprimer le coefficient de la plus haute puissance de x dans la forme la plus générale de ϕ, on obtient, pour le système spécial dont il s'agit,

$$Rl = a + 3c\beta - d, \quad Rm = A + 3C\beta - D,$$
$$Rr_1 = a + 3c\beta + d, \quad Rs_1 = A + 3C\beta + D,$$
$$Rr_2 = a + 12c\beta - 4d, \quad Rs_2 = A + 12C\beta - 4D;$$

donc
$$Rmr_1 = (a + d + 3c\beta)(A - D + 3C\beta),$$
$$Rmr_2 = (a - 4d + 12c\beta)(A - D + 3C\beta),$$
$$Rls_1 = (a - d + 3c\beta)(A + D + 3C\beta),$$
$$Rls_2 = (a - d + 3c\beta)(A - 4D + 12C\beta).$$

Rs_3 possédera évidemment le terme e^2.

Rt_1, en négligeant les termes contenant β, sera formé au moyen des deux séries de coefficients

$$\begin{matrix} a & b & c & d & e, \\ 1 & 0 & 0 & 0 & 0 \end{matrix}$$

et sera égal à e, et de même, sous la même supposition, Rt_2 sera formé au moyen des deux séries

$$\begin{matrix} a & b & c & d, \\ 0 & 0 & 1 & 0, \end{matrix}$$

et sera égal à b.

De plus, $R(u)$ est absolument zéro, et

$$n = ae - 4bd + 3c^2.$$

On voit donc que $R(ls_3)$, seul des huit produits, contiendra le terme de^2, et conséquemment ne peut pas entrer dans une équation numérique quelconque entre ces produits. En le mettant de côté, on voit que, des sept produits qui restent, $R(nt_1)$ et $R(nt_2)$ contiendront, le premier, à lui seul, le terme c^2e, le second, à lui seul, le terme c^2b; conséquemment, en se souvenant que $R(pu) = 0$, ce n'est qu'entre Rmr_1, Rmr_2, Rls_1, Rls_2 qu'une liaison numérique (s'il y en a aucune) peut exister. Quant à ces quatre quantités, si même on ne tenait nul compte de β, une seule combinaison linéaire existe entre elles, pour laquelle la valeur est zéro, c'est-à-dire

$$3R(ls_1) - 2R(ls_2) - 3R(mr_1) + 2R(mr_2),$$

laquelle, en ayant égard à β, devient

$$(a - d + 3c\beta)(5A - 5D - 15C\beta) - (A - D + 3C\beta)(5a - 5d - 15c\beta),$$

c'est-à-dire $$30[(A - D)c - (a - d)C]\beta,$$

qui, évidemment, n'est pas zéro. Donc les huit covariants réductibles du type $3.4.2$, mr_1, mr_2, ls_1, ls_2, ls_3, nt_1, nt_2, pu pour le système spécial qu'on a considéré, et *à plus forte raison* pour le système cubico-biquadratique général, sont linéairement indépendants.

Trouvons le nombre total des covariants linéairement indépendants de ce type. En général, pour deux formes dont les ordres sont i, i', les covariants du type j, j', ϵ linéairement indépendants sont en nombre égal à $S - S'$, où

$$S = \sum_{m=w}^{m=0} (m:i, j)(w - m:i', j') \quad \text{et} \quad S' = \sum_{m=w'}^{m=0} (m:i, j)(w' - m:i', j'),$$

$$w = \frac{ij + i'j' - \epsilon}{2}, \quad w' = w - 1,$$

$(m:i, j)$ représentant le nombre des compositions qu'on peut effectuer de m avec j chiffres (zéro y compris) dont nul ne surpasse i, ou bien avec i chiffres dont nul ne surpasse j.

Dans le cas actuel,

$$w = \frac{4.3 + 3.4 - 2}{2} = 11, \quad w' = 10,$$

$$i = j' = 4, \quad j = i' = 3.$$

En donnant à m les valeurs successives de 0 jusqu'à 11, on trouve pour $(m:3, 4)$ ou bien $(m:4, 3)$ les valeurs

1, 1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1,

et, en faisant la progression dans le sens inverse,

1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1, 1.

On a conséquemment

$$\begin{aligned} S &= 1 + 2 + 6 + 12 + 16 + 20 + 20 + 16 + 12 + 6 + 2 + 1, \\ S' &= \quad\ 2 + 3 + \ 8 + 12 + 20 + 16 + 20 + 12 + 8 + 3 + 2 \\ \text{et} \quad S - S' &= 1 \quad + 3 + \ 4 + \ 4 \quad\ + \ 4 - \ 4 \quad\ - 2 - 1 - 1 \\ &= 8. \end{aligned}$$

Conséquemment le nombre *total* des covariants linéairement indépendants du type 3.4.2 n'est pas plus grand que le nombre des covariants de ce même type linéairement indépendants et *réductibles*: il n'y a donc pas de place *in rerum natura* pour les deux covariants quadratiques *irréductibles* du type 3.4.2 imaginés par M. Gundelfinger.

Dans une prochaine Communication j'entreprendrai l'examen de la seule forme qui reste à discuter, c'est-à-dire le covariant linéaire des degrés 4, 5 dans les coefficients, qui se trouve dans la Table de M. Gundelfinger, mais en dehors de la mienne. On sait déjà que le nombre des formes irréductibles pour le système en question est ou 61 ou 62. Il me semble peu douteux que c'est le premier de ces nombres qui sortira victorieux de la discussion du type 4.5.1.

21.

DÉTERMINATION DU NOMBRE EXACT DES COVARIANTS IRRÉDUCTIBLES DU SYSTÈME CUBO-BIQUADRATIQUE BINAIRE.

[*Comptes Rendus*, LXXXVII. (1878), pp. 477—481.]

Le seul type donné par M. Gundelfinger qui reste à discuter est le covariant linéaire des degrés 4 et 5 dans les coefficients de la biquadratique et la cubique respectivement. Un type quelconque étant représenté par $\alpha.\beta.\gamma$ quand ce type est monadelphique, je me servirai de $\alpha.\beta.\gamma$ indifféremment pour signifier le type et la forme qui y appartient et de $[\alpha.\beta.\gamma]$ pour signifier le coefficient de la plus haute puissance de x dans cette forme. On trouvera que le type $4.5.1$ qui est à discuter peut être produit de douze manières diverses, par la combinaison entre eux des types inférieurs déjà reconnus comme appartenant à des formes irréductibles, et j'écrirai les douze produits correspondants sous la forme

$$\begin{aligned}
&Z_1 = (3.0.6,\ 0.2.6)^6(1.0.4,\ 0.3.3)^3, && X = (3.0.0)(1.0.4,\ 0.5.5)^4,\\
&Z_2 = (3.0.6,\ 0.2.6)^6(1.0.4,\ 0.3.5)^4, && Y_2 = (2.0.0)(2.0.8,\ 0.5.9)^8,\\
&U_1 = (1.1.1)(3.0.0)(0.4.0), && Y_1 = (2.0.0)(2.0.4,\ 0.5.5)^4,\\
&U_2 = (1.1.1)(3.0.6,\ 0.4.6)^6, && J_1 = (2.1.1)(2.0.4,\ 0.4.4)^4,\\
&U_3 = (1.1.1)(3.0.8,\ 0.4.8)^8, && J_2 = (2.1.1)(2.0.0)(0.4.0),\\
&U_4 = (1.1.1)(3.0.12,\ 0.4.12)^{12}, && J_3 = (2.1.1)(2.0.8,\ 0.4.8)^8.
\end{aligned}$$

Ecrivons

$$0.1.3 = (1,\ 0,\ 0,\ 1)(x,\ y)^3,\quad 1.0.4 = (a,\ b,\ c,\ d,\ e)(x,\ y)^4,$$

on aura

$$2.0.4 = (A,\ B,\ C,\ D,\ E)(x,\ y)^4,$$

où $A = ac - b^2,\ \ B = \dfrac{ad - be}{2},\ \ C = \dfrac{ae + 2b - 3c^2}{6},\ \ D = \dfrac{be - cd}{2},\ \ E = ce - d^2,$

$$3.0.6 = (L,\ M,\ N,\ P,\ Q,\ R,\ S)(x,\ y)^6,$$

où $L = a^2d - 3abc + 2b^3,\ \ 2P = b^2e - d^2a,\ \ S = -e^2b + 3edc - 2d^3,$

$$[1.1.1] = [(1.0.4,\ 0.1.3)^3] = a - d,\quad [2.1.1] = [(2.0.4,\ 0.1.3)^3] = A - D,$$

$$0.2.2 = xy,\ \ 0.3.3 = x^3 - y^3,\ \ 0.5.5 = x^4y - xy^4,\ \ 0.3.5 = x^4y + xy^4.$$

Donc $[(2.0.4,\ 0.5.5)^4] = A + 4D$, $[(1.0.4,\ 0.5.5)^4] = a + 4d$,

$$0.2.6 = x^6 + 2x^3y^3 + y^6,$$

donc $$[(3.0.6,\ 0.2.6)^6] = L - 2P + S,$$

$$0.4.6 = x^6 - y^6,$$

donc $$[(3.0.6,\ 0.4.6)^6] = L - S.$$

Faisons $$a = 1, \quad c = b^2, \quad e = bd;$$

alors $A = 0$, $D = 0$.

Donc $$Y_1 = 0, \quad J_1 = 0, \quad J_2 = 0, \quad J_3 = 0.$$

Je vais démontrer que nulle liaison linéaire ne subsistera entre les coefficients de la plus haute puissance de x dans les huit covariants X, Y_2, Z_1, Z_2, U_1, U_2, U_3, U_4. $3.0.12$ représente $(1.0.4)^3$, et $0.4.12$ représente $(0.1.3)^4$; donc U_4 contiendra a^4, c'est-à-dire 1, et, comme on va voir, sera la seule des huit formes nommées qui le contient; donc la liaison, si elle existe, ne peut pas contenir U_4.

$$2.0.0 = ae - 4bd + 3c^2 = 3(b^4 - bd),$$

$$0.5.9 = (0.1.3)^2(0.3.3) = (x^3 + y^3)^2(x^3 - y^3) = x^9 + x^6y^3 - x^3y^6 - y^9,$$

$$2.0.8 = (1.0.4)^2 = e^2y^8 + \ldots.$$

Donc $[(2.0.8,\ 0.5.9)^8]$ contiendra le terme e^2, et Y_2, par conséquent, le terme b^4e^2 ou b^6d^2.

$$[(1.0.4,\ 0.3.3)^3] = a + d, \quad [(1.0.4,\ 0.3.5)^4] = a - 4d;$$

ainsi on peut remplacer (Z_1), (Z_2) par les combinaisons linéaires T_1, T_2, où

$$T_1 = L - 2P + S, \quad T_2 = d(L - 2P + S),$$

et $$L = d - b^3, \quad 2P = b^3d - d^2, \quad S = 2b^3d^2 - 2d^3,$$

$$(X) = (1 + 4d)\Delta, \quad (U_1) = (1 - d)\Delta,$$

où $$\Delta = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = \begin{vmatrix} 1 & b & b^2 \\ b & b^2 & d \\ b^2 & d & bd \end{vmatrix},$$

de sorte qu'on peut substituer, au lieu de (X) et (U_1), Δ et $d\Delta$,

$$(U_2) = (1 - d)(L - S),$$

$$3.0.8 = (a, b, c, d, e)(x, y)^4 . (A, B, C, D, E)(x, y)^4,$$

$$0.4.8 = xy(x^3 + y^3)^2 + x^7y + 2x^4y^4 + xy^7.$$

Donc $(U_3) = (1 - d)\Lambda$, où Λ est une fonction linéaire de aB, bA, cb, dB, bD, aE, eA, dE, eD, c'est-à-dire, puisque $A = 0$, $D = 0$, Λ est une fonction linéaire de

$$d - b^3;\ b^3d + 2b^3 - 3b^4;\ d^2 - b^3d;\ b^3d - d^2;\ b^3d^2 - d^3.$$

On voit que b^6d^2 n'entre comme terme dans aucune des quantités

$$T_1,\ dT_1,\ \Delta,\ d\Delta,\ (1-d)(L-S),\ (1-d)\Lambda;$$

donc la liaison dont on discute l'existence ne peut pas contenir.

Quant aux six quantités qui restent, Δ seul contient b^6, $d\Delta$ seul db^6, et Λ seul b^4; donc la liaison, si elle existe, doit avoir lieu entre

$$T_1,\ dT_1,\ (1-d)(L-S),$$

et conséquemment entre les trois quantités

$$L-2P+S,\ (1-d)(L-P),\ (1-d)(S-P),$$

dont la dernière seule contient d^4 et les deux premières, c'est-à-dire

$$(1-d)[d+2d^2-(1-d)b^3],\quad \tfrac{1}{2}(1-d)[2d+d^2-(2+d)b^3],$$

ne sont pas l'une un multiple de l'autre. Donc il n'y a nulle liaison linéaire entre des coefficients du même rang des douze covariants qu'on considère pour le cas où $1.0.4$ et $0.1.3$ sont de la forme

$$(1,\ b,\ b^2,\ d,\ bd)(x,\ y)^4,\quad (1,\ 0,\ 0,\ 1)(x,\ y)^3$$

respectivement, et conséquemment, dans le cas général, une telle liaison, si elle existe, ne peut avoir lieu qu'entre les quatre dont les coefficients en question s'évanouissent pour le cas spécial, c'est-à-dire entre Y_1, J_1, J_2, J_3, mais cela est inadmissible; car, sur cette supposition, on aurait

$$\lambda(2.0.0)(2.5.1)+\mu(2.1.1)(2.4.1)=0,$$

où les quatre facteurs sont irréductibles. Il y a donc douze covariants réductibles, mais linéairement indépendants, du type $4.5.1$.

Or le nombre total des covariants de ce type linéairement indépendants est $S-S'$, où

$$S=\sum_{q=0}^{q=w}(q:4,\ 4)(w-q:3,\ 5)\quad\text{et}\quad w=\frac{4.4+3.5-1}{2}=15,$$

et S' est ce que S devient quand on substitue $w-1$ (c'est-à-dire 14) à w. Or, en donnant à q les valeurs successives de 0 jusqu'à 15, $(q:4,\ 4)$ prend les valeurs

1, 1, 2, 3, 5, 5, 7, 7, 8, 7, 7, 5, 5, 3, 2, 1

et $(q:3,\ 5)$

1, 1, 2, 3, 4, 5, 6, 6, 6, 6, 5, 4, 3, 2, 1, 1.

On a donc

$$\begin{aligned} S&=1+\ 1+\ 4+\ 9+20+25+42\\ &\quad+42+48+42+35+20+15+\ 6+2+1,\\ S'&=1+\ 2+\ 6+12+25+30\\ &\quad+42+42+48+35+28+15+10+3+2 \end{aligned}$$

et

$$S-S'=1+2+3+8+12+6-6-8-4-1-1=12,$$

c'est-à-dire le nombre total des covariants linéairement indépendants du type $4.5.1$ est entièrement épuisé par les covariants réductibles et linéairement

indépendants de ce type. Donc il n'y a nul covariant irréductible du type 4.5.1, et conséquemment le montant des *grundformen* pour le système cubo-biquadratique binaire est 61, comme j'ai trouvé, et non pas 64 comme M. Gundelfinger avait pensé.

Je conclus par l'observation importante que ma méthode serait parfaitement démontrée *à priori* si l'on pouvait démontrer le théorème suivant:

Soit σ le nombre total de formes linéairement indépendantes d'un type donné appartenant à un système donné de quantics, c'est-à-dire $\sigma = S - S'$ pour les formes binaires obtenues par composition des formes irréductibles de types inférieurs, et σ' le nombre de formes du même type; alors, si σ n'est pas plus petit que σ', le nombre des formes irréductibles du type sera $\sigma - \sigma'$ et dans le cas contraire zéro: c'est-à-dire que, dans le premier cas, il n'existera nulle liaison linéaire entre les formes composées et, dans le cas contraire, seulement $\sigma' - \sigma$ telles liaisons. Ce principe, indubitablement vrai pour les quantics binaires, s'étend probablement à des quantics en général et, puisque j'ai donné la règle universelle pour trouver le nombre total des formes linéairement indépendantes d'un type donné, il s'ensuit que, si l'on possède la connaissance d'une assemblée de formes ou plus simplement la connaissance des types numériquement exprimes qui figurent dans une assemblée, parmi lesquels se trouvent toutes les formes irréductibles, on a le moyen de trouver par un calcul purement arithmétique quels sont les types qui correspondent à des formes irréductibles et combien il y en a pour chaque type.

On aurait donc la solution arithmétique et sans tâtonnement du problème qui vient à la fin de la méthode de M. Gordan, dont la difficulté a créé tant d'embarras dans l'application de cette méthode et produit des erreurs tellement graves dans les résultats obtenus et jusqu'à ce jour acceptés comme vrais.

22.

SUR LES COVARIANTS IRRÉDUCTIBLES DU QUANTIC DU SEPTIÈME ORDRE *.

[*Comptes Rendus*, LXXXVII. (1878), pp. 505—509.]

M. CAYLEY a eu la bonté de calculer pour moi, par une méthode propre à lui, la fraction génératrice pour le quantic $(x, y)^7$ dans sa forme réduite. Il trouve que son numérateur est

$$
\begin{aligned}
& a^0 \,.\, 1 \\
&+ a^1 \,(-x - x^3 - x^5) \\
&+ a^2 \,(+x^2 + x^4 + 2x^6 + x^8 + x^{10}) \\
&+ a^3 \,(-x^7 - x^9 - x^{11} - x^{13}) \\
&+ a^4 \,(2x^4 + x^8 + x^{14}) \\
&+ a^5 \,(x + 2x^3 - x^9 - x^{11}) \\
&+ a^6 \,(-1 + 2x^2 - x^4 - x^8 - x^{10} + x^{12}) \\
&+ a^7 \,(4x + 4x^5 - x^7 - x^9 + x^{11} - x^{13}) \\
&+ a^8 \,(2 - x^2 - 3x^6 - 3x^8 - x^{10} - x^{12}) \\
&+ a^9 \,(x + 3x^3 + x^5 - x^7 + 2x^9 + 2x^{13}) \\
&+ a^{10} (-1 + 4x^2 - x^6 - 2x^8 - 2x^{10} - x^{14}) \\
&+ a^{11} (5x + 3x^3 + 2x^5 - x^7 - 2x^9 - x^{11} + x^{13}) \\
&+ a^{12} (5 + x^2 - 4x^6 - 6x^8 - 4x^{10} - x^{12} - 2x^{14}) \\
&+ a^{13} (x - 4x^5 - 4x^7 - x^9 + x^{11} + 4x^{13}) \\
&+ a^{14} (2 + 5x^2 + x^4 + x^6 - 2x^8 + 3x^{12} - x^{14}) \\
&+ a^{15} (3x - x^3 - x^5 - 7x^7 - 5x^9 - x^{11} - x^{13}) \\
&+ a^{16} (6 + 3x^2 + 3x^4 - 4x^6 - 3x^8 - x^{12} + 5x^{14}) \\
&+ a^{17} (-x - 2x^3 - 9x^5 - 8x^7 - 4x^9 - 3x^{11} + 4x^{13}) \\
&+ a^{18} (2 + 6x^2 + x^4 + 2x^6 + 2x^8 + x^{10} + 6x^{12} + 2x^{14})
\end{aligned}
\qquad
\begin{aligned}
&+ a^{36} x^{14} \\
&+ a^{35} (-x^9 - x^{11} - x^{13}) \\
&+ a^{34} (x^4 + x^6 + 2x^8 + x^{10} + x^{12}) \\
&+ a^{33} (-x - x^3 - x^5 - x^7) \\
&+ a^{32} (1 + x^6 + 2x^{10}) \\
&+ a^{31} (-x^3 - x^5 + 2x^{11} + x^{13}) \\
&+ a^{30} (x^2 - x^4 - x^6 - x^{10} - 2x^{12} - x^{14}) \\
&+ a^{29} (-x + x^3 - x^5 - x^7 + 4x^9 + 4x^{13}) \\
&+ a^{28} (-x^2 - x^4 - 3x^6 - 3x^8 - x^{12} + 2x^{14}) \\
&+ a^{27} (2x + 2x^5 - x^7 + x^9 + 3x^{11} + x^{13}) \\
&+ a^{26} (-1 - 2x^4 - 2x^6 - x^8 + 4x^{10} - x^{14}) \\
&+ a^{25} (x - x^3 - 2x^5 - x^7 + 2x^9 + 3x^{11} + 5x^{13}) \\
&+ a^{24} (2 - x^2 - 4x^4 - 6x^6 - 4x^8 + x^{10} + 5x^{14}) \\
&+ a^{23} (+4x + x^3 - x^5 - 4x^7 - 4x^9 + x^{13}) \\
&+ a^{22} (-1 + 3x^2 - 2x^4 + x^8 + x^{10} + 5x^{12} + 2x^{14}) \\
&+ a^{21} (-x - x^3 - 5x^5 - 7x^7 - x^9 - x^{11} + 3x^{13}) \\
&+ a^{20} (5 - x^2 - 3x^6 - 4x^8 + 3x^{10} + 3x^{12} + 6x^{14}) \\
&+ a^{19} (4x - 3x^3 - 4x^5 - 8x^7 - 9x^9 - 2x^{11} - x^{13})
\end{aligned}
$$

Quant au dénominateur, on sait d'avance qu'il est

$$(1 - ax)(1 - ax^3)(1 - ax^5)(1 - ax^7)(1 - a^4)(1 - a^6)(1 - a^8)(1 - a^{10})(1 - a^{12}).$$

Pour obtenir la fraction génératrice sous sa forme canonique, je multiplie le numérateur et le dénominateur de cette forme réduite chacun par

$$(1 + a^6)(1 + a^{10})(1 + ax)(1 + ax^3)(1 + ax^5).$$

[* See below, p. 144.]

Alors le dénominateur devient évidemment

$$(1-a^4)(1-a^8)(1-a^{12})^2(1-a^{20})(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})(1-ax^7)$$

et le numérateur devient $P+Q$ où, pour trouver Q, on n'a qu'à substituer, pour un terme quelconque Ka^jx^ϵ, le terme $Ka^{j'}x^{\epsilon'}$, avec la condition que

$$j+j'=55 \quad \text{et} \quad \epsilon+\epsilon'=23.$$

On voit que la fraction sera alors sous sa forme canonique, par la raison qu'on ne trouvera ni a^4, ni a^8, ni a^{12}, ni a^{20} dans le numérateur affecté du signe $-$. On comprend qu'en effectuant le développement de l'une ou l'autre expression, selon les puissances ascendantes de a et de x, le coefficient de a^jx^ϵ exprimera le nombre total des covariants du degré j dans les coefficients du quantic du septième ordre et de l'ordre ϵ dans les variables.

Je trouve alors, pour la valeur de P, l'expression suivante:

$$\begin{aligned}
&a^0.1\\
&+a^3(x^3+x^5+x^7+x^9+x^{11}+x^{15})\\
&+a^4(2x^4+x^6+2x^8+x^{10}+x^{14})\\
&+a^5(x+2x^3+2x^5+2x^7+2x^9-x^{17}-x^{21})\\
&+a^6(x^2+2x^4+3x^6+2x^8+2x^{12}-x^{14}-x^{16})\\
&+a^7(3x+x^3+5x^5+x^7+x^{11}-x^{13}-2x^{15}-x^{19}+x^{23})\\
&+a^8(2+3x^2+3x^4+6x^6+3x^{10}-2x^{12}-2x^{14}-x^{16}-2x^{18})\\
&+a^9(3x+5x^3+7x^5+2x^7+4x^9-x^{11}-2x^{13}-2x^{15}-3x^{17}-x^{19})\\
&+a^{10}(5x^2+4x^4+6x^6+6x^8-3x^{10}-3x^{12}+x^{14}-4x^{16}-x^{18}-x^{22})\\
&+a^{11}(5x+8x^3+11x^5-4x^7-2x^{11}+x^{13}-3x^{15}-x^{17})\\
&+a^{12}(4+9x^2+9x^4+12x^6+2x^{10}-7x^{12}-5x^{14}-4x^{16}-x^{20}+x^{22})\\
&+a^{13}(9x+8x^3+13x^5+5x^7-x^9\\
&\qquad -3x^{11}-13x^{13}-9x^{15}-3x^{17}-x^{19}+x^{21})\\
&+a^{14}(4+9x^2+12x^4+15x^6-2x^8-3x^{10}\\
&\qquad -10x^{12}-11x^{14}-8x^{16}-3x^{18}+3x^{22})\\
&+a^{15}(9x+12x^3+16x^5+6x^7+6x^9\\
&\qquad -7x^{11}-11x^{13}-9x^{15}-4x^{17}-x^{19}+2x^{21}+2x^{23})\\
&+a^{16}(5+14x^2+15x^4+12x^6+x^8-x^{10}\\
&\qquad -13x^{12}-4x^{14}-10x^{16}-x^{18}+3x^{20}+2x^{22})\\
&+a^{17}(12x+14x^3+17x^5-5x^7-3x^9-17x^{11}\\
&\qquad -16x^{13}-11x^{15}-5x^{17}+2x^{19}+3x^{21})\\
&+a^{18}(9+14x^2+14x^4+14x^6-4x^8-13x^{10}\\
&\qquad -21x^{12}-18x^{14}-18x^{16}-x^{18}+2x^{20}+5x^{22})\\
&+a^{19}(15x+16x^3+18x^5+27x^7-8x^9\\
&\qquad -19x^{11}-20x^{13}-20x^{15}-6x^{17}+2x^{21}+4x^{23})
\end{aligned}$$

$$
\begin{aligned}
&+ a^{20}(6 + 14x^2 + 18x^4 + 12x^6 - 8x^8 - 14x^{10} \\
&\qquad + 2x^{12} - 18x^{14} - 13x^{16} + 2x^{18} + 5x^{20} + 6x^{22}) \\
&+ a^{21}(14x + 17x^3 + 19x^5 - x^7 - 8x^9 - 25x^{11} \\
&\qquad - 23x^{13} - 14x^{15} - 2x^{17} + 4x^{19} + 8x^{21} + 4x^{23}) \\
&+ a^{22}(9 + 17x^2 + 15x^4 + 11x^6 - 8x^8 - 18x^{10} \\
&\qquad - 31x^{12} - 17x^{14} - 13x^{16} + 6x^{18} + 9x^{20} + 9x^{22}) \\
&+ a^{23}(17x + 17x^3 - 20x^5 - 43x^7 - 18x^9 - 32x^{11} \\
&\qquad - 26x^{13} - 22x^{15} - 4x^{17} + 9x^{19} + 9x^{21} + 5x^{23}) \\
&+ a^{24}(8 + 17x^2 + 14x^4 + 9x^6 - 19x^8 - 66x^{10} \\
&\qquad - 37x^{12} - 24x^{14} - 17x^{16} + 8x^{18} + 9x^{20} + 12x^{22}) \\
&+ a^{25}(15x + 15x^3 + 17x^5 - 7x^7 - 27x^9 - 30x^{11} \\
&\qquad - 32x^{13} - 23x^{15} + 3x^{17} + 9x^{19} + 12x^{21} + 9x^{23}) \\
&+ a^{26}(9 + 13x^2 + 14x^4 + 6x^6 - 20x^8 - 23x^{10} \\
&\qquad - 35x^{12} - 19x^{14} - 10x^{16} + 10x^{18} + 14x^{20} + 14x^{22}) \\
&+ a^{27}(14x + 15x^3 + 13x^5 - 15x^7 - 18x^9 - 37x^{11} \\
&\qquad - 31x^{13} - 17x^{15} + 3x^{17} + 14x^{19} + 14x^{21} + 6x^{23}).
\end{aligned}
$$

Pour effectuer le tamisage, en observant qu'en vertu des formules de M. C. Jordan on peut négliger toute puissance de x dont l'exposant excède 15, on obtient, pour les termes positifs de P et de Q qu'on doit obtenir, la table suivante:

$$
\begin{aligned}
&+ x^0(1 + 2a^8 + 4a^{12} + 4a^{14} + 5a^{16} + 9a^{18} + 6a^{20} + 9a^{22} \\
&\qquad + 8a^{24} + 9a^{26} + 6a^{28} + 9a^{30} + 5a^{32} + 4a^{34} + 4a^{36} + 2a^{40} + a^{48}) \\
&+ x\ (a^5 + 3a^9 + 5a^{11} + 9a^{15} + 12a^{17} + 15a^{19} + 14a^{21} + 17a^{23} + 15a^{25} \\
&\qquad + 14a^{27} + 14a^{29} + 12a^{31} + 9a^{33} + 6a^{35} + 5a^{37} + 2a^{39} + 3a^{41} + a^{43}) \\
&+ x^2(a^6 + 5a^8 + 5a^{10} + 4a^{12} + 9a^{14} + 14a^{16} \\
&\qquad + 14a^{18} + 14a^{20} + 17a^{22} + 17a^{24} + 13a^{26} + 14a^{28} \\
&\qquad + 12a^{30} + 9a^{32} + 8a^{34} + 2a^{36} + 3a^{38} + 2a^{40} + a^{42}) \\
&+ x^3(a^3 + 2a^5 + a^7 + 5a^9 + 8a^{11} + 8a^{13} + 12a^{15} \\
&\qquad + 14a^{17} + 16a^{19} + 17a^{21} + 17a^{23} + 15a^{25} \\
&\qquad + 15a^{27} + 14a^{29} + 9a^{34} + 9a^{33} + 5a^{35} + 2a^{37} + 3a^{39}) \\
&+ x^4(2a^4 + 2a^6 + 3a^8 + 4a^{10} + 9a^{12} + 12a^{14} + 15a^{16} + 14a^{18} + 14a^{20} \\
&\qquad + 15a^{22} + 14a^{24} + 14a^{26} + 14a^{28} + 9a^{30} + 9a^{32} + 4a^{34} + 2a^{36}) \\
&+ x^5(a^3 + 2a^5 + 5a^7 + 7a^9 + 11a^{11} + 13a^{13} + 16a^{15} + 17a^{17} \\
&\qquad + 18a^{19} + 19a^{21} + 17a^{25} + 13a^{27} + 10a^{29} + 8a^{31} + 6a^{33} + 2a^{35}) \\
&+ x^6(a^4 + 3a^6 + 6a^8 + 6a^{10} + 12a^{12} + 15a^{14} + 12a^{16} \\
&\qquad + 14a^{18} + 12a^{20} + 11a^{22} + 9a^{24} + 6a^{26} + 3a^{28} + 3a^{30}) \\
&+ x^7(a^3 + 2a^5 + a^7 + 2a^9 + 5a^{13} + 6a^{15} + 27a^{19})
\end{aligned}
$$

$$
\begin{aligned}
&+ x^8 (2a^4 + 2a^6 + 6a^{10} + a^{16} + a^{52}) \\
&+ x^9 (a^3 + 2a^5 + 4a^9 + 6a^{15} + a^{45} + a^{51}) \\
&+ x^{10} (a^4 + 3a^8 + 2a^{12} + a^{44}) \\
&+ x^{11} (a^3 + a^7 + 2a^{35} + 2a^{49}) \\
&+ x^{12} (2a^6 + 2a^{20} + a^{48} + a^{52}) \\
&+ x^{13} (a^{11} + 2a^{43} + 3a^{47} + a^{51}) \\
&+ x^{14} (a^4 + a^{10} + 6a^{40} + 4a^{46} + 2a^{50} + a^{52}) \\
&+ x^{15} (a^3 + a^{39} + 6a^{45} + 2a^{49} + 2a^{51}).
\end{aligned}
$$

Le tamisage étant effectué (ce qu'on peut aisément opérer par simple inspection), les termes et les coefficients numériques, qui seuls restent sains et saufs, toute soustraction faite, seront les suivants·

$$
\begin{aligned}
&1,\ 2a^8,\ 4a^{12},\ 4a^{14},\ 5a^{16},\ 9a^{18},\ a^{22}, \\
&a^5x,\ 3a^9x,\ 5a^{11}x,\ 9a^{15}x,\ 2a^{17}x,\ a^{19}x, \\
&a^6x^2,\ 5a^8x^2,\ 5a^{10}x^2,\ 4a^{12}x^2,\ 4a^{14}x^2, \\
&a^3x^3,\ 2a^5x^3,\ a^7x^3,\ 5a^9x^3,\ 5a^{11}x^3, \\
&2a^4x^4,\ 2a^6x^4,\ 3a^8x^4,\ 4a^{10}x^4, \\
&a^3x^5,\ 2a^5x^5,\ 5a^7x^5,\ 2a^9x^5, \\
&a^4x^6,\ 2a^6x^6,\ 3a^8x^6, \\
&a^3x^7,\ 2a^5x^7, \\
&2a^4x^8,\ a^6x^8, \\
&a^3x^9,\ 2a^5x^9, \\
&a^4x^{10}, \\
&a^3x^{11},\ a^7x^{11}, \\
&a^4x^{14}, \\
&a^3x^{15}.
\end{aligned}
$$

En ajoutant à ces termes ceux qui sont fournis par le dénominateur, c'est-à-dire

$$a^4,\ a^8,\ 2a^{12},\ a^{20},\ a^2x^2,\ a^2x^6,\ a^2x^{10},\ ax^7,$$

on a le tableau complet des invariants et covariants irréductibles du quantic du septième ordre, sous la convention qu'on comprend, par Ka^jx^ϵ, K covariants du degré j et de l'ordre ϵ. De même $2a^8$, a^8 signifiera trois invariants du degré 8; $4a^{12}$, $2a^{12}$ six invariants du degré 12. Le covariant dénoté en haut par a^3x^{15} démontre que la limite inférieure pour l'ordre des covariants d'un système illimité de quantics, chacun d'ordre inférieur à n, est actuellement atteinte quand $n=7$, et même quand le système illimité se réduit à un seul quantic, ce qui aussi a lieu pour $n=8$ et pour tous les ordres inférieurs, sauf pour $n=3$, dans lequel cas la limite 4, il est vrai, est atteinte; mais le système doit contenir au moins deux quantics. L'apparence des invariants, dont les degrés sont 14, 18 et 22 (nombres nécessairement pairs), est aussi digne d'observation. On en conclut (et même un seul de ces covariants servirait à établir la même conclusion) que $1-a^7$ paraîtra comme facteur dans la partie invariantive du dénominateur de la fraction génératrice pour tout quantic dont l'ordre est pair et plus grand que 10.

23.

SUR LA FORME BINAIRE DU SEPTIÈME ORDRE.

[*Comptes Rendus,* LXXXVII. (1878), pp. 899—903.]

IL y a une erreur dans la Table pour la fraction réduite sur laquelle j'ai basé mon calcul des covariants irréductibles de la forme binaire du septième ordre. Le terme qui multiplie a^7, au lieu de

$$4x + 4x^5 - x^7 - x^9 + x^{11} - x^{13},$$

doit être écrit $4x + x^3 + 3x^5 - x^9 + x^{11}$, et, conséquemment, le terme complémentaire qui multiplie a^{29}, au lieu d'être

$$4x^{13} + 4x^9 - x^7 - x^5 + x^3 - x,$$

doit être écrit $4x^{13} + x^{11} + 3x^9 - x^5 + x^3$. Mais, de plus, pour ne pas parler d'erreurs de multiplication, le calcul a besoin d'être modifié, par suite d'une circonstance qui s'est présentée ici pour la première fois dans l'application de ma méthode: c'est que l'existence d'un invariant irréductible du degré 20 a été présumée, tandis qu'il y a toute raison de croire qu'il n'existe nul invariant dont le degré soit 20 ou même un multiple quelconque de 10, appartenant à la forme du septième ordre.

Voici la marche à suivre, à cause de cette circonstance. La fraction réduite a pour dénominateur

$$(1 - a^4)(1 - a^6)(1 - a^8)(1 - a^{10})(1 - a^{12})(1 - ax)(1 - ax^3)(1 - ax^5)(1 - ax^7).$$

Je multiplie le numérateur et le dénominateur par

$$(1 + a^6)(1 + ax)(1 + ax^3)(1 + ax^5).$$

Cela me donne une Table dont celle qui suit est la moitié:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1		0		0		0		0		0		0		0		0		0		0		0	
a^1		0		0		0		0		0		0		0		0		0		0		0		0
a^2	0		0		0		0		0		0		0		0		0		0		0		0	
a^3		0		1		1		1		1		1		0		1		0		0		0		0
a^4	0		0		2		1		2		1		0		1		0		0		0		0	
a^5		1		2		2		2		2		0		0		0		−1		0		−1		0
a^6	0		3		2		3		3		0		2		−1		−1		0		0		0	
a^7		3		2		4		4		0		1		0		−2		0		−1		0		1
a^8	2		3		4		6		1		3		−1		−2		0		−1		0		0	
a^9		3		5		7		1		4		0		−2		−1		−2		0		1		0
a^{10}	−1		5		8		6		4		1		−4		0		−3		−1		0		0	
a^{11}		5		8		8		8		4		−4		−1		−5		−1		0		0		0
a^{12}	4		9		9		12		4		−1		−3		−5		−6		0		−1		1	
a^{13}		9		8		11		5		−2		−4		−8		−10		−3		−1		1		0
a^{14}	4		9		11		10		−3		−4		−9		−11		−7		−2		0		3	
a^{15}		8		10		14		1		0		−10		−11		−8		−2		0		4		2
a^{16}	5		11		13		9		−2		−5		−18		−8		−8		−1		3		3	
a^{17}		9		13		12		2		−3		−18		−13		−13		−5		3		3		−1
a^{18}	7		11		11		8		−4		−16		−19		−13		−15		3		2		5	
a^{19}		12		11		11		−1		−12		−18		−18		−18		−1		3		4		4
a^{20}	7		9		10		6		−14		−17		−21		−19		−9		3		5		9	
a^{21}		9		9		11		−9		−12		−23		−24		−11		−1		4		8		4
a^{22}	5		8		10		−1		−12		−17		−28		−12		−9		6		10		8	

Pour la compléter, on n'a qu'à se rappeler que, pour chaque terme $ka^\alpha x^\lambda$ dans la moitié donnée, il faut suppléer un terme $ka^\beta x^\mu$ dans la partie supprimée, où $\alpha + \beta = 45$, $\lambda + \mu = 23$; ainsi, toutes les colonnes de chiffres dans la partie donnée se répéteront en sens inverse, par rapport en même temps à la direction verticale et à la direction horizontale, dans la partie supprimée. Je suppose ce numérateur multiplié par

$$1 + a^{10} + a^{20} + \ldots$$

à l'infini, et le facteur $1 - a^{10}$ chassé du dénominateur, qui ne contiendra alors que les facteurs

$$1 - a^4,\ 1 - a^{12},\ 1 - a^8,\ 1 - a^{12},\ 1 - a^2x^2,\ 1 - a^2x^6,\ 1 - a^2x^{10},\ 1 - ax^7,$$

dont chacun représente par ses indices le degré et l'ordre d'un covariant irréductible; c'est-à-dire, au lieu de multiplier le numérateur et le dénominateur par $1 + a^{10}$, je divise chacun par $1 - a^{10}$.

Alors j'opère par tamisage successivement sur les séries qui multiplient les puissances successives de x dans le numérateur, ce qui, nonobstant le nombre infini des termes dans ces séries, est très-facile à faire, à cause de la récurrence constante des mêmes chiffres. En combinant avec les restes du tamisage ainsi opéré les invariants et les covariants représentés par les facteurs du dénominateur, j'obtiens la Table suivante, où l'on remarquera que nul invariant du degré 20 ne figure:

Table des 124 *covariants irréductibles de la forme binaire du septième ordre.*

Degré dans les coefficients	Ordre dans les variables													
	0	1	2	3	4	5	6	7	8	9	10	11	14	15
1...........								1						
2...........			1				1				1			
3...........				1		1		1		1		1		1
4...........	1				2		1		2		1		1	
5...........		1		2		2		2		2				
6...........			3		2		2		2					
7...........		3		2		4		2						
8...........	3		3		3		3							
9...........		3		5		2								
10...........			4		3									
11...........		5		3										
12...........	6		6											
13...........		7												
14...........	4													
15...........		3												
16...........	2													
17...........		2												
18...........	9													
22...........	1													

Ce qui est absolument démontré, c'est qu'il existe les 124 covariants irréductibles indiqués par cette table. Ce qui est assujetti au doute métaphysique dont j'ai fréquemment parlé, c'est la possibilité de l'existence d'autres irréductibles en dehors de la Table. Si le cas est ainsi, il sera en contradiction avec le *postulatum* qu'il ne faut jamais supposer l'existence de plus de rapports syzygétiques entre les irréductibles qu'il n'est nécessaire pour satisfaire aux valeurs connues du nombre total des covariants linéairement indépendants pour chaque degré et ordre, ou, ce qui revient à la même chose, que des covariants irréductibles et des syzygies indécomposables ne peuvent pas coexister pour le même ordre et degré. En faisant l'énumération des invariants de tous les degrés jusqu'à 20, on trouvera facilement que, selon ce principe, on n'avait pas le droit d'admettre préalablement l'existence d'un invariant irréductible du degré 20. C'est pour la première fois, dans tous les cas si nombreux que j'ai discutés, que cette difficulté

s'est présentée, c'est-à-dire l'impossibilité de trouver une fraction canonique avec un numérateur fini, équivalente à la fraction réduite. Mais les résultats que j'obtiens ne sont nullement moins certains, à cause de cette difficulté que j'ai trouvé le moyen sûr et commode de vaincre. Les détails du calcul seront donnés dans une prochaine partie de l'*American Journal of Mathematics*.

Je terminerai ici par une observation qui me paraît très-significative: c'est qu'il résulte du calcul qui a été fait que l'effet du tamisage est précisément le même que si l'on avait multiplié le numérateur de la forme réduite par $1+a^{10}$ au lieu de le diviser par $1-a^{10}$, de sorte qu'on aurait pu agir précisément comme si l'invariant irréductible du degré 20 existait; seulement, au bout du compte, on aurait exclu cet invariant de la Table des formes irréductibles.

Quant à ce qui se rapporte au tamisage que j'ai appliqué aux séries simplement infinies, il est bon de se rappeler que l'usage qu'on fait de la fraction génératrice (pour un quantic binaire) mise sous une forme canonique n'est qu'une méthode abrégée, et pour ainsi dire artificielle, pour obtenir le même résultat qu'on pourrait obtenir, mais avec beaucoup plus de difficulté, en opérant directement le tamisage sur la série de nombres, doublement infinie, qu'on obtient en développant cette fraction en série de puissances de a et x, de laquelle série les coefficients représenteront le nombre des covariants linéairement indépendants pour chaque degré et chaque ordre, de zéro jusqu'à l'infini. Cette remarque fait voir aussi que la distinction entre les irréductibles primaires et secondaires ne tient à aucune différence essentielle de nature entre les deux, mais seulement à la méthode qu'on emploie pour les obtenir, et, en variant cette méthode, les irréductibles peuvent changer leur nom de *primaires* en *secondaires*, et *vice versâ*.

24.

ON AN APPLICATION OF THE NEW ATOMIC THEORY TO THE GRAPHICAL REPRESENTATION OF THE INVARIANTS AND COVARIANTS OF BINARY QUANTICS,—WITH THREE APPENDICES.

[*American Journal of Mathematics* I. (1878), pp. 64—125.]

[The figures are given on p. 163.]

BY the *new* Atomic Theory I mean that sublime invention of Kekulé which stands to the *old* in a somewhat similar relation as the Astronomy of Kepler to Ptolemy's, or the System of Nature of Darwin to that of Linnæus;—like the latter it lies outside of the immediate sphere of energetics, basing its laws on pure relations of form, and like the former as perfected by Newton, these laws admit of exact arithmetical definitions.

Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern algebra to a mixed society, mainly composed of physicists, chemists and biologists, interspersed only with a few mathematicians, to which I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for "Grundformen" or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view, as I will now proceed to explain.

To those unacquainted with the laws of atomicity I recommend Dr Frankland's *Lecture Notes for Chemical Students*, vols. 1 and 2, London (Van Voorst), a perfect storehouse of information on the subject arranged in the most handy order and put together and explained with true scientific accuracy and precision. On the algebraical side of the subject my readers may consult Salmon's *Lessons on Higher Algebra*, Clebsch's *Binären Formen*

or Faà de Bruno's treatise more elementary than the former, *Sur les formes binaires* (Turin, 1876). I propose also to run a course of articles on the Invariantive Theory, beginning from the beginning, through the pages of this Journal, from my own particular point of view, which will be found, I hope, considerably to simplify the subject.

Any binary quantic may be denoted by a single letter with a number attached corresponding to its degree, and may therefore be adumbrated by a chemical symbol with corresponding *valence*. Thus hydrogen, chlorine, bromine, or potassium will serve to denote so many distinct binary linear forms; oxygen, zinc, magnesium, &c., binary quadrics; boron, gold, thallium, cubics; carbon, lead, silicon, tin, quartics; nitrogen, phosphorus, arsenic, antimony, &c., quintics; sulphur, iron, cobalt, nickel, &c., sextics. The sixth appears to be the highest degree of valency at present recognizable in natural substances.

The factors of any algebraical form may be regarded as in some sense the analogues of the rays of atomicity in the equivalent chemical atom—these rays being what Dr Frankland, according to his nomenclature, would have to designate as free bonds; such rays between two consecutive atoms in a molecule are conceived as blending in some manner so as to represent some unknown kind of special relation existing between them; they may then with propriety be called bonds or lines of connexion.

An invariant of a form or system of algebraical forms must thus represent a saturated system of atoms in which the rays of all the atoms are connected into bonds. Thus, for example, O_2 (oxygen combined with itself) will represent a quadratic invariant of a quadric. Its graph is seen in Fig. 1 (a). Potash, a combination of potassium, oxygen and hydrogen, having for its graph that of Fig. 2, will represent the invariant to a system of one quadratic and two linear forms which is linear in each set of coefficients. This is in fact the *Connective* between the given quadratic and another obtained by taking the product of the two linear forms. Phosphorus and arsenic are quinquivalent, but form "tetratomic molecules." An isolated element of phosphorus may possibly, therefore, be represented by the graph of Fig. 3, which will correspond, if the figure is indecomposable (which requires examination to determine), to the quart-invariant of a quintic, and the same for arsenic. So too the graph to nitric anhydride (Fig. 4) may possibly serve to express the resultant of a binary quadric and quintic, or this blended with any other invariant of the system included under the same type [10: 5, 2; 2, 5]*. And in general, the Jacobian to any two quantics will be completely expressed by their two corresponding atoms connected by a pair of bonds. Nitric acid has for its graph that of Fig. 5. This will

* 10 is the weight; 5, 2 the degree and order in the coefficients of the quintic; 2, 5 the degree and order in the coefficients of the quadric. See p. [151].

correspond to an invariant of a quintic, quadric and linear form of the first order in the coefficients of each extreme and of the third order in those of the middle form. Such an invariant as is well known (by virtue of a general principle about to be stated), is, in substance, the same thing as a lineo-cubic linear covariant of a quintic and quadric. The general arithmetical rule (also hereafter to be set forth) for determining the number of asyzygetic derivatives of a given type, enables us to see that such a covariant exists and is monadelphic. It may readily be obtained by making the given quintic (after substituting $\frac{d}{dy}$ and $-\frac{d}{dx}$ for x and y respectively) operate on the cube of the given quadratic.

The general principle above referred to, which is extremely easily proved from the partial differential equation (but which I believe I was the first to enunciate), is that every covariant of one quantic or several simultaneous quantics may be transformed into an invariant of the same quantic or set of quantics enlarged by the addition thereto of one additional linear form; the degree in the variables becoming replaced by the order in the new set of coefficients, and the orders in the original sets of coefficients remaining unchanged.

Thus, covariants might altogether be dispensed with and invariants alone made the object of study. But algebraists have found and will continue to find it more convenient to dispense with the additional linear form and to retain in use covariants as well as invariants. With me, covariants are to be regarded as simple emanations, so to say, from differentiants which are functions of the coefficients alone, and of which invariants are merely a particular species satisfying a certain condition of maximum; this is why the properties of invariants can with difficulty be made out so long as they are studied alone; it was only by contemplating the whole group of differentiants simultaneously, that I was enabled, after a suspense of more than a quarter of a century, to set on an irrefragable basis Professor Cayley's fundamental arithmetical theorem for calculating the number of asyzygetic invariants and covariants to a given quantic, and also the more general theorem which I have shown applies to a system of quantics*.

I will here give this rule, as it may be useful to us in the sequel. First, for a single quantic.—Let i be its degree, j the order of any covariant, w its weight (that is, the weight of its root-differentiant). Then we may call its type $[w: i, j]$. Now let us, in general, employ $(m: i, j)$ to signify *the number of ways* in which m can be made up with j parts of which each is either 0, 1, 2, 3, &c. up to i, and let us use the symbol $\Delta\,(m: i, j)$ to denote $(m: i, j) - \{(m-1): i, j\}$; then $\Delta\,(w: i, j)$ is the number of arbitrary

* The demonstration is given in a paper inserted in the *Philosophical Magazine* for March of this year [p. 117, above].

numerical parameters in the most general covariant or invariant answering to the type $[w: i, j]$. It is a known theorem in partitions of numbers that $(m: i, j) = (m: j, i)$, from which it follows that the number of arbitrary parameters remains unaltered when the degree of the primitive and the order of the derivative are interchanged. It is sometimes more convenient to use the degree of the derivative in lieu of the weight to express its type; let then ϵ be the degree, so that $\epsilon = ij - 2w$; then I shall employ, when desirable, $[i, j: \epsilon]$ to signify the same thing as $[w: i, j]$. If there be several quantics, the type may be expressed in like manner by $[w: i, j; i', j';$ &c.$]$, or by $[i, j; i', j';$ &c.$: \epsilon]$. The rule for finding the number of independent parameters, or the most general covariant or invariant corresponding to either of these types, then becomes as follows. Let $(m: i, j; i', j';$ &c.$)$ denote the number of ways in which m can be made up of j elements each comprised between 0 and i, combined with j' elements each comprised between 0 and i', and so on, and let $\Delta(m: i, j; i', j';$ &c.$)$ denote $(m: i, j; i', j';$ &c.$) - (m-1: i, j; i', j';$ &c.$)$. The number of parameters in question is $\Delta(w: i, j; i', j';$ &c.$)$ and I may observe that the value of Δ remains unaltered when *any one* i is interchanged with the corresponding j, and consequently when any number of i's are interchanged, each respectively with its corresponding j. This theorem of reciprocity for a single quantic is due to M. Hermite. The above statement, applicable to a quantic system, constitutes a notable and important generalization of it. In Note D to Appendix 2, it will be shown that this theorem still further generalized by employing the method of Emanation (virtually the same thing as Regnault's law of substitution) admits of the following simple chemico-algebraical statement. *In an algebraical compound (in an algebraical sense) m n-valent atoms may be replaced by n m-valent ones.* But it should be observed that this replacement involves an entire reconstruction of the representative graph and conveys the notion of respondence or contraposition rather than similarity of type. (See Appendix 2.)

It may be well here (as it will be useful in the sequel) to say a few words more on these differentiants in their relation to covariants. Every covariant may be regarded as arising from either of two differentiants, as from a root. One, the coefficient of the highest power of x, is called a differentiant in x; the other, the coefficient of the highest power of y, a differentiant in y. It is not, for ordinary purposes such as present themselves in this study, requisite to consider more than one of these at a time, and for greater brevity it will be understood that, unless I give notice to the contrary, a *differentiant* will always be understood to mean one in x. I shall also suppose, when dealing with a single binary quantic, that the successive coefficients beginning with the highest power of x, are $a, b, c, \ldots h, k, l$ multiplied successively by the binomial coefficients proper to the degree of the form.

A differentiant, D, may then be defined as a rational integer function of the coefficients of equal weight in all its terms in respect to either variable subject to satisfy the equation

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \dots\right) D = 0.$$

An invariant again may be regarded as a rational integer isobaric function of the coefficients which is a differentiant both in regard to x and y, but it may be best defined as a differentiant (meaning in one of the variables as x) to a given form or form-system whose weight (in respect of the selected variable) is the greatest possible that its order in the coefficients admits of. [The doubleness of the character and the symmetry, direct or skew, of a differentiant satisfying this condition of maximum then become matter of deduction from the definition.] To each covariant corresponds but one differentiant (in a given variable), and *vice versa*, to each differentiant will correspond only one covariant. In fact, D being the differentiant in x, the covariant taking its rise in D is

$$Dx^{\epsilon} + \Omega\,.\,Dx^{\epsilon-1}y + \frac{1}{1\,.\,2}(\Omega\,.)^2 Dx^{\epsilon-2}y^2 + \dots,$$

where $\Omega.$ represents the operator,

$$\left(l\frac{d}{dk} + 2k\frac{d}{dh} + 3h\frac{d}{dg} + \dots\right)$$

if D belongs to a simple quantic, and

$$\Sigma\left(l\frac{d}{dk} + 2k\frac{d}{dh} + \dots\right)$$

if it belongs to a quantic system, and where ϵ is $ij - 2w$ for a single quantic, and $\Sigma ij - 2w$ for a quantic system, i representing the degree of any one form in the variables, j the order of the differentiant in the corresponding set of coefficients, and w the weight of the differentiant. As ϵ can never become negative, we see that the maximum value of w, when each i and its corresponding j is given, will be $\frac{1}{2}ij$ for one form, and $\frac{1}{2}\Sigma ij$ for a form system. By the weight of any covariant I shall understand the weight of the differentiant in which it may be regarded as originating. Precisely as algebraists find their advantage in using covariants when invariants alone might be made to suffice, chemists find theirs in the use of organic or inorganic compound radicals, as unsaturated forms capable of becoming saturated by the addition of the right number of monad elements to the unsatisfied atoms, that is, those through which a sufficient number of *bonds* do not pass to exhaust their valency. Thus, for example, Hydroxyl H – O – is the linear covariant of the quadratic form oxygen, and the linear form hydrogen; this, combined with the linear form potassium, expresses the invariant potash denoted by H – O – K.

As the free valence of a single atom corresponds to the degree of a single quantic, so the free valence of a molecule formed by an aggregate of atoms will express the degree of the corresponding covariant. Let us understand by the *toti-valence* of a molecule the sum of the absolute valences of the separate atoms of which it is composed. This toti-valence will obviously correspond to the sum, Σij, above mentioned. Since every bond or connecting line in the graph passes through two atoms, this toti-valence must be equal to the free valence of the molecules increased by twice the number of bonds; but Σij is the toti-valence, and ϵ (the degree of the covariant) is the number of unsatisfied bonds, and we have already stated in effect that ϵ increased by twice the weight of the root differentiant (which for brevity we call the weight of the covariant) is equal to Σij; hence the weight of a covariant (meaning that of its root differentiant), represented by any chemicograph, is the number of bonds or connecting lines between the atoms.

Let us consider an invariant or a covariant belonging to a type containing only one numerical parameter, which I shall call a monadelphic form*. Then this is either decomposable into factors or not; in the former case it may be termed composite, in the latter case prime. When prime its graph will also be prime, when composite its graph will be composite in a sense which will be made more clear by one or two examples. Let us take as a first example a graph composed of four triadic atoms of the same name, as in Fig. 6, where each atom, for instance, represents boron and in ordinary chemical symbolism would be denoted by the same letter B, but where for facility of reference I use four different letters to mark the positions of the several atoms. This corresponds to a covariant of a cubic for which the complete type, if we use the weight or number of bonds, is [4 : 3, 4], or, if we use the free valency, is [3, 4 : 4]. Now for a cubic the fundamental types, expressed in terms of the order and degree alone, omitting the constant number 3, which refers to the given degree, are

$$\begin{array}{c} 1\,.\,3 \\ 4\,.\,0 \\ 2\,.\,2 \\ 3\,.\,3\,. \end{array}$$

Consequently, there is but one covariant corresponding to the given graph, and that is the product of the primitive by the covariant whose order and degree are each 3, the well-known skew covariant of $(a, b, c, d \between x, y)^3$ whose root or base is the differentiant $ad^2 - 3abc + 2b^3$.

* The type itself may also be termed a monadelphic type: so I shall speak when necessary of diadelphic, triadelphic, &c. types and designate any forms contained under such types as diadelphic, triadelphic, &c. forms. A family comprising many brothers, or any member of such a family, may each without doing violence to the laws or usage of language be termed *polyadelphic*.

It must be well understood that the bonds are not rigid, but capable of being curved or bent into any desired form. In this case the mode of decomposition is self-evident; for the skew covariant is represented by the triangle of Fig. 7, and we have only to draw out the elastic bond AC into the position ADC and place the atom D anywhere upon it to obtain the given graph. On the contrary the skew covariant itself is indecomposable and its graph ABC is obviously so too. Now let us consider the graph of Fig. 8. If the atoms at the angles are all triadic, there is no free valency, and the figure represents the invariant to a cubic form corresponding to 4.0 in the above table. It will be found, on trial, impossible to decompose it. But now suppose the atoms to be tetradic, the graph will represent a covariant of the fourth order and of the fourth degree to a quartic, each atom having one degree of valency unsatisfied. The fundamental derivatives of a quartic, of which all others are algebraical combinations, are represented in the following table of order and degree

$$\begin{array}{c} 1.4 \\ 2.0 \\ 3.0 \\ 2.4 \\ 3.3. \end{array}$$

The complete covariant answering to the graph will therefore be $\lambda U + \mu V$, where, λ, μ being arbitrary numbers, U is the product of the primitive (1.4) by the cubinvariant 3.0, and V the product of the Hessian 2.4 by the quadrinvariant 2.0. Since, on making either $\lambda = 0$ or $\mu = 0$, the covariant breaks up and in two different ways into factors, we ought to expect that the graph should be capable of two corresponding modes of decomposition, and such we shall easily see is the case. For 1°, the invariant 3.0 may be represented by the graph of Fig. 9. Now imagine the three points E, F, G to come together and blend at D, and at D place a fourth atom. The given graph is thus recovered. Observe that this could not be done for the case of triads (corresponding to a cubic form) because, in the figure last referred to, the valence at each atom A, B, C is quadrivalent. Next, for the decomposition corresponding to the case of $\lambda = 0$ where the covariant breaks up into 2.0 multiplied by 2.4, the decomposition will be more easily followed by considering the graph to be pulled out into the form seen in Fig. 10. We may conceive this as the superposition of two carbon graphs, one in which the carbon atoms are at A and B connected by the *four* bonds AB, ACB, BDA, $ACDB$ denoting the quadrinvariant, and another in which the carbon atoms C, D are connected by the *two* bonds CAD, CBD, leaving two degrees of valence free at each atom and thus representing the quadro-quart-invariant or Hessian of the primitive.

I will now pass to the very interesting case which corresponds to one of the proposed graphs for benzole (or rather for the compound radical obtained by striking off its hydrogen atoms), a sextivalent hexad molecule of carbon—not the one proposed by Kekulé and which I believe still commands the general assent of chemists, but that suggested by Ladenburg* and put by him under the form of a wedge or prism. As, however, the question is one purely of colligation or linkage in the abstract, it is sufficiently described as a hexagon in which the three pairs of opposite angles are joined, or, if we please, as two triangles in which each angle of one is connected with a corresponding angle of the other. In regard of the atomicity theory, all these modes of colligation are identical, and the supposition that there is any real difference between them, or that figures in space are distinguishable from figures in a plane (as I heard suggested might be the case by a high authority at a meeting of the British Association for the Advancement of Science, where I happened to be present), is a departure from the cautious philosophical views embodied in the theory as it came from the hands of its illustrious authors and continued to be maintained by their sober-minded successors and coadjutors, and affords an instructive instance of the tendency of the human mind to the worship, as if of self-subsistent realities, of the symbols of its own creation.

The order (or number of atoms) being 6 and the unexhausted valences (one at each atom) also 6, we must turn to our table of fundamental derivatives to the quartic and shall find that the combination 6.6 is not amongst them, but that it can be obtained, and in only one way, by composition of the combinations therein contained. It is, in fact, the product of the cubic invariant 3.0 by the skew covariant 3.6, which has the very same *root* $a^2d - 3abc + 2b^3$ as the skew covariant to the cubic and accordingly has the same graph, namely a simple triangle. (It may be well to remark here incidentally, that it follows as an immediate consequence from the conditioning partial differential equation, that a root-differentiant to any quantic or system of quantics of given degree or degrees remains such to every other system in which one or more of those degrees is augmented.) On the other hand the cubic invariant has for its graph a triangle in which each line is doubled or looped. I shall show that Ladenburg's graph for the radical to benzole may be obtained by the superposition of these two forms. Let $ABC\gamma\beta\alpha$ represent a sextivalent tetradic hexad (Fig. 11); ABC, with the three loops $A\alpha\gamma C$, $C\gamma\beta B$, $B\beta\alpha A$, will represent a saturated triple atom of carbon, or the cubinvariant of a binary quartic. Again, $\alpha\gamma\beta$ taken alone will represent a sextivalent compound atom, or the fundamental skew covariant of the quartic, and the superposition of the two figures obviously gives the graph as it stands.

Another form of the product of the same two graphs would be a triangle inscribed in another, as in Fig. 12. Here $\alpha\beta\gamma$, as before, is the sextivalent

* *Berichte der deutschen chemischen Gesellschaft*, 1869, 141. I am indebted for this reference to my able colleague, Professor Ira Remsen.

molecule and ABC with the additional bonds $A\beta C$, $B\gamma A$, $C\alpha B$, the saturated one.

A simple hexagon of triadic atoms (Fig. 13) being sextivalent will serve to represent a derivative from a cubic of the sixth order and sixth degree. Such a covariant, in its most general form, will contain two parameters and be represented by $\lambda U^3 + \mu V^2$ where U is the Hessian 2 . 2 and V the skew cube covariant 3 . 3, and it is easy to see that this figure may be decomposed either into 3 bivalent, or 2 trivalent graphs. Thus AB, CD, EF, with the additional bonds $BCDEFA$, $DEFABC$, $FABCDE$, will represent the former; two atom groups such as A, C, E (with the bonds ABC, $AFEDC$, CDE, $CBAFE$, EFA, $EDCBA$) and B, D, F (with the bonds BCD, $BAFED$, DEF, $DCBAF$, FAB, $FEDCB$) the other. The first method of regarding the hexagon as a combination of three dyads may perhaps be admitted to throw some light on what Dr Frankland styles the two distinct molecular weights of sulphur. When two atoms of sulphur, regarded as bivalent, are combined by two loops, we have a representation of an isolated element of it as "a diatomic molecule." When three of these letters, regarded now as submolecules, are combined, or multiplied together into the hexagon, we have a representation of the isolated element as "a hexatomic molecule." More generally, let μ be the number of solutions of the equation in positive integers $2x + 3y = m$, then μ arbitrary parameters will enter into the most general representation of a covariant to a cubic of the order m in the coefficients and the degree m in the variables. Its graph will be a simple polygon of m sides and this will be capable of being decomposed, in μ essentially distinct ways, into elementary graphs consisting either, of binary groups or, ternary groups exclusively or, the two sorts of groups intermixed.

It may be easily shown (see Appendix 3) that every covariant of a binary form multiplied by a suitable power of its primitive, is capable of being represented by a rational integer function of covariants consisting, in addition to the primitive, of covariants exclusively of the second and third orders in the coefficients. I have already given an example of the mode in which a graph may be augmented by an additional atom corresponding to the multiplication of a covariant by the primitive.

The important proposition above referred to (given in Clebsch's *Binären Formen*) amounts then to affirming that any homogeneous graph augmented by a suitable number of atoms of the same, may be decomposed, in one or more ways, into bilooped dyads and single-sided triangles. Such a proposition ought to admit of graphical proof. The theorem has considerable graphical importance because it enables us, in some cases at least, to discriminate the true from the spurious graphs, or as we might say, pseudographs, representing a given type. Thus, it serves to show that Fig. 14 and not Fig. 15 is the

graph to the discriminant of a cubic; for, in accordance with Clebsch's theorem, this discriminant, namely

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

multiplied by a^2 becomes equal to the square of $a^2d - 3abc + 2b^3$, together with four times the cube of $ac - b^2$, and consequently its graph, after combination with two additional points, should be decomposable, at will, into 3 double-looped lines, or into 2 single-lined triangles, which is the case with Fig. 14, inasmuch as its combination with two points gives rise to a simple hexagon, but not with Fig. 15.

If we call the apices of the two figures, 14, 15, a, b, c, d, the true graph (on substituting negative signs for bonds and prefixing a sign of summation) reads as

$$\Sigma\,(a-b)^2(c-d)^2(a-c)(b-d),$$

which is the cubinvariant of the quartic whose roots are a, b, c, d, so that a graph to an invariant of the type [3, 4 : 0] gives the algebraical expression in terms of the roots of an invariant of the reciprocal type [4, 3 : 0]. On the other hand, the pseudograph treated in the same way reads as

$$\Sigma\,(a-b)(b-c)(c-d)(d-a)(a-c)(b-d),$$

the value of which is zero; a similar remark may probably be found to be true of reciprocal graphs of invariants in general. This is abundantly confirmed by subsequent investigation; see remarks at end of Appendix 1.

So again, if we take the graph of Fig. 42, which represents an invariant to the type [3, 2; 1, 2 : 0], it reads off into

$$\Sigma\,(B_1-B_2)^2(B_1-H_1)(B_2-H_2),$$

belonging to the reciprocal type [2, 3; 2, 1 : 0], and the Σ is in fact the discriminant of one binary quadratic multiplied by the connective between it and another.

So if we take the graph represented in (a), Fig. 45,

$$\Sigma\,(O_1-O_2)(O_1-H)(O_2-K)$$

will represent an invariant to the type [2, 2; 1, 1; 1, 1 : 0]. If, however, we were to substitute H_1, H_2 in lieu of H and K, so as to form the hydroxyl graph of Fig. 45 (b), it would not be true that $\Sigma\,(O_1-O_2)(O_1-H_1)(O_2-H_2)$ would represent an invariant to the type [2, 2; 2, 1 : 0]; on the contrary it would be zero. But hydroxyl is *not an invariant*, for to the combination of a quadratic and a linear form there appertains no invariant of the second degree in the coefficients of each of them. This may be easily proved by the rule I have given at the commencement of this paper. I have gone through this calculation for the benefit of those new to the subject and to show how the arithmetical "rule of multiplicity" is to be applied. Had I been writing

solely for algebraists it would have been unnecessary to prove so familiar a fact. We have here

$$i=2,\quad j=2,\quad i'=1,\quad j'=2,\quad w=\frac{ij+i'j'}{2}=3.$$

To find $(w: i, j; i', j')$ we have to count the combinations

$$\begin{array}{c|c} 2.1 & 0.0 \\ 2.0 & 0.1 \\ 1.1 & 0.1 \\ 1.0 & 1.1; \end{array}$$

the number of these is 4. Again to find $(w-1: i, j; i'; j')$ we have to count the combinations

$$\begin{array}{c|c} 2.0 & 0.0 \\ 1.1 & 0.0 \\ 1.0 & 0.1 \\ 0.0 & 1.1, \end{array}$$

of which the number is also 4. Hence

$$\Delta(3: 2, 2; 1, 2) = 4 - 4 = 0.$$

So that hydroxyl, being of the type [3: 2, 2; 1, 2], cannot be an invariant.

So far then the supposed law is safe; but I think I see other difficulties in the way of its application to heteronymous types, so that if it shall be capable of being made universally applicable, other parts of the graphical theory, as it has been laid down, will possibly require reconsideration. What I advance is to be regarded not as dogmatic but as tentative and open to correction.

It is obvious that not every chemico-graph, potential or even actual, corresponds to an invariantive derivative. Of this I have already given examples. Were the case otherwise we should have surprised the secret of nature, for, as we know how to obtain all possible fundamental forms to binary quantics, we should know *à priori* all possible compound radicals. As a matter of fact the cases of algebraical invariance in nature seem to be rare and rather the exception than the rule. Thus while muriatic acid (H – Cl), is an invariant, self-saturating hydrogen (H – H), is a non-invariant, there being a linear invariant to two linear forms but not to a single one. In like manner ozone (Fig. 16) is also non-invariantive, there being no cubic invariant to a quadratic form. But there is an essential difference to be observed between the two cases. A graph consisting of a single or an odd number of bonds between two atoms of the same kind can *never*, for any species of such atoms, be invariantive, because no covariant of the second order in the coefficients can have an *odd* weight. If that were possible, then, by the theorem

of reciprocity, a quadratic function could have an invariant or covariant of an odd weight, which is, of course, not true. Whereas a triangle of n-ads, although it does not picture an invariant when $n = 2$, does do so when $n = 3$ or any higher number. When an homonymous graph is given in weight (the number of bonds) and in order (the number of atoms) two of the elements of its type $(w: i, j)$ say w, j are known and the third i is left indeterminate. For all values of i which make $\Delta(w: i, j)$ greater than zero, there will be one or a plurality of such graphs according to the value of Δ. If no value of i makes Δ greater than zero, there will be no such graph possible, but it is not necessary, to ascertain this, to make an indefinite number of trials, for it is obvious that for all values of i equal to or greater than w, Δ has the same value, namely $\Delta(w: \infty, j)$, since the condition that a number w shall not be made up of numbers greater than i, when i is equal to w, becomes nugatory.

It will be instructive to consider the case of $w = 5, j = 3$, and consequently the free valence $\epsilon = 3i - 10$; this implies that i must be at least equal to 4. But if we take $i = 4$, $\epsilon = 2$, as there is no covariant to a binary quartic whose order is 3 and degree 2, we may be sure that $\Delta(5: 4, 2) = 0$. Hence we have only to consider the case of $i = w = 5$, $\epsilon = 5$. $\Delta(5: 5, 3)$ is the number of covariants of the fifth order and fifth degree to a cubic of which there is but *one*, formed by the multiplication together of the Hessian and skew-covariant. If now we proceed to form the graph corresponding to the type $[5: 5, 3]$, we have the choice of two figures, 17, 18. In the former figure there are three degrees of vacancy from saturation at A and one at each of the points B, C. In the latter, one at A and two at each of the points B and C. The graph, we must recollect, is to correspond to a cubic covariant of the fifth degree to a fifthic which is unique and indecomposable. This enables us to fix upon the true representation. It cannot be the graph of Fig. 17, for that may be considered as generated by the combination of one isolated nitrogen atom with two atoms of nitrogen, B, C, connected by five bonds; two of these being subsequently welded together and bent out into the angle having A at its vertex. [The hypothetical nitrogen pair exists in chemistry but not as an algebraical invariant.] Hence the true figure can but be that given in Fig. 18, where the free valence is separated into the parcels 2, 1, 2, and not as in Fig. 17 into the parcels 1, 3, 1. And it should be observed that, for all higher values of i beyond 5, this will continue to be the one and only true graph to the corresponding covariant. It thus appears that every given homogeneous graph has an intrinsic character of capability or incapability of respondence to algebraical in- or co-variance, irrespective of the particular valence assigned to its atoms, and it is natural to suppose that there must be some immediate intrinsic criterion for determining this character, so as to dispense with the necessity of any algebraical considerations to establish it; but if such criterion exists, I have not yet been able to make

out what it is*. In common with this view we may consider the theory of reciprocity of algebraical derived forms. It has already been stated that to every m-ad of n-ad atoms having a given number of bonds corresponds an n-ad of m-ad atoms with the same number of bonds. As for example, to a quasi carbon-ad (so to say) of sulphur will correspond a quasi sulphur-ad of carbon, the number of bonds and consequently the amount of free atomicity remaining the same in the two molecules. This suggests the possibility of there being some mode of passing from a graph to its reciprocal (this reciprocity being seemingly of quite a different kind from that which connects correlated girders or frameworks in graphical statics). I offer the subjoined instance of such transformation tentatively and with a view to stimulate inquiry, rather than as possessing any assurance of the validity of the process employed.

Suppose the case of $i = 4$, $j = 2$, $w = 4$; the one and only corresponding graph will be a system of 4 bonds connecting two atoms A, B. If now we take a pair of these bonds, stretch them out, weld them together and form a knot between them at C, and in like manner convert the other pair of bonds into a pair knotted at D, we shall have a graph consisting of a simple quadrilateral which will correspond to the case of $i = 2$, $j = 4$.

Again, suppose $i = 6$, $j = 4$, $w = 12$. We may consider either of the graphs quasi in Figures 19, 20. In the first of these figures we may take four bonds connecting respectively AC, CB, AD, DB, stretch and weld them together and form a knot between them at a new point E which will then be attached by four bonds to the atom $ABCD$. I mean that we may stretch out AC, CB, to meet in E (Fig. 21) and have EC common, and in like manner stretch out AD, DB to E and have ED common and then knot together the four bonds of the strings at E. In like manner we may form another knot F with bonds through AB, BC, AD, DC, and shall thus obtain the reciprocal graph of Fig. 21, where now $i = 4$, $j = 6$, $w = 12$. So again it will be found that we may distort Fig. 20 (if I can trust to my recollection of the result of previous work) in two different ways into a reciprocal graph.

At the risk of provoking the ire or ridicule of my chemical friends and the chemical public, I will venture to throw out a few remarks on the substructure, so to say, of the accepted theory of atomicity and to offer a suggestion as to a possible mode of getting rid of some imperfections under which it appears at present to labour. First there is the inconsistency of admitting the isolated existence of single atoms of mercury, cadmium and zinc, as monads with their bonds or tails absorbed or suppressed or else swinging loose and unsatisfied in direct opposition (as it seems to me) to the fundamental postulate of the theory. Next, one cannot get over a somewhat uncomfortable feeling at the representation of isolated oxygen in the state

* The law of reciprocity, however, exemplified above can obviously be made to supply the criterion in question.

of ozone by a triangular graph, which, although conceivable, is supported by no analogous case unless that of baric peroxide, or any similar graph, be regarded as such. Thirdly, there is the vague and unsatisfactory (not to say unthinkable) explanation of the variability of the valence of a given atom by what Dr Frankland calls "the very simple and obvious assumption that one or more pairs of bonds belonging to the atom of an element can unite and having saturated each other become, as it were, latent."

Now these stumbling-blocks to the acceptance of the theory may be removed by one simple, clear and unifying hypothesis, which will in no wise interfere with any actually existing chemical constructions. It is this: leaving undisturbed the univalent atoms, let every other n-valent atom be regarded as constituted of an n-ad of *trivalent* atomicules arranged along the apices of a polygon of n sides. Thus, sextivalent, quinquivalent and quadrivalent atoms in their state of maximum valence will be represented by Figures 22, 23, 24, where the letters denote *trivalent atomicules*. When the valence is reduced by two we need only conceive any one of the side loops doubled or a new loop as formed by the coalescence of a pair of free bonds or tails, and when in the Figures 22 and 23 the valence is reduced by 4, we may in like manner either suppose existing loops doubled, or fresh ones inserted, or both changes to go on simultanéously, by the coalescence of two pairs of tails. We have thus a conceivable and conformable-to-analogy method of accounting for the variability in question. So likewise, a trivalent atom with maximum state of valence will be represented by Fig. 25, and when univalent by Fig. 26. Again, an isolated zinc element will have for its graph Fig. 1 (b), the two letters Z signifying the zinc atomicules, and so in like manner isolated cadmium and mercury may be represented. On the other hand O_2, isolated oxygen in its ordinary state, will be represented by the graph of Fig. 27, whilst ozone will have for its representative graph the well known Kekuléan hexad (which, in its importance to chemistry, would seem to vie with Pascal's mystic hexagons to geometry) represented in Fig. 28, where as in Fig. 27, each letter O represents an atomicule of oxygen. So an isolated element of carbon would be represented by the graph of Fig. 29.

This hypothesis of atomicules, if unobjectionable on other grounds, would not be open to the charge of having any tendency to disturb or complicate the existing graphology; for we should still be at perfect liberty to substitute for the graphs (a) of Figures 30, 31, 32 the abridged notation (b), and should naturally do so when considering the relations of atoms to each other. The beautiful theory of atomicity has its home in the attractive but somewhat misty border land lying between fancy and reality and cannot, I think, suffer from any not absolutely irrational guess which may assist the chemical enquirer to rise to a higher level of contemplation of the possibilities of his subject. I have therefore ventured to make the above suggestion.

Chemical graphs, at all events, for the present are to be regarded as mere translations into geometrical forms of trains of priorities and sequences having their proper *habitat* in the sphere of order and existing quite outside the world of space. Were it otherwise, we might indulge in some speculations as to the directions of the lines of emission or influence or radiation or whatever else the bonds might then be supposed to represent as dependent on the manner of the atoms entering into combination to form chemical substances. Such not being the case, what follows is to be considered as having relation to mere *algebraical* atoms, or atomicules (quantics) and their bonds which may be regarded as represented by the linear factors of such quantics.

Let us consider a symmetrical trivalent atomicule whose three bonds or rays make angles of 120° with each other. Calling τ, τ', τ'', the tangents of the angles which the axis of y makes with its rays, we have

$$\tau' = \frac{\tau + \sqrt{(3)}}{1 - \sqrt{(3)}\,\tau}, \qquad \tau'' = \frac{\tau - \sqrt{(3)}}{1 + \sqrt{(3)}\,\tau},$$

so that its equation will be easily found to be

$$(1 - 3\tau^2)\,x^3 + (9\tau - 3\tau^3)\,x^2y + (9\tau^2 - 3)\,xy^2 + (\tau^3 - 3\tau)\,y^3 = 0,$$

which may be identified with the standard form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$$

by writing $\qquad a = 1 - 3\tau^2 = -c, \qquad b = 3\tau - \tau^3 = -d.$

Suppose the three atomicules to become condensed into a single atom after the manner of the graph of Fig. 25. The combination will be represented by the cubic covariant (see Tables des Invariants et Covariants, Table V, annexed to Faà de Bruno's *Théorie des Formes Binaires*)

$$(a^2d - 3abc + 2b^3)\,x^3 + (3abd - bac^2 - 3b^2c)\,x^2y$$
$$+ (3bc^3 + 6b^2d - 3acd)\,xy^2 + (3bcd - ad^2 + 2c^3)\,y^3,$$

which, for the present case, becomes

$$2\,(1 + \tau^2)^3\,[(3\tau - \tau^3)\,x^3 + (9\tau^2 - 3)\,x^2y + (3\tau^3 - 9\tau)\,xy^2 + (1 - 3\tau^2)\,y^3].$$

Hence the new ray-directions will have for their equation

$$-dx^3 + 3cx^2y - 3bxy^2 + ay^3 = 0,$$

or the pencil of the atom will be identical with that of each of the separate atomicules, but accompanied with a rotation (whatever that may mean) of the whole pencil of rays through a right angle in its own plane. Again, suppose that only two atomicules are brought into connexion as in (a) of Fig. 30. The quadricovariant which expresses the atom (Faà de Bruno *ante*) is

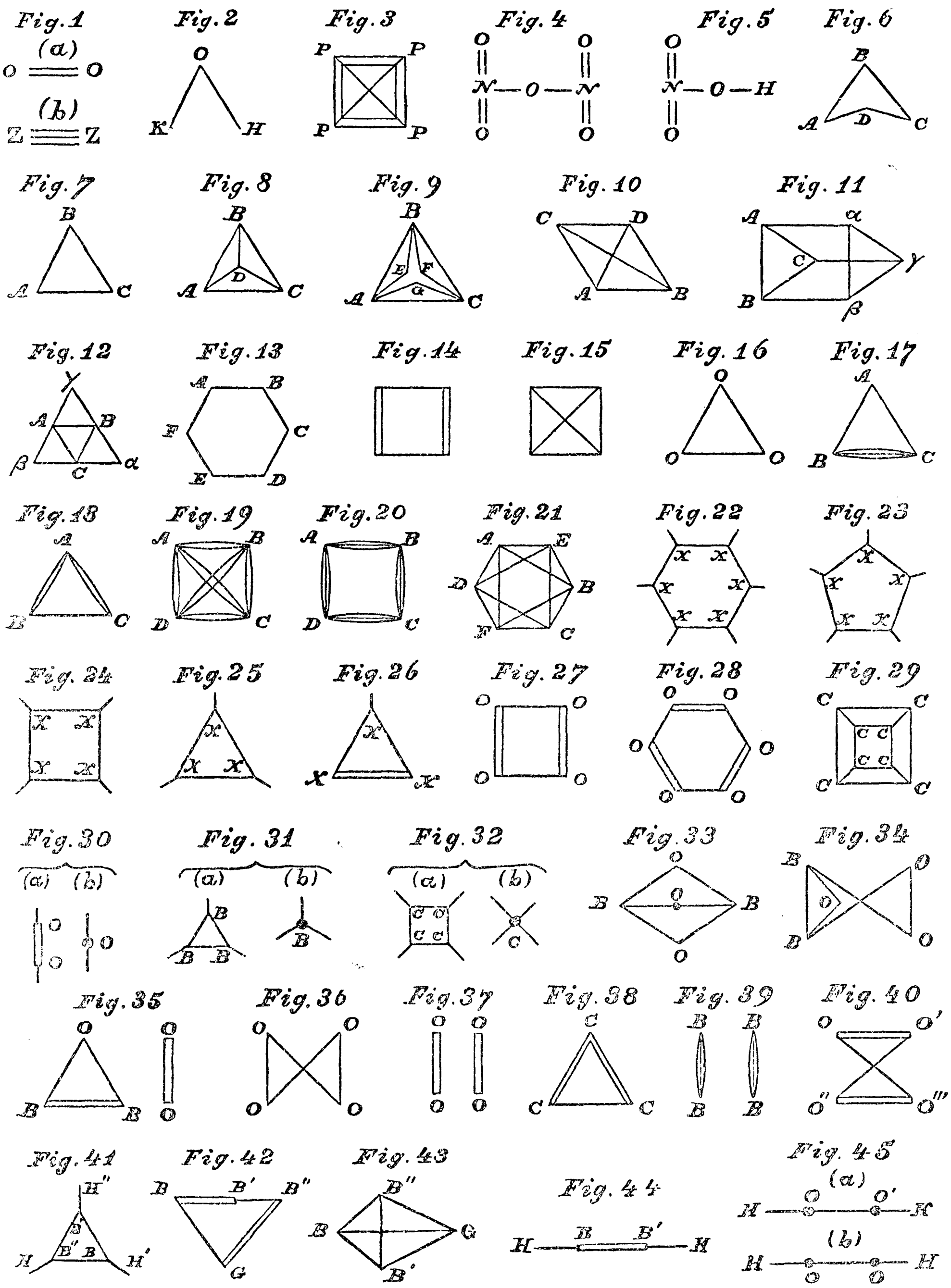
$$(ac - b^2)\,x^2 + (ad - bc)\,xy + (bd - c^2)$$

which here becomes $\qquad -(1 + \tau^2)^3\,(x^2 + y^2).$

Hence the ray-directions will be given by the equation

$$y^2 + x^2 = 0, \qquad y = \pm\, x\sqrt{(-1)},$$

which we may, if we please, according to the usual convention concerning the



square root of minus unity, explain by supposing that the original rays are situated in planes perpendicular to the joining line XX, and that these are

replaced by two rays lying in opposite directions along the line XX, where the atomicules are condensed into one atom. But it would be idle to pursue this speculation further.

The most remarkable point in the theory which I have endeavoured to unfold in the preceding pages is the relation between it and that of reciprocal types.

We have seen that the graph to an invariant of one type read off as it stands (each bond being construed as the sign *minus*) with the sign Σ prefixed expresses an invariant of the reciprocal type.

This rule may be extended from homogeneous to heterogeneous graphs, provided only that the reciprocity be *total*, by which I mean that every i and every j in the type $[i, j; i', j'; i'', j''\ldots: 0]$ are interchanged. It may be observed, in passing, that in the case of types to which resultants belong, the type is identical in form with its total reciprocal. As, for example, boric anhydride (consisting of two of boron and three of oxygen) is of the type [3, 2; 2, 3: 0].

On referring to "System of Cubic and Quadratic," Salmon's Lessons, third edition, p. 179, it will be seen that besides the resultant there is another invariant represented in Dr Salmon's notation by "$\Delta(0, 2) \times I(2, 1)$"; a linear combination of these two with arbitrary multipliers will express the most general form belonging to the type in question.

From the property of these types being their own complete reciprocals, it follows that a complete set of independent graphs of any such type will represent the constitution of a complete set of independent forms belonging to the type. Thus, in the case suggested by boric anhydride we have the two independent graphs of Figures 33, 34. Hence the complete representation of the invariants appertaining to the self-reciprocal diadelphic type [3, 2; 2, 3: 0] is $\lambda U + \mu V$, where U is the resultant

$$(a-\alpha)(a-\beta)(a-\gamma)(b-\alpha)(b-\beta)(b-\gamma)$$

and V is
$$\Sigma(a-\gamma)(a-\beta)(b-\alpha)(b-\gamma)(b-a)(\beta-\alpha).$$

U is derived from the graph of Fig. 33 by replacing the several O's by α, β, γ, and the B's by a, b, and V in like manner from the graph of Fig. 34. This latter graph is replaceable by the disjoined graph of Fig. 35, to which, by the rule for combination of graphs, it is easily seen to be equivalent.

Hence, instead of $\lambda U + \mu V$ we may write $\lambda V + \mu V'$ where

$$V' = \Sigma(\alpha-\beta)^2(a-b)^2(a-\gamma)(b-\gamma);$$

a, b of course will be understood to be the roots of a general quadric and α, β, γ of a general cubic. A very good similar instance of this kind of equivalence is afforded by the quadrinvariant of a quartic whose type is [4, 2: 0]. The reciprocal of this, namely [2, 4: 0], may be represented, either by the connected graph of Fig. 36, or by the disjoined one of Fig. 37,

and accordingly the noted quadrinvariant $ae - 4bd + 3c^2$ may be expressed (to a numerical factor près) either by the symmetrical function

$$\Sigma (a-c)(a-d)(b-c)(b-d)$$

corresponding to the first, or by $\Sigma (a-b)^2 (c-d)^2$ corresponding to the second graph. Again, let us consider the contrary types [4, 3 : 0], [3, 4 : 0]. The former has for its graph Fig. 38, and admits of no other representation. This gives $\Sigma (\alpha-\beta)^2 (\beta-\gamma)^2 (\gamma-\delta)^2$ for the discriminant of the cubic which belongs to the contrary type. The latter may be figured chemically by the graph (consisting of two molecules of boron) of Fig. 39, or by the equivalent Fig. 27 (capable of being derived from it by the mechanical rule for conversion of graphs). These two latter, algebraically speaking, will be pseudographs, because $\Sigma (\alpha-\beta)^3 (\gamma-\delta)^3$ and $\Sigma (\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\alpha)(\alpha-\gamma)(\beta-\delta)$ are each zero. The graph of Fig. 27 may be mechanically converted, in the manner shown in the preceding case, into the graph of Fig. 40; but the type of the colligation remains unaltered by this conversion and whichever of the two we employ, we obtain $\Sigma (\alpha-\beta)^2 (\gamma-\delta)^2 (\alpha-\gamma)(\beta-\delta)$ as the representation in terms of the roots, of the cubic invariant to the quartic, namely to a numerical factor près $ace - b^2e - ad^2 + 2bcd - c^3$.

Thus we see that the graphical method suggested by the theory of atomicity is a real instrument not merely for the representation but also for the calculation and comparison of algebraical results. The important bearing upon it of the principle of contrary or reciprocal graphs, renders it desirable that I should put the algebraical theory or law of reciprocity, in its most complete form, before my readers; it will form the subject of Appendix 2.

I might have noticed explicitly at the commencement of this paper, instead of tacitly assuming it as I have done, that the chemical fact of a compound molecule playing the part of an atom with a valence equal to the free valence of the radical, is the precise homologue to the algebraical fact that every invariant or covariant of a covariant, or set of covariants, to a quantic, or system of quantics, is itself an invariant or covariant to such quantic, or system of quantics; and again that Regnault's chemical principle of substitution and the algebraical one of emanation* are identical; and again, the modern notion of two semi-molecules, simple or compound, combining or uniting to form a chemical substance is tantamount to the construction of an invariant, the connective (or in Professor Gordan's language, the final "Ueberschiebung") of a quantic, or of the derivee of a quantic or a set of quantics,

* By which I mean in this place the operation upon an invariant or covariant of the symbol $(a'\delta_a + b'\delta_b + \ldots)$ performed any number of times in succession; a, b, for instance, may refer to Hydrogen $(ax+by)$ and a', b' to Chlorine $(a'x+b'y)$, and then the emanantive operator, according to a notation used, if I mistake not, by Professor Clerk Maxwell in his theory of poles, might be denoted by $Cl\delta_H$.

with itself. So again, it will hereafter be seen* that Hermite's law of reciprocity applied to quantic systems and stated in its widest terms, amounts to affirming in chemical language that in any compound an arbitrarily selected group of m n-adic atoms may be replaced by a group of n m-adic atoms, but how far this law of replacement has objective validity in the chemical sphere, I am not able to say.

Attention might also have been called to the fact that every chemicograph may, for anything that has been shown to the contrary, and probably in all cases does admit of algebraical interpretation, provided that each given atom however often repeated in a graph counts as a distinct quantic with its own distinct set of coefficients. I do not know whether chemists are of opinion that every chemico-graph exists or is capable of existence in nature; if this is not the case, the condition of the possibility of such existence (should it be discovered) must admit of being stated in mathematical terms. The condition for its existence in algebra may be gathered from what precedes, to be certainly for monadelphic types and probably in all cases, as follows, namely: *if the difference between every two letters of an algebraically existent graph be raised to the power whose index is the number of bonds connecting them, the permutation sum of the product of those powers must not vanish.* Finally, an irreducible covariant is the homologue of a compound radical. Thus we see that chemistry is the counterpart of a province of algebra as probably the whole universe of fact is, or must be, of the universe of thought.

APPENDIX 1.

Remarks on Differentiants Expressed in Terms of the Differences of the Roots of their Parent Quantics.

Since the preceding matter was written, in dwelling upon the law of reciprocal graphs, I came to what appeared to be a formidable difficulty in the way of its reception, a very lion in my path, so formidable that, for a time, I thought that it would be necessary, either to abandon this law, or else to admit the unwelcome conclusion that not every type of invariant was susceptible of graphical representation.

But further consideration has shown me that this apprehension was

* In Note D to Appendix 2. The proposition stated in the text results from the joint effect of the law of substitution or emanation combined with Hermite's law extended to quantic systems.

entirely groundless owing to an algebraical fact on which I had not previously reflected, but which this difficulty forced upon my notice. The difficulty in question arose out of the expressions given by M. Hermite and le père Joubert respectively for the skew invariants of the binary quintic and sextic. I shall first address myself to the consideration of the former. Following Dr Salmon's notation (Lessons, Third Edition, p. 230), let α, β, γ, δ, ϵ be the roots of a quintic, and let

$$F = (\alpha-\beta)(\alpha-\epsilon)(\delta-\gamma) + (\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)$$
$$G = (\alpha-\beta)(\alpha-\gamma)(\epsilon-\delta) + (\alpha-\delta)(\alpha-\epsilon)(\beta-\gamma)$$
$$H = (\alpha-\beta)(\alpha-\delta)(\epsilon-\gamma) + (\alpha-\gamma)(\alpha-\epsilon)(\delta-\beta).$$

Then it will be found as will presently be shown that the product $F.G.H$ is a symmetrical function of the four roots β, γ, δ, ϵ, consequently, on forming four other similar products symmetrical in respect to $\alpha, \gamma, \delta, \epsilon$: $\alpha, \beta, \delta, \epsilon$: $\alpha, \beta, \gamma, \epsilon$: $\alpha, \beta, \gamma, \delta$ respectively, the product of these five products will be symmetrical in respect to α, β, γ, δ, ϵ and being a function of the differences of the roots of order 18 and of weight 45, that is of the type [45 : 5, 18], must be (paying no attention to a mere numerical factor) I, the skew invariant to the quintic.

Now consider the type reciprocal to this [45 : 18, 5] (monadelphic like the preceding), and expressing the invariant of the fifth order to an octodecadic. Suppose this has a graph. It will follow from the law of reciprocal graphs that I may be expressed under the form

$$\Sigma\,(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c(\alpha-\epsilon)^d(\beta-\gamma)^e(\beta-\delta)^f(\beta-\epsilon)^g(\gamma-\delta)^h(\gamma-\epsilon)^k(\delta-\epsilon)^l,$$

where $a+b+c+\ldots = 45$ and each letter α, β, γ, δ, ϵ is conditioned to appear the same number of times, which at first might seem contradictory to what has just been established, but in reality is in perfect accordance with it. For imagine the product of the 15 quantities

$$FGHF'G'H'F''G''H''F'''G'''H'''F^{IV}G^{IV}H^{IV}$$

to be actually written out giving rise to 2^{15}, or 32768 terms, and to each of these terms prefix the sign Σ indicating that the sum is to be taken of the 120 values which it assumes on permuting the five letters α, β, γ, δ, ϵ. The sum of all these partial sums is $120I$; hence some, at least, of them cannot vanish. Let ΣT be any one that does not vanish. Then ΣT is a function of the differences of the roots of the same weight and order as the entire expression; it is therefore to a numerical factor près identical with I, just as every fragment of a mirror is itself a mirror, or as every particle of diamond dust, a diamond.

Thus, as many distinct non-vanishing forms as there may be of ΣT, so many different graphs to the quint-invariant of a binary octodecadic shall

we be able to construct agreeing respectively with the different representations of I of the form

$$\Sigma(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c\ldots$$

and it is probable that the virtual equivalence of all these several graphs may admit of being made out by inspection, as we saw was the case with the two graphs (one dissociated, the other connected) corresponding to the two algebraical representatives of the quadrinvariant of a quartic. Thus, what seemed, at first sight, to be fatal to the admissibility of the algebraico-graphical theory only serves to set in a clearer light its value as an instrument of research.

If we analyse M. Hermite's form of the skew invariant* to the quintic we shall see that it depends upon this simple but not obvious fact, that writing

$$F=(c,\ d)(a-b)+(a,\ b)(c-d)$$

$$G=(b,\ d)(a-c)+(a,\ c)(d-b)$$

$$H=(b,\ c)(a-d)+(a,\ d)(b-c)$$

and interpreting any such quantity as (a, b) to mean either 1 or $(a+b)$ or ab the product FGH is a symmetrical function of a, b, c, d, because on interchanging any two letters (say for example c, d) that one of the three quantities F, G, H (in this example H) in which those two letters are affected with the same sign, will remain unaltered in value whilst the other two (here G and F) change, each into the negative of the other.

Consequently we may interpret (a, b) to mean $(e-a)(e-b)$ and then the product of the five products corresponding to FGH is a function of the coefficients which expressed in terms of the differences of the roots will be of the weight 15 and of the order $1.6+4.3$ or 18 because in one of the five products each letter will enter in six dimensions and in each of the other four products in three dimensions; thus in FGH, e^6 will appear, but in each of the other four products e^3 will be the highest power of e. Hence the quindenary product is the invariant in question. No further step is necessary, the proof is complete as stated.

This remark will enable us to illustrate the process of transformation, which I have compared with grinding a diamond into dust, by an example

* I am wont to compare in my mind this symmetrical and translucent form to the Pitt Diamond and Père Joubert's to the Koh-i-Noor. In Note D to Appendix 2 a method is given whereby these forms may be transmuted into one another subject, however, to the bare possibility that the one, put into the algebraical alembic at a certain stage of the process, instead of passing into the other may, so to say, evaporate and be reduced to nothing. In the theory of forms, all-embracing Zero is the source and reconciler of contradictions, because, algebraically speaking, *everything is contained in nothing*, and so in a morphological sense "nought is everything" though not "everything is nought."

that can be completely pursued to the end. For let us now regard a, b, c, d as the roots of a binary quartic; then

$$\{(a-b)+(c-d)\}\{(a-c)+(d-b)\}\{(a-d)+(b-c)\}$$

will be a differentiant thereto of weight three and order three; it will, in fact, represent the root-differentiant of the skew sextic covariant.

Imagine this multiplied out without disturbing the marks of coupling so as to give eight terms or fragments analogous to the 32768 fragments spoken of in the preceding case. These terms will be of only four different patterns, one of the pattern $(a-b)(a-c)(a-d)$, three of the pattern $(a-b)(a-c)(b-c)$, three of the pattern $(a-b)(b-c)(d-b)$ and one of the pattern $(c-d)(d-b)(b-c)$. Prefixing Σ to each of these pattern terms to signify the sum resulting from the 24 permutations of a, b, c, d, we know *à priori* that not all of these can be zero since a linear function of them will be 24 times the differentiant in question, and on examination we find that the second and fourth Σ will vanish, but that the first and third will not. Accordingly, we shall have two new expressions

$$\Sigma(a-b)(a-c)(b-c), \qquad \Sigma(a-b)(b-c)(b-d),$$

each of which represents a differentiant of the same type as the original one, and this type being monadelphic or henparametric, the original product and these two sums will only be different representations of the same differentiant. Thus we see that each independent form belonging to a given type is susceptible (when expressed as a function of the differences of the roots) of a number of distinct phases, or, as we may express it, an algebraical form, in this theory, is in general polyphasic and accordingly its Icon or linkage exponent will be in general polygraphic, and each phase will have its own appropriate graph. It is a work of some difficulty, in general, to recognize the substantial identity of the different phases of the same algebraical form, and in like manner it may not, in all cases, be easy to recognize the substantial identity of the different graphs of its Icon, but sufficient has been shown to indicate the possibility and method of establishing such identity. The more I study Dr Frankland's wonderfully beautiful little treatise the more deeply I become impressed with the harmony or homology (I might call it, rather than analogy) which exists between the chemical and algebraical theories. In travelling my eye up and down the illustrated pages of "the Notes," I feel as Aladdin might have done in walking in the garden where every tree was laden with precious stones, or as Caspar Hauser when first brought out of his dark cellar to contemplate the glittering heavens on a starry night. There is an untold treasure of hoarded algebraical wealth potentially contained in the results achieved by the patient and long continued labour of our unconscious and unsuspected chemical fellow-workers.

We have seen that M. Hermite's beautiful expression for the skew invariant of the quintic proves its own character. A similar analysis may be applied to père Joubert's equally beautiful and even more remarkable expression for that of the sextic. M. de Bruno's statement of this, Table IV[10], contains two very perplexing typographical errors, namely, 4th line from foot of page, in V_0, $x_1x_2(x_\infty + x_0 - x_3 - x_2)$ should read $x_1x_2(x_\infty + x_0 - x_3 - x_4)$, and 3rd line from foot of page, in W_0, $x_2x_4(x_2 + x_3 - x_\infty - x_0)$ should be $x_2x_4(x_1 + x_3 - x_\infty - x_0)$. Moreover, the form in which the expression is presented in M. de Bruno's pages tends to mask its true nature and to suggest an analogy, which has no existence in fact, between it and M. Hermite's form; the latter is intrinsically a quinary group of triadic products, but such representation in the case of M. Joubert's form is purely conventional and confusing, it really being a single indecomposable quindenary product. Call a, b, c, d, e, f the six roots of a sextic, and let ab; cd; ef be any one of the 15 *duadic synthemes** which can be formed with them, and

$$F = \pm \left\{ \begin{array}{l} \quad ab\,.\,(c+d-e-f) \\ + cd\,.\,(e+f-a-b) \\ + ef\,.\,(a+b-c-d) \end{array} \right\}$$

The external sign is arbitrary, but must be considered as *determined* once for all for each of the 15 values of F. The product of these 15 values is a symmetrical function of the roots. For suppose any two letters, as a, b, to be interchanged; then three of the factors F in which a and b are coupled will undergo no change, but the remaining twelve will evidently be resoluble into six pairs reciprocally related, so that each F of a pair is transformed either into the other or into its negative and on either supposition the product of the pair remains unaltered in value. Also this product is a differentiant, for $\Sigma\delta_a$ operating on any one factor evidently reduces it to zero. It is also of the weight 45 and of the order 15. Hence the product of the fifteen values of F is the skew invariant to the sextic.

It seems desirable to make the *differentiantive* character of the form self-apparent. This may be done by virtue of the remark that $\pm F$ may be replaced by the form

$$\left\{ \begin{array}{l} \quad (a-d)(b-f)(c-e) + (a-f)(b-d)(c-e) \\ + (a-c)(b-e)(d-f) + (a-e)(b-c)(d-f) \\ + (a-c)(b-f)(d-e) + (a-f)(b-c)(d-e) \\ + (a-d)(b-e)(c-f) + (a-e)(d-b)(c-f) \end{array} \right\}$$

* A duadic syntheme of $2n$ letters is a combination of n duads containing between them all the letters. In it the order of the duads and of the letters in each duad is disregarded. Hence the number of such is $\dfrac{\Pi 2n}{2^n \Pi n}$ or $1.3.5\ldots(2n-1)$. For an odd number of letters simple synthemes do not exist but in lieu of them we may construct diplo-synthemes containing every letter taken twice over.

This sum contains 64 terms, of which 48 are the terms in F taken 4 times over, and the other 16 are the 8 quantities *ace, bdf, acf, bde, bce, adf, bcf, ade*, each appearing twice with opposite signs. If we expand the product of the 15 values of F, we shall obtain 35,184392,568832, or upwards of 35 billions of terms distributable among a certain number of patterns; on prefixing Σ to one of each pattern a certain number of such sums will be zero, but the remaining ones of which there must be some (and there will probably be a very large number) will all be (except as to a numerical multiplier) identical with each other and with père Joubert's formula. We see by these examples that there is a sort of polymorphism or pheno-polymorphism, as it may be termed, which is of a much more superficial character than and ought to be carefully distinguished from true polymorphism, eteo-polymorphism as we may call it, and this distinction as it has a marked bearing upon the theory of algebraical linkages, it is reasonable to expect may not be without importance in the study and construction of chemical graphs. Although I have been dealing, in what precedes, with particular cases, the reasoning is general in its nature and leads to conclusions which I will proceed to express in exact terms.

Let us understand by a permutation-sum of a function of letters belonging to one or more sets (n, n', n'', ... being the number of letters in the respective sets) the sum of the $\Pi n\,\Pi n'\,\Pi n''\ldots$ values which the function assumes when the letters in each several set are permuted *inter se*; and let us understand by a monomial differentiant one which (with the usual convention as to $a=1$) may be expressed as a permutation-sum of a single product of differences of roots of the parent quantic, or quantic system; then in the first place it has virtually been proved, in what precedes, and is undoubtedly true that every monadelphic differentiant is monomial, and it may easily be proved in like manner that a differentiant of multiplicity k may be represented by the sum of k monomial differentiants.

For greater simplicity let us confine ourselves to the case of monadelphic invariants and let us consider any two such belonging to reciprocal types; then the algebraical value of either one, in terms of the roots of its parent quantic or quantic system, will be represented by the permutation-sum of the product of the differences of every two letters in the other taken as many times as there are connecting bonds between them, such letters being for this purpose regarded as the roots in question. Hence also we may derive the rule previously given for determining whether or not any given graph, in which the number of bonds is equal to half the toti-valence, represents or not an algebraical invariant—the condition of its doing so being that the permutation-sum of the product of the differences between the connected letters (each bond giving one such difference) shall be other than zero. This rule will stand good whether the type of the graph be monadelphic or not.

A very simple instance occurs to me of the monomial law for monadelphic types. Let α, β, γ be the roots of a cubic. It will easily be found that the type (4 : 3, 4) to which

$$\{(\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\}^2$$

belongs is monadelphic; prefix to it the sign of summation, which is merely equivalent to multiplying it by 6. It will not be a monomial permutation-sum as it stands, but it may be replaced by $2\Sigma(\alpha-\beta)^2(\alpha-\gamma)^2$ or $\Sigma(\alpha-\beta)^4$ each of which monomial sums is a half of

$$\{(\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\}^2.$$

POSTSCRIPT. Subsequently to the printing of the foregoing sheets I have seen in an editorial notice in the English Journal *Nature* (Feb. 14, 1878) a statement of the claims of Dr Frankland to be the discoverer and first promulgator of the law of atomicity, and I appear unconsciously to have done injustice to this great English chemist by attributing the discovery to Kekulé. I derived my impression on the subject from the popular belief and from the account of it given by Wurz in his *Histoire des doctrines chémiques.* If the facts of the case are as set forth in *Nature* and admit of no qualifying statements, I am unable to understand how such a discovery as that of valence or atomicity, which furnishes the master-key to our knowledge of the transformations of matter and raises chemistry to the rank of a mathematical and predictive science (it was previously only arithmetical), can have escaped receiving the award of a Copley Medal from the society in whose Transactions it appeared. I can hardly imagine that, if the first announcement and proof of universal gravitation or the circulation of the blood had been communicated to the world in a paper inserted in the *Philosophical Transactions* in these days, its author would have failed to receive for it the highest mark of recognition in the power of the Royal Society of London to bestow, and in my humble judgment the law of atomicity in its far-reaching importance and the labour, and mental acumen required for its discovery, stands fully on a level with either of these great landmarks in the history of natural science. It seems also from the same article in *Nature* that my distinguished friend, Professor Crum Brown, to whose personal teaching at Edinburgh I owe the very slight acquaintance with the subject I can lay claim to, was the first to use the admirable method of chemico-graphs.

The conception of hydro-carbon graphs as "trees with nodes, branches and terminals" and the indispensable notion of constructing them by starting from "an intrinsic central node or pair of nodes, so as to get rid of the otherwise unsurmountable difficulty of having to recognize equivalent forms appearing several times over in the same construction," are exclusively my own and were used by me in my communications with Professor Crum Brown on the subject and stated by me in a letter to Professor Cayley, who has

adopted them as the basis of his own isomerical researches. In the account of this method given in German chemical journals I am informed that all reference (or at least all adequate reference) to my name as the author of it "fine by degrees and beautifully less," has at length entirely evaporated. M. Camille Jordan was led by quite a different order of considerations and with quite a different object in view to a discovery of the same centres before me, but I was not acquainted with this fact when I rediscovered them and made the application above mentioned. The idea of this application stands in the same relation to Professor Cayley's perfected use of it, as his idea of the use to be made of the equation $\Delta(w: i, j) =$ the number of linearly independent covariants of the type $[i, j: ij-2w]$ stands to my completed method founded thereon, for obtaining the scale and connecting syzygies of the irreducible covariants to a quantic, laying me thereby under an obligation which I should take it in very ill part if any translator of my papers on the subject failed to acknowledge in unmistakable terms.

The hydro-carbon graphs, it may be noticed, belong to the limiting case of chemico-graphs; where no cyclical system of bonds connects any groups of atoms in a graph, it becomes an arborescence.

I have found it a profitable exercise of the imagination, from a philosophical point of view, to build up the conception of an *infinite* arborescence and to dwell on the relations of time and causality which such a concept embodies. An example of the good to be gained by these limitless mental constructions (new tracts and highways, so to say, opened out in the all-embracing "grand continuum" which we call space) is afforded by the valuable applications to the theory of local probability and the integral calculus in general made by Professor Crofton (my successor at Woolwich) of his new idea of an infinite reticulation (warp and woof), every finite portion of which contains an infinite number of meshes, being formed by the crossings of two sets of parallel lines all infinitely extended in both directions and those of the same set equidistant and infinitely near to each other. So the largest idea of an arborescence is that of an infinite number of nodes with an infinite number of branches proceeding from each of them.

APPENDIX 2.

NOTE ON M. HERMITE'S LAW OF RECIPROCITY.

I take for granted that the treatise of M. Faà de Bruno represents this theory as it at present stands, in which case it seems to have made no advance since it was first promulgated by M. Hermite in his well known paper in the *Cambridge and Dublin Mathematical Journal*, 1854. It will be seen, however, I think from what follows, that it admits of being presented in a somewhat

simpler and more general form. It rests essentially on the proposition of reciprocity in the theory of partitions that $(w: i, j) = (w: j, i)$, from which it follows as an immediate consequence that the number of arbitrary constants in the general covariant (or invariant) whose type is $[w: i, j]$, is the same as that whose type is $[w: j, i]$ since that number will be $\Delta(w: i, j) = \Delta(w: j, i)$ for each. Let now $\phi(a, b, c, \ldots l)$ be any differentiant of the order j in the coefficients, and of the weight w to a binary quantic $F(x, y)$ of the degree i in the variables; then ϕ is the root of a single covariant whose order is j and degree in the variables $ij - 2w$. Let ϕ be expressed (as from the definition of a differentiant must necessarily be possible) as a function of the differences of the roots $\alpha_1, \alpha_2, \ldots \alpha_i$ of F when y is made unity. For any difference $\alpha_p - \alpha_q$ substitute $\frac{d}{dx_p} \cdot \frac{d}{dy_q} - \frac{d}{dx_q} \cdot \frac{d}{dy_p}$, and let ϕ be converted into $\dot{\phi}$ by this substitution. Now operate with $\dot{\phi}$ upon the product of the i forms $G(x_1, y_1)$, $G(x_2, y_2), \ldots G(x_i, y_i)$, $G(x, y)$ signifying the general form of the degree j in the variables, and after the operation has been performed turning each subscript x into x and each subscript y into y, after the manner of Professor Cayley's original method of generating invariants or covariants as "Hyperdeterminants;" we shall thus obtain an in- or co-variant to a form of the degree j which will be of the order i in the coefficients and of the degree $ij - 2w$ in the variables, for there are w factors in $\dot{\phi}$ and each factor is of the second dimension in two of the x's and the corresponding two y's. Thus we shall have passed from a form of the type $[i, j: ij - 2w]$ to another of the type $[j, i: ij - 2w]$, or which is the same thing, from one of the type $[w: i, j]$ to another of the type $[w: j, i]$.

This latter may be called the *image* of the first. For facility of reference, let the number of arbitrary parameters in the one and the other type be called the multiplicity. If we repeat upon this image the process by which it was deduced from its primitive, we shall obviously get back the original type, but it by no means follows that if the multiplicity exceed unity, we shall get back the primitive form itself. It may be possible to revert to the same type without reverting to the same individual specimen of it*; and such, we shall presently see, is what in general happens.

Before proceeding further I shall give a very simple methodical rule for finding the image to any given invariantive form. Since, for any given value of i, the form and its image are each given when their root-differentiants are respectively given, it will be sufficient to assign the law for passing from the differentiant of the primitive to that of its image.

* Just as, if I rightly understand the explanation given of fluorescence, a ray of light may give birth to some other form of motion and that again to another ray of light but of a different colour from the first. The theory of reciprocity treated of in the text is, in fact, a theory of alternate generation.

For this purpose, let the given in- or co-variant be expressed in terms of symmetrical functions of the roots of the quantic when the leading coefficient (a), is made equal to unity. Then it will consist of terms, any one of which, apart from its numerical coefficient, will be of the form

$$\Sigma\,(\alpha_1\alpha_2\ldots\alpha_\lambda)^0\,(\beta_1\beta_2\ldots\beta_\mu)^1\,(\gamma_1\gamma_2\ldots\gamma_\nu)^2\,(\delta_1\delta_2\ldots\delta_\pi)^3\ldots$$

$\alpha_1\alpha_2\ldots\alpha_\lambda$, $\beta_1\beta_2\ldots\beta_\mu$, $\gamma_1\gamma_2\ldots\gamma_\nu$, &c. being all distinct and comprising between them *all* the i roots and of course $\mu+2\nu+3\pi+$&c. will be equal to the weight; to pass from a differentiant expressed in terms of roots of a given quantic to the expression in terms of coefficients of the allied quantic of its image it will be found that the only thing necessary is to change any such factor as α^λ (where α is any root of the given quantic) into C_λ, the coefficient of the term containing y^λ in the allied one. This rule is a consequence (obtainable by ordinary algebraical processes) from the method above explained, where it is to be borne in mind that in order to obtain the image from the given form we have only to substitute for each root α_κ which occurs in ϕ, the fraction $\frac{dx_\kappa}{dy_\kappa}$ and to multiply the result by such a power of $\frac{d}{dy_1}\cdot\frac{d}{dy_2}\cdots\frac{d}{dy_\kappa}$, as will just serve to make it integral. A much simpler demonstration of this rule will be given in the sequel, and it will be shown that it not only holds good for deriving the leading term of the reciprocal (in the case of a covariant) from that of the primitive (that is, the root-differentiant of the one from the root-differentiant of the other) but that it is applicable to deriving the whole of one expression from the whole of the other.

As an example, take the differentiant whose type is [3: 3, 3], the root or base of the skew covariant to a cubic $(a, b, c, d \between x, y)^3$. Its value is $a^2d-3abc+2b^3$; expressed in terms of the roots α, β, γ, making $a=1$, this becomes

$$\alpha\beta\gamma-\frac{3\,(\alpha+\beta+\gamma)\,(\alpha\beta+\alpha\gamma+\beta\gamma)}{9}+2\,\frac{(\alpha+\beta+\gamma)^3}{27},$$

$$\text{or } \frac{1}{27}\left\{27\alpha\beta\gamma-9\,(\alpha+\beta+\gamma)\,(\alpha\beta+\alpha\gamma+\beta\gamma)+2\,(\alpha+\beta+\gamma)^3\right\},$$

$$\text{or } \frac{1}{27}\left\{2\Sigma\alpha^3-3\Sigma\alpha^1\beta^2+12\alpha\beta\gamma\right\},\text{ that is, } \frac{1}{27}\left\{2\Sigma\alpha^0\beta^0\gamma^3-3\Sigma\alpha^0\beta^1\gamma^2+12\alpha^1\beta^1\gamma^1\right\}.$$

Applying the rule, this becomes converted into

$$\frac{1}{27}\left\{6C_0{}^2C_3-18C_0C_1C_2+12C_1{}^3\right\},$$

or, reverting to the letters a, b, c, d, the image becomes the primitive affected with the factor $\frac{6}{27}$ and may be seen to be its own conjugate. Or again, let the primitive be the discriminant of a cubic, that is,

$$\frac{1}{27}\,(\alpha-\beta)^2\,(\alpha-\gamma)^2\,(\beta-\gamma)^2 \text{ or } (\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha-\alpha\beta^2-\beta\gamma^2-\gamma\alpha^2)^2;$$

this is equal to

$$\Sigma\,(\alpha^2\beta^4 + 2\Sigma\alpha\beta^2\gamma^3 - 2\Sigma\alpha^3\beta^3 - 6\alpha^2\beta^2\gamma^2 - 2\Sigma\alpha\beta\gamma^4).$$

Hence, by our rule, the image will be

$$\frac{1}{27}\,(6c_0c_2c_4 + 12c_1c_2c_3 - 6c_0c_3^2 - 6c_2^3 - 6c_1^2c_4),$$

or, using a, b, c, d, e in lieu of c_0, c_1, c_2, c_3, c_4, we obtain the form

$$-\frac{6}{27}\,(ace + 2bcd - ad^2 - c^3 - b^2e),$$

that is, $-\frac{2D}{9}$, where D is the well known quadrinvariant to a quartic

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Treating this quadrinvariant as a function of the roots of a biquadratic form and proceeding as before to form its image, we shall obtain a second image which will be a numerical multiple of the original invariant.

But now let us consider the case of polyadelphic forms belonging to reciprocal types and for greater brevity, as the calculations are necessarily long, take a quantic of the self-contrary type $[w : i, i]$, as, for example $[6 : 4, 4]$ which belongs to the covariant of the fourth order and fourth degree to a quartic. This will be diadelphic; its general form is a linear combination of two products, one of the quartic itself by its cubinvariant, the other of the Hessian by the quadrinvariant. It will therefore have for its leading coefficient the differentiant

$$\lambda a\,(ace + 2bcd - ad^2 - c^3 - b^2e) + \mu\,(ac - b^2)\,(ae - 4bd + 3c^2),$$

say $\lambda U + \mu V$. Let us first find the image of U. Expressed in terms of the roots $\alpha, \beta, \gamma, \delta$, it is

$$\begin{aligned}
&\frac{1}{6}\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,(\alpha\beta\gamma\delta) \\
&+ \frac{1}{48}\,(\alpha + \beta + \gamma + \delta)\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) \\
&- \frac{1}{16}\,(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)^2 - \frac{1}{216}\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^3 \\
&- \frac{1}{16}\,(\alpha + \beta + \gamma + \delta)^2\,\alpha\beta\gamma\delta,
\end{aligned}$$

which is

$$\frac{[6]\,(\alpha\beta\gamma^2\delta^2)}{6} + \frac{[24]\,(\alpha\beta^2\gamma^3) + [48]\,(\alpha\beta\gamma^2\delta^2) + [12]\,(\alpha^2\beta^2\gamma^2) + [12]\,(\alpha\beta\gamma\delta^3)}{48}$$

$$-\frac{[4]\,(\alpha^2\beta^2\gamma^2) + [12]\,(\alpha\beta\gamma^2\delta^2)}{16}$$

$$-\frac{[6](\alpha^3\beta^3)+[90](\alpha\beta\gamma^2\delta^2)+[72](\alpha\beta^2\gamma^3)+[24](\alpha^2\beta^2\gamma^2)+[24](\alpha\beta\gamma\delta^3)}{216}$$

$$-\frac{[4](\alpha\beta\gamma\delta^3)+[12](\alpha\beta\gamma^2\delta^2)}{16},$$

where any term, as for example $[48](\alpha\beta\gamma^2\delta^2)$, means the sum of the quantities of the type $\alpha\beta\gamma^2\delta^2$ each taken a sufficient number of times to make up 48 combinations, so that it is identical in meaning with $8\Sigma(\alpha\beta\gamma^2\delta^2)$ in the common notation. This convention is useful in saving the unnecessary labour of performing divisions in this first part of the process which have to be exactly reversed by multiplications in the transformation process which follows. The value of the above sum is, for purposes of transformation, equivalent to

$$\frac{1}{36}\left\{3\alpha\beta\gamma^2\delta^2+6\alpha\beta^2\gamma^3-4\alpha^2\beta^2\gamma^2-4\alpha\beta\gamma\delta^3-\alpha^3\beta^3\right\},$$

which gives for the image of U

$$\frac{1}{36}(3b^2c^2+6abcd-4ac^3-4db^3-a^2d^2)$$

or $\frac{1}{36}(U-V)$, where it will be observed that $(V-U)$ is identical with the discriminant to $(a, b, c, d \between x, y)^3$. Let us now proceed to find the image of $(U-V)$. Using σ to denote the sum of the combinations of $\alpha, \beta, \gamma, \delta$ taken i and i together, where $\alpha, \beta, \gamma, \delta$ are the roots of the general quartic, we have

$$U-V=\frac{\sigma_1^2\sigma_2^2}{192}+\frac{\sigma_1\sigma_2\sigma_3}{16}-\frac{\sigma_2^3}{54}-\frac{\sigma_3\sigma_1^3}{64}-\frac{\sigma_3^2}{16}$$

$$=\frac{1}{1728}(9\sigma_1^2\sigma_2^2+108\sigma_1\sigma_2\sigma_3-32\sigma_2^3-27\sigma_3\sigma_1^3-108\sigma_3^2).$$

Expanding and transforming, it will be found that the image of $(U-V)$ is $\left(\frac{21}{432}U-\frac{1}{432}V\right)$ and the second image of U which is $\frac{I(U-V)}{36}$ does not revert to the form U.

As a simpler example we may take the covariant to a quartic, still of the fourth order in the coefficients as before, but of the eighth degree in the variables. This will have for its root-differentiant

$$\lambda a^2(ae-4bd+3c^2)+\mu(ac-b^2)^2, \text{ say } \lambda U+\mu V.$$

Here $$U=\sigma_4-\frac{\sigma_1\sigma_3}{4}+\frac{\sigma_2^2}{12}=\frac{1}{12}(12\sigma_4-3\sigma_1\sigma_3+\sigma_2^2),$$

and for the purpose of transformation is equivalent to

$$\frac{1}{12}\left\{12\alpha\beta\gamma\delta-3(4\alpha\beta\gamma\delta+12\alpha^2\beta\gamma)+6\alpha\beta\gamma\delta+6\alpha^2\beta^2-12\alpha^2\beta\gamma\right\}$$

$$=\frac{1}{12}\left\{6\alpha\beta\gamma\delta+6\alpha^2\beta^2-12\alpha^2\beta\gamma\right\}.$$

Hence, using I to denote "image of,"

$$IU = \frac{1}{2}\left\{b^4 + a^2c^2 - 2acb^2\right\} = \frac{1}{2}V.$$

Again $$V = \left(\frac{\sigma_2}{6} - \frac{\sigma_1^2}{16}\right)^2$$

$$= \frac{1}{48^2}\left\{3\alpha^2 + 3\beta^2 + 3\gamma^2 + 3\delta^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\right\}^2,$$

which, for purposes of transformation, will be found equivalent to

$$\frac{1}{48^2}\left\{36\alpha^4 + 132\alpha^2\beta^2 - 48\alpha^2\beta\gamma - 144\alpha^2\beta + 24\alpha\beta\gamma\delta\right\}.$$

Consequently

$$IV = \frac{1}{192}\left\{3a^3e + 11a^2c^2 - 4ab^2c - 12a^2bd + 2b^4\right\}$$

$$= \frac{1}{192}\left\{3a^2(ae - 4bd + 3c^2) + 2(b^2 - ac)^2\right\}$$

$$= \frac{1}{192}(3U + 2V).$$

Let now $\lambda : \mu$ be so chosen that

$$I(\lambda U + \mu V) = \rho(\lambda U + \mu V).$$

This gives $$\frac{\mu U}{64} + \left(\frac{\lambda}{2} + \frac{\mu}{96}\right)V = \rho(\lambda U + \mu V),$$

or $$\frac{\mu^2}{64} - \frac{\lambda\mu}{96} - \frac{\lambda^2}{2} = 0,$$

that is, $$3\mu^2 - 2\lambda\mu - 96\lambda^2 = 0.$$

The two values of $\frac{\mu}{\lambda}$ derived from this equation are 6 and $-\frac{16}{3}$. The corresponding values of ρ will be 6 and $-\frac{1}{12}$. There are thus two definite systems of $\lambda : \mu$, and no more, which will make $\lambda U + \mu V$ self-conjugate and it is obvious that there will be no other values of $\lambda : \mu$ which will make

$$I^2(\lambda U + \mu V) = \rho(\lambda U + \mu V),$$

for, I^2U and I^2V being determinate linear functions of U, V, we shall have a quadratic equation for determining $\lambda : \mu$, but the two values of $\lambda : \mu$ which make $\lambda U + \mu V$ self-conjugate must satisfy this equation, and hence there can be no others. Reverting to the preceding example of the type [6 : 4, 4], we have found

$$IU = \frac{1}{36}U - \frac{1}{36}V$$

$$I(U - V) = \frac{21}{432}U - \frac{1}{432}V.$$

Hence $$IV = -\frac{9}{432}U - \frac{11}{432}V,$$

and making $$I(\lambda U + \mu V) = \rho(\lambda U + \mu V),$$

the equation for finding ρ will be

$$\begin{vmatrix} \frac{12}{432} - \rho \;, & -\frac{12}{432} \\ -\frac{9}{432} \;, & -\frac{11}{432} - \rho \end{vmatrix} = 0,$$

whence $$\rho_1 = -\frac{1}{27}, \quad \rho_2 = \frac{5}{144};$$

also, since $$\left(\frac{12}{432}\lambda - \frac{9}{432}\mu\right) = \rho\lambda,$$

we shall have $$\frac{\lambda_1}{\mu_1} = -\frac{9}{28}, \quad \frac{\lambda_2}{\mu_2} = 3.$$

What intrinsic peculiar properties are possessed by the principal forms* is a question as to which we are at present quite in the dark, as are we also with regard to the general character of the equation in ρ. It were much to be wished that some one would work out the case of a triadelphic type, as for example the type of covariants of the 6th order in the coefficients and the 6th degree in the variables, to a sextic. It might be supposed from the two preceding examples that the values of ρ are necessarily rational, but it will be shown hereafter that such is not the case.

It is easy to see that the relation between any form belonging to a given type of multiplicity 2 or 3 and its second image may be geometrically represented by means of a quadric curve or surface. Thus suppose the multiplicity is three, and that the three values of ρ are A, B, C. Construct an ellipsoid or hyperboloid whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$. Draw r any radius vector making angles α, β, γ with the principal axes, p a perpendicular from the centre upon the tangent plane at the point where r meets the quadric, making angles λ, μ, ν with these axes. Then if

$$K(\cos\alpha U + \cos\beta V + \cos\gamma W)$$

be any given form of the system for which U, V, W are the principal forms,

$$\frac{K}{pr}(\cos\lambda U + \cos\mu V + \cos\nu W)$$

will be its second image. And we may say that, if a form lies in the

* By a principal form (in general), as hereafter stated in the text, I mean one which is the reciprocal of its first image in the sense that it bears a numerical ratio to its second image. The numerical quantity by which it must be multiplied to give the second image, I call a principal multiplier.

direction of the axis of instantaneous rotation, its second image will lie in the perpendicular upon the invariable plane: or more simply if by the direction of a form $\lambda U + \mu V + \nu W$ we understand that of a straight line whose direction cosines are as $\lambda : \mu : \nu$ and by its modulus $\sqrt{(\lambda^2 + \mu^2 + \nu^2)}$, we may say that if a radius vector of the ellipsoid (or other quadric) represent the direction and modulus of an in- or co-variant the corresponding radius vector of the polar reciprocal to the quadric will represent the direction and modulus of its second image.

The true nature of the reciprocity theorem, in the general case where i, j have any values whatever, is now obvious. Let $U_1, U_2, \ldots U_q$ be independent forms belonging to the type $[w: i, j]$, whose multiplicity is q, and $V_1, V_2, \ldots V_q$ as many forms belonging to the reciprocal type $[w: j, i]$. We may, by virtue of the transformation process, express each IU in terms of linear functions of the forms V and *vice versa*, so that each I^2U will be a known linear function of all the U's. For clearness sake suppose $q = 3$ and let

$$I^2 U_1 = a U_1 + b U_2 + c U_3$$
$$I^2 U_2 = a' U_1 + b' U_2 + c' U_3$$
$$I^2 U_3 = a'' U_1 + b'' U_2 + c'' U_3.$$

Now make

$$I^2 (\lambda U_1 + \mu U_2 + \nu U_3) = \rho\, (\lambda U_1 + \mu U_2 + \nu U_3).$$

We shall have for finding ρ the equation

$$\begin{vmatrix} (a-\rho), & b, & c \\ a', & (b'-\rho), & c' \\ a'', & b'', & (c''-\rho) \end{vmatrix} = 0,$$

and then the three systems of values of $\lambda : \mu : \nu$, which make the second image of $\lambda U_1 + \mu U_2 + \nu U_3$ coincide to a numerical factor près, with itself, will be rational functions of the respective roots. So, in general, when the multiplicity of the type $[w: i, j]$ is q, there will be in general q special forms, and no more, which have reciprocal forms belonging to the type $[w: j, i]$, and if the interchangeable elements, i, j are equal, then these q forms will all be self-conjugate. It is conceivable that in certain cases the equation in ρ may have equal roots; in that event each such equality would introduce a corresponding indeterminateness in the forms admitting of conjugates. For example, if the multiplicity were 2 and the two roots of ρ equal, that would signify that *every* form belonging to the type would have a conjugate—a fact analogous to an ellipse becoming a circle, or an ellipsoid a spheroid—and so in general.

A form having a conjugate, that is, whose second image is a numerical multiplier of itself, may be called a principal form. If the multiplicity of the

type is q, there will be q such. All but these will give rise to an endless succession of images such that any $q+1$ of an even order (the form itself included among these) will be connected by a linear equation. That the succession is endless is clear from the consideration that if an image, say of the $(2p)$th rank, is identical (to a numerical factor près) with the form, we have an equation of the qth degree for finding the values of the systems of multipliers λ, μ, ν of U, V, W; therefore there are only q such systems, but the systems which satisfy $I^2F=\rho F$ must also satisfy $I^{2p}F=\rho' F$, and consequently there are no others.

To illustrate this, suppose

$$I^2U = aU + bV$$
$$I^2V = cU + dV;$$

then

$$I^4U = (a^2+bc)\,U + (ab+bd)\,V$$
$$I^4V = (ca+ad)\,U + (cb+d^2)\,V.$$

If now we put

$$\begin{vmatrix} a-\rho, & b \\ c \quad , & d-\rho \end{vmatrix} = 0,$$

to find the values of $\lambda : \mu$ which make $I^2(\lambda U + \mu V) = \rho(\lambda U + \mu V)$ we have

$$(a-\rho)\,\lambda + c\mu = 0.$$

In like manner, if we make

$$\begin{vmatrix} a^2+bc-\rho, & ab+bd \\ ca+ad \quad , & cb+d^2-\rho \end{vmatrix} = 0,$$

to find the values of Λ and M which make $I^4(\Lambda U + \mathrm{M}V) = R(\Lambda U + \mathrm{M}V)$, we have

$$(a^2+bc-R)\,\Lambda + (ca+ad)\,\mathrm{M} = 0,$$

and it will be found that

$$a-\rho = \frac{a-d}{2} \pm \frac{1}{2}\sqrt{\{(a-d)^2+4bc\}}$$

$$a^2+bc-R = \frac{a^2-d^2}{2} \pm \frac{a+d}{2}\sqrt{\{(a-d)^2+4bc\}},$$

so that the values of $\lambda : \mu$ and $\Lambda : \mathrm{M}$ are the same, and such we know *à priori* must be the case.

It ought to be noticed that the method explained in the preceding pages furnishes a complete solution of the problem following. Given any in- or co-variant, say of the jth order in the coefficients to a form Q of the ith degree, to find the process of differentiation which performed upon the product

$$Q(x_1, y_1) \,.\, Q(x_2, y_2) \ldots\ldots Q(x_j, y_j)$$

shall produce the j-partite-emanant of the in- or co-variant so given, and it proves incidentally that every binary in- or co-variant may be represented as

a hyperdeterminant. To make this clear, let us call the above product, or rather that product divided by $(\Pi i)^j$, the j-ary norm of Q and denote it by NQ. Again, let G be any given differentiant to the type $[w: j, i]$, say $G(\rho_1, \rho_2, \ldots \rho_j)$ which is necessarily identical with

$$G\{0;\ (\rho_2 - \rho_1);\ (\rho_3 - \rho_1);\ \ldots (\rho_j - \rho_1)\}.$$

For $\rho_\kappa - \rho_1$ write $\dfrac{d}{dx_\kappa} \cdot \dfrac{d}{dy_1} - \dfrac{d}{dy_\kappa} \cdot \dfrac{d}{dx_1}$ and let the quantity so formed be called the hyperdeterminant to G and be denoted by HG. Then if E be any principal form to the type $[w: i, j]$, of the multiplicity q and belonging to a quantic Q, and G be its first image, we shall have

$$(HG)\,(N_jQ) = \rho F,$$

where ρ is one of the roots of a known equation of the qth degree in ρ. Consequently, since any form belonging to the given type is a linear function of its q principal forms, every such form may be expressed by means of the hyperdeterminant

$$\Sigma_{\lambda=q}^{\lambda=1} \frac{c_\lambda}{\rho_\lambda} (HG_\lambda)\, NQ,$$

the given form being supposed to be expressible by $\Sigma_{\lambda=q}^{\lambda=1} c_\lambda F_\lambda$, where F is any one of the q principal forms.

It follows from what has been shown above that in general from any one particular given form belonging to a type of multiplicity q may be deduced the $(q-1)$ others (by taking the successive second images) and thus the general form obtained; the exception is when the given form happens to be a linear function of less than q of the principal forms. A further consequence is that any in- or co-variant given in terms of the roots of its quantic may be converted by explicit processes into a function of the coefficients. Thus, for example, suppose that the multiplicity of the type is 3; call the given form R_0 and the successive second images R_1, R_2, R_3, R_4. These latter will be all known by the rule of transformation and we shall have R_4 a known linear function of the three preceding forms, say equal to

$$\alpha R_1 + \beta R_2 + \gamma R_3.$$

Hence if we put $$R_0 = \lambda R_1 + \mu R_2 + \nu R_3,$$

we must have $$R_1 = \lambda R_2 + \mu R_3 + \nu(\alpha R_1 + \beta R_2 + \gamma R_3);$$

hence $$\nu = \frac{1}{\alpha}, \quad \mu = -\frac{\gamma}{\alpha}, \quad \lambda = -\frac{\beta}{\alpha}$$

and thus R_0, given in terms of the roots, becomes known in terms of the coefficients of its quantic. And so in general, q being the multiplicity, $(q+1)$ forms deduced from the given function of the roots will serve to determine its value as a function of the coefficients. In fact by regarding R_0 as a linear function of the principal forms, it is easy to see it and all its

successive secondaries (that is, second images) form a recurring series, the scale of relations being

$$R_0 - \Sigma \frac{1}{\rho} R_1 + \Sigma \frac{1}{\rho^2} R_2 - \Sigma \frac{1}{\rho^3} R_3 + \ldots = 0,$$

where $1 : \rho$ is the ratio of any principal form to its immediate secondary. Thus E_0 being given in terms of the roots and consequently $E_1, E_2, \ldots E_q$, in terms of the coefficients, E_0 becomes known in terms of the coefficients and of the quantities $\Sigma \frac{1}{\rho}, \Sigma \frac{1}{\rho^2}, \ldots$; these latter are identical with the quantities previously mentioned and furnish the simplest means of forming the equation in ρ, which (if we agree to call $\rho_1, \rho_2, \ldots \rho_q$ the moduli of the several principal forms $F_1, F_2, \ldots F_q$, that is, the ratios of their respective second images to themselves) may be termed the modular equation for any given type*.

It might have been useful, had I thought of it in time, and may be useful when the subject comes again under consideration, to treat a form and its second image, in which the type is restored as *antecedent* and *consequent*, and to describe the first image as the *alternate* form to the primitive, inasmuch as we pass, by what biologists term alternate generation, from one type to the other. It has been shown, in what precedes, that the transformation by images at each second step leads back to the original type, but, contrary to what might have been supposed, does not in general imply the resuscitation of the individual form.

The theorem of reciprocity has been seen to be, in its essence, a theorem of differentiants, and ought therefore to admit of being proved by means of the necessary and sufficient partial differential equation to which differentiants are subject. This may be done as follows. If we call $\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_r$ the successive elements to a binary quantic expressed in its customary form, so that ϵ_r is the coefficient of the term containing y^r divested of its numerical binomial coefficient, and if we write

$$U = \frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots,$$

where $\alpha, \beta, \gamma, \ldots$ are the roots of the quantic, it is very easily proved that

$$U\epsilon_r = -r\epsilon_{r-1}\dagger.$$

Let $C\Sigma\alpha^r\beta^s\gamma^t \ldots$ be any term in a given differentiant F, the indices $r, s, t, \ldots$ being any whatever with no condition as to their being distinct from each

* But it will be better to adhere to the previous convention and to designate the ρ's as the principal multipliers and the equation in ρ as the principal equation.

† In fact it may easily be proved by the ordinary rule for the change of one system of independent variables into another that, if $a_1, a_2, \ldots a_i$ be the roots of $(\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_i \between x, y)^i$,

$$\Sigma \frac{d}{da} = -\sum_{q=0}^{q=i} q\epsilon_{q-1} \frac{d}{d\epsilon_q}.$$

other, and let $N(r, s, t, \ldots)$ signify the number of combinations comprised in Σ; also let $CN(r, s, t, \ldots) \,.\, \epsilon_r \epsilon_s \epsilon_t \ldots$ be called the image of the term above written and G the image of F, that is, the sum of the images of the several terms in F; where it must be observed that the ϵ quantities do not necessarily refer to roots the same in number or name as the roots $\alpha, \beta, \gamma, \ldots$. Now suppose that we have any term, such as $Q\Sigma\alpha^l\beta^m\gamma^n \ldots$ in UF, where U refers to the given roots $\alpha, \beta, \gamma, \ldots$ and means $\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots$. This term must arise from terms of the several forms

$$\left.\begin{array}{llll} A\,\Sigma\alpha^{l+1} & \beta^m & \gamma^n & \ldots \\ B\,\Sigma\alpha^l & \beta^{m+1} & \gamma^n & \ldots \\ C\,\Sigma\alpha^l & \beta^m & \gamma^{n+1} & \ldots \end{array}\right\} \text{in } F;$$
$$\&\text{c. } \&\text{c.} \ldots$$

corresponding to these there will be the images

$$\left.\begin{array}{llll} AN(l+1, m, n, \ldots)\, \epsilon_{l+1} & .\, \epsilon_m & .\, \epsilon_n & \ldots \\ BN(l, m+1, n, \ldots)\, \epsilon_l & .\, \epsilon_{m+1} & .\, \epsilon_n & \ldots \\ CN(l, m, n+1, \ldots)\, \epsilon_l & .\, \epsilon_m & .\, \epsilon_{n+1} & \ldots \end{array}\right\} \text{in } G,$$
$$\&\text{c. } \&\text{c.} \ldots$$

where G belongs to a quantic whose type is reciprocal to that of F, and it is clear that the effect of operating upon F with U will be to give

$$Q = A\rho N(l+1, m, n, \ldots)(l+1) + B\rho N(l, m+1, n, \ldots)(m+1)$$
$$+ C\rho N(l, m, n+1, \ldots)(n+1) + \&\text{c.} \ldots$$

ρ being a number easily determinable, but which there is no occasion to express. Again if $R\epsilon_l \,.\, \epsilon_m \,.\, \epsilon_n \ldots$ be the correlative term in G, we have by virtue of the formula $U\epsilon_r = -r\epsilon_{r-1}$, where the operator U refers to the roots of the quantic of reciprocal type,

$$(-)^w R = AN(l+1, m, n, \ldots)(l+1) + BN(l, m+1, n, \ldots)(m+1)$$
$$+ CN(l, m, n+1, \ldots)(n+1) + \&\text{c.} \ldots$$

Consequently, since on account of the identity $F=0$, we must have $Q=0$ for every term $Q\Sigma\alpha^l \,.\, \beta^m \,.\, \gamma^n \ldots$, we must also have $R = \rho^{-1} Q = 0$ and therefore, this being true for all the arguments $\epsilon_l \,.\, \epsilon_m \,.\, \epsilon_n \ldots$, we must have $UG = 0$. Hence, when any quantity F is a differentiant of a given quantic, its image (as defined in the text) is also a differentiant to a quantic of reciprocal type to the given one. This is the simplest method of establishing the theorem, but still the method originally employed in the note is valuable as serving to establish the important proposition that every in- or co-variant of a binary quantic is a hyperdeterminant.

I will proceed to show that for a system of two or more quantics of degrees $i, i', i'', \ldots$, we may pass from a covariant of the type $[w\colon i, j\,;\, i', j'\,;\, i'', j''\,; \ldots]$

to one of the type $[w:j, i;\ i', j';\ i'', j'';\dots]$ by taking its image in respect to the quantic whose indices, i, j, are to be interchanged precisely according to the same rule as if there were no other quantic present. As regards the law of reciprocity, a combination of quantics is analogous to a mixture of gases, according to Dalton's view, each playing the part, as it were, of a vacuum in respect to the other.

Let $[w: i, j;\ i', j';\dots]$ be the type, $[w: j, i;\ i', j';\dots]$ one of the antitypes, $(\epsilon_0, \epsilon_1, \epsilon_2, \dots \epsilon_j \between x, y)^j$ the general form of the jth degree, α, β, γ, ... its roots when $\epsilon_0 = 1$. Let $\eta_r = (-)^r \epsilon_r$; then, since

$$\Sigma \frac{d}{d\alpha} \epsilon_r = - r\epsilon_{r-1}$$

$$\Sigma \frac{d}{d\alpha} \eta_r = r\eta_{r-1}.$$

Let D be any differentiant of the given type, a, b, c, ... the roots of the quantic of degree i, a', b', c', ... the roots of the quantic of degree i', with the usual convention as to the leading coefficients becoming unities. Let $\Sigma a^l b^m \dots$, $\Sigma a'^{l'} b'^{m'} \dots \Sigma \dots$ be the arguments of any term in

$$\left(\Sigma \frac{d}{da} + \Sigma \frac{d}{da'} + \dots\right) D,$$

say UD, then the coefficient of the term last written will arise from operating with U upon

$$\left.\begin{array}{llllll} & A\,.\,\Sigma a^{l+1}\,.\,b^m & \dots\ \Sigma a'^{l'} & b'^{m'} & .\ \&c.\ \dots \\ + & B\,.\,\Sigma a^l & .\,b^{m+1}\,.\ \dots\ \Sigma a'^{l'} & b'^{m'} & .\ \&c.\ \dots \\ + & \dots\dots\dots\dots\dots\dots & & & \\ + & A'\ \Sigma a^l & .\,b^m\ \dots\ \Sigma a'^{l'+1} & b'^{m'} & .\ \&c.\ \dots \\ + & B'\ \Sigma a^l & .\,b^m\ \dots\ \Sigma a'^{l'} & b'^{m'+1} & .\ \&c.\ \dots \\ & & \&c.\ \dots & & \&c.\ \dots \end{array}\right\}$$

and the value of the coefficient will be

$$\left.\begin{array}{l} \quad A\ (l\ +1)\ N\,(l+1, m, \dots)\ N\,(l', m', \dots\dots\dots)\dots \\ + B\ (m\ +1)\ N\,(l, m+1, \dots)\ N\,(l', m', \dots\dots\dots)\dots \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + A'\ (l'\ +1)\ N\,(l, m, \dots\dots)\ N\,(l'+1, m', \dots)\dots \\ + B'\ (m'+1)\ N\,(l, m, \dots\dots)\ N\,(l', m'+1, \dots)\dots \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\}$$

$$\div N\,(l, m, \dots)\ N\,(l', m', \dots)\dots$$

To these feeders or contributory terms will correspond, in the image,

$$\begin{array}{l} \quad AN\ (l+1, m, \dots)\ \eta_{l+1}\,.\,\eta_m\ \dots\ \Sigma a'^{l'}\ .\ b'^{m'}\ \dots \\ + BN\ (l, m+1, \dots)\ \eta_l\ .\ \eta_{m+1}\,.\ \dots\ \Sigma a'^{l'}\ .\ b'^{m'}\ \dots \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + A'N\,(l, m, \dots\dots)\ \eta_l\ .\ \eta_m\ \dots\ \Sigma a'^{l'+1}\,.\ b'^{m'}\ \dots \\ + B'N\,(l, m, \dots\dots)\ \eta_l\ .\ \eta_m\ \dots\ \Sigma a'^{l'}\ .\ b'^{m'+1}\dots \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}$$

and it is obvious that by operating upon this with the U corresponding to its roots we shall obtain the argument $\eta_l . \eta_m \dots \Sigma a'^{l'} . b'^{m'} . \&c. \dots$ affected with the very same coefficient as that above written, except that in its denominator the factor, $N(l, m, \dots)$, will not appear. Hence, when D is a differentiant of the given type, its image (obtained by expressing the i set of coefficients in terms of roots and then replacing every power, ρ^q, of any such root, ρ, by η_q, leaving all the other coefficients unchanged) will also be a differentiant of the type transformed by interchanging i with its conjugate j*.

When there is but one quantic the effect of substituting ϵ_q instead of η_q will evidently only be to introduce a common factor $(-)^w$ into each term, which is immaterial and we may accordingly in that case reflect ρ^q into ϵ_q. Of course, in the general case, if all the letters i are simultaneously interchanged with the letters j, a similar conclusion follows.

As an example, let us take the two quadratics,

$$ax^2 + 2bxy + cy^2,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

their resultant $(a\gamma - c\alpha)^2 + 4(a\beta - b\alpha)(c\beta - b\alpha)$, belongs to the type $[4: 2, 2; 2, 2]$ which is its own reciprocal whichever of the interchangeable elements we permute. This resultant, treating a as unity, will be equal to

$$(\alpha\rho_1^2 + 2\beta\rho_1 + \gamma)(\alpha\rho_2^2 + 2\beta\rho_2 + \gamma)$$

$$= \alpha^2\rho_1^2\rho_2^2 + 2\beta\alpha(\rho_1^2\rho_2 + \rho_1\rho_2^2) + 4\beta^2\rho_1\rho_2 + \alpha\gamma(\rho_1^2 + \rho_2^2) + 2\beta\gamma(\rho_1 + \rho_2) + \gamma^2$$

the image of which will be

$$\alpha^2\epsilon_2^2 - 4\alpha\beta\epsilon_1\epsilon_2 + 4\beta^2\epsilon_1^2 + 2\alpha\gamma\epsilon_0\epsilon_2 - 4\beta\gamma\epsilon_0\epsilon_1 + \gamma^2\epsilon_0^2,$$

or as we may write it,

$$\alpha^2c^2 - 4\alpha\beta bc + 4\beta^2b^2 + 2\alpha\gamma ac - 4\beta\gamma ab + a^2\gamma^2,$$

which is $(c\alpha - 2b\beta + a\gamma)^2$, the square of the well known connective. Again, if we combine $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with $\alpha x + \beta y$, we have the invariant

$$a\beta^3 - 3b\alpha\beta^2 + 3c\alpha^2\beta - d\alpha^3, \text{ say } I,$$

* Thus the rule of images for passing from a differentiant of a given type belonging to a single quantic to one of the opposite type is extended to the case of passing from a differentiant of a given type belonging to a system of quantics to any associated type, that is, to any type in which one or more of the numbers i chosen at discretion is or are interchanged with the corresponding numbers j, and it will presently be seen that this implies the extension of the rule without any alteration from differentiants or invariants to covariants of a quantic or system of quantics. In Note A it will further be shown that for any inversions whatever (or, to speak more accurately, for any cycle of inversions leading back to the original type), although the principal multipliers change their values as the cycle of inversion changes, the *principal forms themselves remain the same*,—a surprising conclusion but very easily proved. In other words, however many quantics there may be in the parent system, there is never more than one single set of principal forms of derivatives to it of a given type. A cycle of arbitrarily intercalated pairs of reversals (here of successive i's and j's), by which a type returns to itself, comes under the category, "Verschlingung," or "Knotting" of Gauss, Listing and Tait.

belonging to the type [3: 3, 1; 1, 3]. Write $\alpha = 1$, $\beta = -\rho$; this becomes

$$-a\rho^3 - 3b\rho^2 - 3c\rho - d,$$

of which an image, say J, belonging to the type [3: 3, 1; 3, 1],

$$a\epsilon_3 - 3b\epsilon_2 + 3c\epsilon_1 - d\epsilon_0$$

is the connective of

$$\left\{\begin{array}{l} ax^3 + 3bx^2y + 3cxy^2 + dy^3 \\ \epsilon_0 x^3 + 3\epsilon_1 x^2 y + 3\epsilon_2 xy^2 + \epsilon_3 y^3 \end{array}\right\}.$$

Similarly $\quad (a^2d - 3abc + 2b^3)\beta^3 \ldots + \ldots + (d^2a - 3dbc + 2c^3)\alpha^3$,

say I, belonging to the type [6: 3, 3; 1, 3], will have for a reciprocal

$$(a^2d - 3abc + 2b^3)\epsilon_3 + \ldots (d^2a - 3dbc + 2c^3)\epsilon_0,$$

say J, belonging to the type [6: 3, 3; 3, 1]. The graph of I will be that of Fig. 41 and the graph of J, that of Fig. 42, where I use B and G (the initials of boron and gold, instead of Au for the latter) and H (the initial of hydrogen) to represent the algebraical atoms (that is quantics) of valencies (that is degrees) 3, 3, and 1 respectively. Prefixing Σ to the I graph and substituting G_1, G_2, G_3, the three roots of G, for H, H', H'' and B_1, B_2, B_3 for B, B', B'' we obtain

$$\Sigma (B_1 - B_2)(B_1 - B_3)(B_2 - B_3)(B_1 - G_1)(B_2 - G_2)(B_3 - G_3),$$

which by inspection is the root representative of J, and prefixing Σ to the J graph and substituting H for G, we obtain in like manner

$$\Sigma (B_1 - B_2)^2 (B_2 - B_3)(H - B_1)(H - B_3)^2,$$

as the root representative of I.

It may be observed that Fig. 43 is, algebraically speaking, a pseudograph of J, for its reading would give zero for the value of I.

It follows as an immediate consequence from the preceding extension of the law of images to quantic-systems, that the rule for deducing the first term of the reciprocal to a covariant from that of the covariant itself by writing η_r for α^r holds good as a rule for deducing each term of the one from the corresponding term of the other. To see this we have only to recall that every covariant to a quantic or quantic system may be regarded as an invariant of a new system containing the given quantic or system augmented by a linear quantic whose coefficients are y and $-x$.

Note A to Appendix 2.

Completion of the Theory of Principal Forms.

In the case of a derivative from a system of k parent quantics, it at first sight would seem that since reversion (the act of forming the second image, or process, as we may term it, of double reflexion) may be effected in regard to each system of coefficients separately, the method in the text ought to

furnish in general k distinct systems of principal forms, but this is a mere mirage of the understanding which disappears as soon as the question is submitted to close examination. There is always an unique set of μ forms (μ being the multiplicity of the type) which revert unchanged (barring a numerical multiplier) whichever system of coefficients undergoes double reflexion. But a caution is necessary for the right interpretation of this statement. U, V, W... may be the principal forms in regard to one set of coefficients, $\lambda U+\mu V$, W..., or $\lambda U+\mu V+\nu W$..., where λ, μ, ν are indeterminate, in regard to some other. In any such case we may still say that U, V, W... is the principal system in regard to both sets and so in general. We have an example of this if we take any covariant to a single quantic Q and translate it into an invariant of Q and a linear form L. If U, V, W... are principal forms in respect to Q, $\lambda U+\mu V+\nu W+\ldots$ (that is the absolutely general form of the type) may be easily shown to undergo reversion in respect to L unaltered. U, V, W... may consequently still be seen to be a principal form system in respect to Q and L, as each of these quantities is unaltered by reversion in respect either to Q or to L.

Suppose now a diadelphic system of which U, V are the principal forms quâ one set of coefficients. Let R denote a reversion quâ this set, R' quâ some other set. Let $RU=aU$, $RV=bV$ and suppose $R'U=\alpha U+\beta V$. Then

$$R'RU=a\alpha U+b\beta V \text{ and } RR'U=a\alpha U+b\beta V.$$

But by the nature of the process of reversion $RR'=R'R$; hence $a\beta=b\beta$. If $a=b$, every linear combination of U, V is a principal form quâ R. Hence the principal form quâ the R' set, is such for both sets. But if a is not equal to b, we must have $\beta=0$. Hence U will be a principal form quâ R' as well as R, and the same will be true of V. For if

$$R'V=\gamma U+\delta V$$
$$RR'V=a\gamma U+b\delta V$$
$$R'RV=R'bV=b\gamma U+b\delta V.$$

Therefore $a\gamma=b\gamma$ and $\gamma=0$. Thus U, V will each of them be common as principal forms to each set. I have gone through the same somewhat tedious process of proof for triadelphic forms and with the same result. The very beautiful conclusion follows that whatever the multiplicity of a type may be and whatever number of sets of coefficients it involves, there is always *a single system of principal forms* common to all the sets*.

* Suppose there are k quantics in the parent system and that a derivative type μ is given. Each simple inversion of a pair of permutable indices (i, j) will give rise to a distinct principal equation; there will therefore be k such equations. Let ρ be a root of one of these, σ a root of any other. Then a principal form may be expressed as a linear function of any μ independent special forms connected by coefficients which are rational integer functions of ρ. Hence σ may be found as a rational function of ρ; but in like manner ρ may be found as a rational function of σ. Hence ρ, σ must be related by an equation of the form

$$A\rho\sigma+B\rho+C\sigma+D=0,$$

Note B to Appendix 2.

Additional Illustrations of the Law of Reciprocity.

Acetic aldehyde contains two atoms of carbon, one of oxygen and four of hydrogen*. It thus corresponds to the quartic covariant of a quadratic and quartic, linear and quadratic in respect to the coefficients of the first and second respectively; such a form exists algebraically (*Higher Algebra*, third edition, p. 200) and may easily be proved to be monadelphic. Let us treat it as an invariant: if we were to take for its graph a triangle of which C, C, O were the apices and attach two atoms of hydrogen to each C, the permutation-sum of the product of the differences of the connected letters is zero; this then is a pseudograph. A true graph of it is given by the figure

$$\begin{array}{c} H \cdot C \cdot O \cdot H \\ : \\ H \cdot C \cdot H \end{array} ,$$

where each single dot between two letters means a single bond and the two dots between the upper and lower C's stand for a pair of bonds between them. This belongs to the invariantive type [4, 2; 2, 1; 1, 4: 0], the complete reciprocal to which is [2, 4; 1, 2; 4, 1: 0]. The constitution of the latter in terms of the roots is found from the above graph by writing O for C, C for H and H for O and is accordingly

$$\Sigma (O-O')^2 (O-C)(O-O')(O'-C'')(O'-H)(H-C),$$

where the factor $(O-O')^2$ may be put outside the sign of summation. We may therefore take for its graph a detached molecule of oxygen + a molecule of formic acid, which latter contains two of oxygen, one of carbon and two of hydrogen

$$\begin{array}{c} H \cdot C \cdot O \cdot H \\ : \\ O \end{array}$$

---

and thus we see that all the k principal equations are homographically related, that is, that each may be obtained from any other by a substitution of the form

$$\rho = \frac{C\sigma + D}{A\sigma + B}.$$

In a word, the multiplicity μ (whatever the *diversity* k) determines the number of principal forms; and the k sets of principal multipliers are given by k algebraical equations of the μth degree, homographically transformable into one another.

* I originally took chloral as the subject of this investigation, being interested in examining its algebraical constitution in consequence of having had personal experience of its use as an escharotic. But for greater simplicity I have substituted acetic-aldehyde of which chloral is a third emanant, three hydrogen atoms of the former being replaced by three of chlorine in the latter.

will be a graph of it, from which, turning O into C, H into O and C into H we obtain

$$\Sigma\,(C-O')^2\,(C''-H)\,(C'''-O')\,(C'''-H)\,(H-O)$$

as the value, in terms of its roots, of the algebraical equivalent to acetic aldehyde. The graph for formic acid, it may be noticed, exists algebraically (*Higher Algebra*, p. 300).

Instead of the dissociated molecules of oxygen and formic acid, we may exhibit them combined in the graph

$$\begin{array}{c} C\cdot O\cdot O\cdot O\cdot H \\ : \\ O \end{array}$$

which will give another form to the value of the reciprocal in question, namely

$$\Sigma\,(C-H)^2\,(H-O)\,(H-C')\,(C'-C'')\,(C''-C''')\,(C'''-O)$$

which, not being zero and the type being monadelphic*, must be in a pure numerical ratio to the sum above written.

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

A peculiar case might possibly arise in applying the theory of principal forms to a self-reciprocal type $[w: i, i]$ which it is proper to mention. For greater simplicity suppose the type to be diadelphic and let M, N be forms of the type which satisfy the equations

$$IM=\rho M, \qquad IN=\rho' N\,;$$

* As an exercise the reader may satisfy himself that this type is monadelphic by the direct application of the rule for finding the multiplicity. It corresponds to a quadratic covariant of the type [2, 4 ; 4, 1 : 2], which is the same (introducing the weight $\frac{2\,.\,4+4\,.\,1-2}{2}$ in lieu of the degree) as the type [5 : 2, 4 ; 4, 1] and has the same multiplicity μ by the law of reciprocity as the type [5 : 4, 2 ; 4, 1], namely, the difference between the number of modes of composing 5 and of composing 4 with two of the numbers 0, 1, 2, 3, 4 and with one of a *distinct* set of the same numbers. The arrangements for the weight 5 will be

4. 1 : 0, 4. 0 : 1, 3. 2 : 0, 3. 1 : 1, 3. 0 : 2, 2. 2 : 1, 2. 1 : 2, 2. 0 : 3, 1. 1 : 3, 1. 0 : 4,

and for the weight 4,

4. 0 : 0, 3. 1 : 0, 3. 0 : 1, 2. 2 : 0, 2. 1 : 1, 2. 0 : 2, 1. 1 : 2, 1. 0 : 3, 0. 0 : 4.

The numbers of the combinations in the two sets of arrangements are respectively 10 and 9. Hence $\mu=10-9=1$, or the type is monadelphic. The same result of course follows from the known fundamental scale for a quadro-biquadratic system.

the M and N have tacitly been defined to be the principal forms for such a type. Now in general this definition merges into and is coincident with the definition of principal forms for the general case, namely, that I^2M and I^2N must be multiples of M and N and the latter condition might be substituted for the former. But this is not always true, for if $\rho + \rho' = 0$, we shall have

$$I^2M = \rho^2 M, \qquad I^2N = \rho^2 N,$$

and consequently, $$I^2(M + \lambda N) = \rho^2(M + \lambda N),$$

so that if we were to follow the general definition the principal forms might become indeterminate, whereas by following the definition special to the self-reciprocal case they are determinate. Thus for example, suppose that P, Q, two particular forms of the type, satisfy the equations

$$IP = \rho Q, \qquad IQ = \sigma P;$$

the principal forms will then be

$$\surd(\sigma)\,P + \surd(\rho)\,Q \text{ and } \surd(\sigma)\,P - \surd(\rho)\,Q,$$

and the two principal multipliers become $\surd(\rho\sigma)$ and $-\surd(\rho\sigma)$, so that the principal forms according to the general definition would be indeterminate, but according to the definition proper to self-reciprocal forms strictly determinate.

Let us, as a final example of self-reciprocal type, consider the type [10: 5, 5] which is the same as [5, 5: 5] and corresponds to the covariant of the fifth order in the coefficients and of the fifth degree in the variables to a quintic. This is diadelphic, as may be found by consulting the table of irreducible forms for the quintic, which will show that it can arise only from the multiplication of the parent quantic itself by its quartinvariant or from that of the quadratic quadricovariant by the cubic cubo-covariant or from a linear combination of the two products. But without this, the same conclusion may be arrived at by direct calculation of the value of (10: 5, 5) − (9: 5, 5) and the multiplicity will be found to be 18 − 16, or 2 as premised. Let us take as our special forms,

$$P = (ae - 4bd + 3c^2)(ace + 2bcd - ad^2 - c^3 - b^2e),$$
$$Q = a(a^2f^2 - 10abef + 4acdf + 16ace^2 - 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f$$
$$- 76bcde + 48bd^3 + 48c^3e - 32c^2d^2),$$

where $\dfrac{Q}{a}$ is the quartinvariant J given by Salmon, p. 207 (third edition), being in fact the discriminant of the quadricovariant whose root-differentiant is $ae - 4bd + 3c^2$. Call $\alpha, \beta, \gamma, \delta, \epsilon$ the five roots of the quintic and make $a = 1$. Q contains the term f^2 which is the image of $\alpha^5\beta^5$ which can only arise from combinations of the coefficients into which d, e, f none of them enter. But all the terms of Q contain d, e, or f, moreover P has no term containing f^2, therefore IQ does not contain Q but is simply a multiple of P. Again ce^2, which enters into P, is the image of combinations of the form

$\alpha^2\beta^4\gamma^4$, and the only term in Q which can give rise to such combinations is $-32c^2d^2$, or

$$-\frac{32}{10^4}(\Sigma\alpha\beta)^2(\Sigma\alpha\beta\gamma)^2,$$

and each such combination will have unity for its coefficient and their number is 30. Hence

$$IQ = -\frac{30 \,.\, 32}{10000}P = -\frac{12}{125}P.$$

Again, Q contains $-10bef$, and bef is the image of such root-combinations as $\alpha^5\beta^4\gamma$ (60 in number) the only terms in P capable of producing which are $10bc^3d$ and $-3c^5$ or $\frac{1}{5000}\Sigma\alpha(\Sigma\alpha\beta)^3\Sigma\alpha\beta\gamma - \frac{3}{100000}(\Sigma\alpha\beta)^5$. And bef does not appear in P, hence one part of IP will be

$$\left(\frac{60}{-50000} + \frac{3 \,.\, 5 \,.\, 60}{1000000}\right)Q, \quad \text{or } -\frac{3}{10000}Q.$$

Again, ce^2 is the image of such combinations as $\alpha^4\beta^4\gamma^2$ (30 in number) and the only terms in P giving rise to such are $-3c^5 - 8b^2cd^2 + 10bc^3d - 3c^2d^2$; $-3c^5$ is $-\frac{3}{100000}(\Sigma\alpha\beta)^5$ and will give rise to $-\frac{3 \,.\, 20 \,.\, 30}{100000}ce^2$ in IP; $-8b^2cd^2$ is $-\frac{8}{25000}(\Sigma\alpha)^2(\Sigma\alpha\beta)(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{2 \,.\, 8 \,.\, 30}{25000}ce^2$ in IP; $10bc^3d$ is $\frac{10}{50000}\Sigma\alpha(\Sigma\alpha\beta)^3\Sigma\alpha\beta\gamma$ and will give rise to $\frac{7 \,.\, 10 \,.\, 30}{50000}ce^2$ in IP; $-3c^2d^2$ is $-\frac{3}{10000}(\Sigma\alpha\beta)^2(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{3 \,.\, 30}{10000}ce^2$ in IP. Hence the total coefficient of ce^2 in IP is

$$-\frac{9}{500} - \frac{12}{625} + \frac{21}{500} - \frac{9}{1000} = \frac{-90 - 96 + 210 - 45}{5000} = -\frac{21}{5000},$$

and consequently, since P contains the term ce^2 and Q the term $16ce^2$, if $IP = \theta P - \frac{3}{10000}Q$,

$$\theta - \frac{3 \,.\, 16}{10000} = -\frac{21}{5000}, \text{ so that } \theta = \frac{3}{5000},$$

and therefore
$$IP = \frac{3}{5000}P - \frac{3}{10000}Q,$$

and thus the equation for finding the principal multipliers ρ is

$$\begin{vmatrix} \frac{3}{5000} - \rho, & -\frac{3}{10000} \\ -\frac{12}{125}, & -\rho, \end{vmatrix} = 0,$$

or, if
$$\rho = \frac{3\sigma}{10000}, \quad \begin{vmatrix} 2-\sigma, & -1 \\ -320, & -\sigma \end{vmatrix} = 0.$$

Thus $\sigma^2 - 2\sigma - 320 = 0$, the roots of which are irrational. I have thought it advisable to set out the work in this example with some explicitness in order to remove an impression that might otherwise arise from the examples which precede, that the principal multipliers and consequently the principal forms, for self-reciprocal types, necessarily contain only rational numbers.

The work is very much longer for the case of non-self-reciprocal types. The simplest example of such that presents itself to my mind is that of the sextinvariant of a quartic and the quartinvariant of a sextic, for either of which the type is diadelphic. The discussion of this case forms the subject of the annexed Note, for all the calculations of which I am indebted to the labour and skill of Mr F. Franklin, Fellow of Johns Hopkins University. For the sake of brevity the steps of the work have been suppressed and only the final results set out.

Note C to Appendix 2.

On the Principal Forms of the General Sextinvariant to a Quartic and Quartinvariant to a Sextic.

Let

$$L = (ae - 4bd + 3c^2)^3 = \left[\frac{1}{2^3 \cdot 3} \Sigma (\alpha - \beta)^2 (\gamma - \delta)^2\right]^3,$$

$$M = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}^2 = (ace + 2bcd - ad^2 - b^2e - c^3)^2 = \left[\frac{1}{2^4 \cdot 3^3} \Sigma (\alpha - \beta)^2 (\gamma - \delta)^2 (\alpha - \gamma)(\beta - \delta)\right]^2,$$

$$P = (ag - 6bf + 15ce - 10d^2)^2 = \left[-\frac{1}{2^4 \cdot 3 \cdot 5} \Sigma (\alpha - \beta)^2 (\gamma - \delta)^2 (\epsilon - \phi)^2\right]^2,$$

$$Q^* = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = \begin{cases} aceg - acf^2 - ad^2g + 2adef \\ - ae^3 - b^2eg + b^2f^2 + 2bcdg \\ - 2bcef - 2bd^2f + 2bde^2 - c^3g \\ + 2c^2df + c^2e^2 - 3cd^2e + d^4 \end{cases}$$

$$= \frac{1}{2^5 \cdot 3^3 \cdot 5^3} \Sigma (\alpha - \beta)^4 (\gamma - \delta)^4 (\epsilon - \phi)^4 - \frac{71}{2^{10} \cdot 3^4 \cdot 5^4}\left[\Sigma (\alpha - \beta)^2 (\gamma - \delta)^2 (\epsilon - \phi)^2\right]^2.$$

* M. Faà de Bruno, in the tables at the end of his *Théorie des Formes Binaires*, designates Q and $\Sigma (\alpha - \beta)^4 (\gamma - \delta)^4 (\epsilon - \phi)^4$ by the same symbol I_4; a misleading circumstance which gave rise in this instance, and might in others to a large amount of useless labour. As can easily be seen from the above, the true value of $\Sigma (\alpha - \beta)^4 (\gamma - \delta)^4 (\epsilon - \phi)^4$ is

$$120\,(71P + 900Q) = 120\,(71a^2g^2 - 852abfg + 3030aceg - 900b^2eg - 2320ad^2g + 1800bcdg - 900c^3g - 900acf^2 + 3456b^2f^2 + 1800adef - 14580bcef + 6720bd^2f + 1800c^2df - 900ae^3 + 1800bde^2 + 16875c^2e^2 - 24000cd^2e + 8000d^4).$$

It should also be observed that in the expression for Q (the catalecticant) given in the same table, the signs of the terms $-2bd^2f + 2bde^2$ have been interchanged.

Then
$$IL = \frac{P - 6Q}{2^5 \cdot 3^2}, \qquad IM = \frac{P - 33Q}{6^5},$$
$$IP = \frac{L + 2M}{2^4 \cdot 5}, \qquad IQ = \frac{9L - 142M}{2^6 \cdot 3^2 \cdot 5^3},$$
$$I^2L = \frac{7614L + 23868M}{2^{11} \cdot 3^6 \cdot 5^3}, \qquad I^2M = \frac{201L + 2162M}{2^{11} \cdot 3^6 \cdot 5^3}.$$

In order that $\lambda L + \mu M$ shall be a principal form we must have
$$(7614 - 2^{11} \cdot 3^6 \cdot 5^3\rho)\lambda + 201\mu = 0,$$
$$23868\lambda + (2162 - 2^{11} \cdot 3^6 \cdot 5^3\rho)\mu = 0,$$
$$\begin{vmatrix} 7614 - 2^{11} \cdot 3^6 \cdot 5^3\rho, & 201 \\ 23868 \quad , & 2162 - 2^{11} \cdot 3^6 \cdot 5^3\rho \end{vmatrix} = 0,$$
or, putting $\sigma = 2^8 \cdot 3^6 \cdot 5^3\rho$,
$$\sigma^2 - 1222\sigma + 182250 = 0,$$
where it may perhaps be worth noticing that the last term is $2 \cdot 3^6 \cdot 5^3$ and the coefficient of the second term $2 \cdot 13 \cdot 47$. We obtain from this equation
$$\rho = \frac{611 \pm \sqrt{(191071)}}{2^8 \cdot 3^6 \cdot 5^3} *.$$

The principal forms in L and M will then be found to be
$$201L + \{-2726 + 8\sqrt{(191071)}\}M, \qquad 201L + \{-2726 - 8\sqrt{(191071)}\}M;$$
and those in P and Q
$$101P + \{-11436 + 24\sqrt{(191071)}\}Q, \qquad 101P + \{-11436 - 24\sqrt{(191071)}\}Q.$$
Or, if we please, the principal forms in the two cases may be taken as the factors of
$$201L^2 - 5452LM - 23868M^2 \text{ and } 101P^2 - 22872PQ + 205200Q^2$$
respectively†. The question, what reduced quadratic forms can appear in the theory of diadelphic types, may one day or another become the subject of *à priori* investigation and form a new connecting link between the Calculus of Invariants and the Theory of Numbers. The linear functions of L and M and of P and Q, corresponding to the reduced forms of the above expressions might perhaps be termed the principal *rational* forms of the two types respectively.

* The number under the radical sign is, I believe, a prime number, but I have not within reach the tables necessary for verifying this. Professor Newcomb, by an exceedingly ingenious combination of a table of squares with Crelle's table of multipliers (a real stroke of genius), was able to ascertain by an inspection (the work of a few minutes) that 191071, if not a prime number, must contain a factor not greater than a certain moderate sized integer (137 if my memory serves me right) which reduces the trials necessary to be made to a very small compass.

† These are reducible to
$(201, 68, -60800\between L', M)^2$, $(101, -23, -1089667\between P', Q)^2$, where $L' = L - 14M$, $P' = P - 113Q$.

It may be well to notice that if $I^2U = \rho U$, then $I^2 \cdot IU = I \cdot I^2U = \rho IU$, and consequently the principal forms for two reciprocal types are images respectively of one another, and the principal multipliers are the same for the two systems.

NOTE D TO APPENDIX 2.

On the Probable Relation of the Skew Invariants of Binary Quintics and Sextics to one another and to the Skew Invariant of the same Weight of the Binary Nonic.

The law of reciprocity extended, as it has already been in these pages, to systems of quantics, admits of an additional important generalization.

We know that Regnault's law of substitution holds good for algebraical forms, and in fact when transferred to the algebraical sphere becomes identical with the method which I believe I was the first to employ (now familiar to algebraists through the use made of it by Professors Clebsch and Gordan) to which I gave the name of emanation (Faà de Bruno, p. 198).

The principle, stated in chemico-algebraical language, is that in algebraical compounds any number of atoms of a given valence may be replaced by the same number of *new* equi-valent atoms. [In algebra it is essential to lay a peculiar stress on the word *new*; for if the substituted atoms should be homonymous with the remaining atoms, there is a possibility of the transformed compound reducing to zero. As for instance in the algebraical compound $ab' - a'b$ (the representative, say, of potassic iodide), if the atom of potassium should be changed into another of iodine (or *vice versa*), the compound, viewed algebraically, would disappear.]

The law of reciprocity as I have previously given it, translated into chemico-algebraical language amounts to saying that the total number of atoms of one kind (say m n-valent of one kind) may be replaced by n m-valent atoms of another kind; but by applying the rule of substitution first and then that of reciprocity we may see that the condition of *totality* may be done away with and the proposition reduced to the simplified form that in any algebraical compound *m n-valent atoms may be replaced by n m-valent ones.* Whether this law has any application in the chemical sphere, I must leave to chemists to determine.

In addition to the well known fact that a quintic possesses an invariant of the 18th order, and a sextic one of the 15th order, having obtained a complete scheme of the irreducible invariants for the binary quantic of the 10th degree, I was put in possession of the new fact that this last form

possesses an invariant of the 9th order and consequently that the nonic possesses an invariant of the 10th order*.

Now the weight of each of these skew invariants is the same number 45, and I was thus led to suspect that they coexisted in virtue of some secret connexion. What that connexion is I think that I am now (very unexpectedly) in a position to explain and to show (with a high degree of probability) how the values of these three invariants may be actually deduced and calculated from one another. This follows as a consequence of the combined laws of reciprocity and substitution otherwise called emanation. For suppose we have an invariant of a quantic of the mth degree, of the order np in the coefficients. By the principle of emanation we may transform this into an invariant to a system of n quantics, each of the degree m and of the order p in each set of coefficients, and by the generalized law of reciprocity this may be again transformed into an invariant to a system of n quantics, each of degree p and of the order m in each set of coefficients. If now finally these n quantics be all made identical with one another, then the transformed invariant, *provided it does not vanish,* becomes an invariant of the order mn to a single quantic of the degree p, and accordingly we may pass in certain

* I have calculated, with the kind assistance of Mr Halsted, the expression in its canonical form of the generating function to a binary quantic of the 10th degree. The coefficient of t^m in this fraction developed, represents the number of parameters in the general invariant of the mth order of the given decadic. Its denominator is

$$(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)(1-t^9)(1-t^{10})(1-t^{14})$$

and its numerator is the rational integer function

$$1+2t^6+\ldots+2t^{42}+t^{48},$$

the successive coefficients being

1, 0, 0, 0, 0, 0, 2, 0, 4, 2, 7, 6, 15, 13, 16, 25, 22, 31, 34, 40, 41, 47, 46, 49, 48, 49, 46, 47, 41, 40, 34, 31, 22, 25, 16, 13, 15, 6, 7, 2, 4, 0, 2, 0, 0, 0, 0, 0, 1,

showing that the primary fundamental invariants are of the orders 2, 4, 6, 6, 8, 9, 10, 14, and that (by the law of "Tamisage" *anglice* siftage) the secondary (or as they might be better termed the auxiliary) ones are of the orders 6, 8, 9, 10, 11, 12, 13, 14, 15, 17 taken 2, 4, 2, 7, 6, 12, 13, 18, 21, 11 times respectively. Any other invariant of the decadic can be represented as a linear function of a limited number of combinations of the secondaries, having for its coefficients some combination of powers of the primaries.

Suppose that the same numerical order occurs among the primaries and secondaries, as for example 6, which occurs twice among the former and twice among the latter. This will indicate in the first place that, calling A and B the quadric and quartic invariants, the general sextic one will be of the form

$$\lambda A^3+\mu AB+\nu_1Q_1+\nu_2Q_2+\nu_3Q_3+\nu_4Q_4$$

and that any two independent special values of $\nu_1Q_1+\nu_2Q_2+\nu_3Q_3+\nu_4Q_4$ may be taken as primaries and any other independent two as secondaries, and so in general; I mention this to prevent the false suggestion, which might otherwise arise, that the secondaries and primaries are different in internal constitution. This remark receives a beautiful illustration in an algebraical theory (recently developed by me) of chemical isomerism, which gives rise to a generating function precisely similar in character to that applicable to in- and co-variants and is subject to a similar law of interpretation, graphs taking the place of algebraical forms, and atomicules and the numbers of grouped atoms, of degrees and orders.

cases from the type $[m, np: 0]$ to the type $[p, mn: 0]$. So in all probability we may pass from the type $[5, 18: 0]$ to the type $[6, 15: 0]$ and to the type $[9, 10: 0]$. As there is only one invariant of the type $[6, 15: 0]$, or of the type $[9, 10: 0]$, it follows that, if the passage from type to type is real and not nugatory, the three invariants of these second types may be deduced, any one from any other, by the explicit processes above described. There is nothing at all doubtful in the course of the transformation except what arises from the possibility that in the last step of it the effect of rendering identical the different sets of coefficients—that is of finding the counter-emanant, so to say, of the invariant containing n sets of variables—may be to render the whole expression null. This of course would happen if we attempted to pass from the type $[5, 18: 0]$ to the type $[3, 30: 0]$, or to the type $[2, 45: 0]$, which we know are void of forms. But there is no reason why we should expect this to happen when we pass from the given type to other types known to contain one or more forms. It would require no impracticable amount of labour to actually verify the fact of the transformation being effectual between the skew invariants of the sextic and quintic forms. The survival of a *single* known term in either of them, in the process of attempting to deduce it from the other, would be sufficient to establish the effectualness of the method, and that it will be found to be effectual, for reasons too long to dwell upon here, I scarcely entertain a doubt. The process to be employed may be summarily comprehended under the three rubrics of diversification, reciprocation and unification. The first is one of differentiation alone; the second involves the expansion of functions of the coefficients of an equation in terms of roots and the substitution of η_i for α^i; the third consists merely in replacing distinct sets of letters (η) by a single set. In practice the two latter processes would be of course combined into one. It will be instructive to consider some simple example of this method of transformation of types.

Let us take $(ac - b^2)^3$ regarded as belonging to the type $[2, 6: 0]$. I shall show how to pass from this to a form of the type $[3, 4: 0]$. Taking a third emanant of the given form, that is the result of the operation upon it of $\frac{1}{1\cdot 2\cdot 3}(a'\delta_a + b'\delta_b)^3$, we obtain

$$(ac' + a'c - 2bb')^3 + 2(ac - b^2)(a'c' - b'^2)(ac' + a'c - 2bb').$$

Let us call $\alpha, \beta, \alpha', \beta'$ the roots of the two forms $[1, b, c], [1, b', c']$ respectively; then the emanant last found (multiplied by 8) becomes

$$(2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta')$$
$$\{(2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta')^2 + (\alpha - \beta)^2 \cdot (\alpha' - \beta')^2\}.$$

After performing all the multiplications and introducing the zero powers

of $\alpha, \alpha', \beta, \beta'$ in such terms as do not contain one or more of these letters, all that remains is to substitute

$$\begin{aligned} \alpha^0 = \alpha'^0 = \beta^0 = \beta'^0 &= \quad a, \\ \alpha = \alpha' = \beta = \beta' &= -b, \\ \alpha^2 = \alpha'^2 = \beta^2 = \beta'^2 &= \quad c, \\ \alpha^3 = \alpha'^3 = \beta^3 = \beta'^3 &= -d, \end{aligned}$$

the letters a, b, c, d for greater simplicity being used instead of $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$, that is $\eta_0, -\eta_1, \eta_2, -\eta_3$. The result will not vanish. To show this consider the group of terms which change into a^2d^2. These are the binary combinations of $\alpha^3, \alpha'^3, \beta^3, \beta'^3$. $2\alpha\beta$ and $2\alpha'\beta'$ in the first factor give rise to $8\alpha^3\beta^3$, $8\alpha'^3\beta'^3$ and the remaining four terms to $-2\alpha^3\alpha'^3$, $-2\alpha^3\beta'^3$, $-2\beta^3\alpha'^3$, $-2\beta^3\beta'^3$ respectively. Hence the term a^2d^2 will survive with the multiplier $8+8-2-2-2-2$, that is, 8. So again the only terms introducing ac^3 will be the ternary combinations of $\alpha^2, \alpha'^2, \beta^2, \beta'^2$. $2\alpha\beta$ and $2\alpha'\beta'$ will be found to produce as many positive as negative terms of this kind, but $-\alpha\alpha'$ will produce $4\alpha^2\alpha'^2\beta^2 + 4\alpha^2\beta^2\beta'^2$, giving rise to $8ac^3$, and as the same will be true for $-\alpha\beta', -\beta\alpha', -\beta\beta'$, we see that $32ac^3$ will emerge in the result. Hence the given invariant becomes converted into

$$(a^2d^2 + 4ac^3 + \ldots),$$

that is the discriminant of the cubic whose type is [3, 4: 0] as was to be shown.

I think it is little doubtful that wherever there exist forms contained under each of two types, the product of whose rank and order is identical, we may pass from the one to the other by means of the combined processes of emanation and reciprocation, as in the foregoing example*. [The case is much the same as with transvection. That process may produce a null form, but any actually existent form may be produced by it and exhibited as a transvect.] To pass from Hermite's to Cayley's skew form, we must first by emanation change [5, 18: 0] into [5, 6; 5, 6; 5, 6: 0] and then this latter into [6, 15: 0]; by means of the process last exemplified.

* Call

$$(b^2-ac)^3 = A,\ a^2d^2+4ac^3+\ldots = B,\ a'\delta_a + b'\delta_b + c'\delta_c = E,\ a\delta_{a'} + b\delta_{b'} + c\delta_{c'} + d\delta_{d'} = H^{-1}.$$

Then it follows from the text that

$$B = \tfrac{1}{12} H^{-2} I E^3 A,$$

where it may be observed that E^3A is diadelphic, for it will be proved that (6: 3, 2; 3, 2) = 16, and (5: 3, 2; 3, 2) = 14, so that any form whatever coming under the same type as E^3A is a linear function of $(ac'+a'c-2bb')^3$ and $(ac'+a'c-2bb')(ac-b^2)(a'c'-b'^2)$, say L and M (whose difference, $L-M$, is $\frac{1}{6}E^3A$), and operated on by $H^{-2}I$ would produce a multiple of B (whose type is monadelphic) with the sole exception of $\lambda L - 2\mu M$, the result of operating upon which would be zero. Similarly we may see that in any given case the chances are infinitely in favour of the expectation that the process will *not* be nugatory by which it has been shown we may pass from one known type $[m, np: 0]$ to another known one $[p, nm: 0]$.

APPENDIX 3.

On Clebsch's Theory of the "Einfachstes System associirter Formen" (*vide Binären Formen, p.* 330) and its Generalization.

Let $(a, b, c, \ldots k, l \between x, y)^n$ be any binary quantic. Let the provector symbol $(l\delta_k + 2k\delta_h + 3h\delta_g + \ldots)$ be denoted by Ω, and the revector symbol $(a\delta_b + 2b\delta_c + 3c\delta_d + \ldots)$ by $\mho$. Let Q_{2i} represent the quadrinvariant of the above form when $n = 2i$. Now let Ω and $\mho$ be made to comprise the $2i+1$ letters $a, b, c, \ldots l, m$; then $a\Omega Q_{2i} - 2bQ_{2i}$* will be nullified by the operation of $\mho$ and will therefore be a cubinvariant for the case of $n = 2i+1$, which we may call Q_{2i+1}. Also let $Q_0 = a$; then $Q_0, Q_1, Q_2, \ldots Q_\mu$ will be differentiants to all binary quantics of degree equal to or greater than μ. The above I call basic differentiants. Their distinguishing characteristic is that the highest letter in each of them enters into it only in the first degree multiplied by a or by a^2 and by no other letter. Now let D be any given differentiant of degree μ and for the moment make $a = 1$. Then it is obvious that D may be expressed—by means of successive substitutions of its ultimate, its penultimate, its antepenultimate, etc. letters up to c inclusive, in terms of the corresponding basic differentiants and the anterior letters,—as a rational integer function of $Q_1, Q_2, \ldots Q_\mu, b$; or, restoring to a its general value, will be a rational integer function of $Q_0, Q_1, Q_2, \ldots Q_\mu, b$, say F, divided by a power of a. But I say that b will have disappeared in the process. For $\mho D = 0$; and $\mho Q_0 = 0$, $\mho Q_1 = 0 \ldots \mho Q_\mu = 0$. Hence, regarding each Q as a constant, $\left(a\dfrac{d}{db}\right) F = 0$, or F does not contain b.

Again, suppose we take a system of two quantics and let $Q_0, Q_1, \ldots Q_\mu$ be the basic differentiants of the one, $Q_0', Q_1', \ldots Q_\nu'$ of the other, and let D be any differentiant of the system. Then by the same method as before we shall find

$$D = \frac{F(Q_0, Q_2 \ldots Q_\mu : Q_0', Q_2' \ldots Q_\nu' : b, b')}{a^m \cdot a'^n}.$$

Also each Q will be nullified by $\mho$, and each Q' by $\mho'$, and therefore each Q and Q' as well as D will be nullified by the operator $\mho + \mho'$. Hence we shall have

$$\left(a\frac{d}{db} + a'\frac{d}{db'}\right) F = 0,$$

or

$$F = \phi\,(ab' - a'b),$$

* For by a well-known formula if D is a differentiant in x of the type $[w : i, j]$,

$$\mho\Omega D = (ij - 2w)\, D.$$

Consequently when Q_{2i} is regarded as a differentiant in x of the type $[2i : 2i+1, 2]$

$$\mho\Omega Q_{2i} = Q_{2i} \text{ also } \mho Q_{2i} = 0 \text{ and } \mho b = a.$$

Hence

$$\mho\,(a\Omega Q_{2i} - 2bQ_{2i}) = 0.$$

ϕ being a rational integral form of function. In like manner for a system of three quantics, regarding the several sets of its basic differentiants as constant, we shall have

$$F = \phi\,(ab' - a'b : ac' - a'c : bc' - b'c),$$

where ϕ is a rational integral form of function, or

$$F = \psi\,(ab' - a'b : ac' - a'c : a,\ a'),$$

and so in general. Hence, remembering that any relation between differentiants must continue to subsist between the covariants of which they are the roots, and now, understanding by base forms the complete covariants of which the basic coefficients are the roots, we may pass from differentiants to in- or co-variants and obtain the following theorems.

(1) For a single quantic of degree i, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of its i base forms and whose denominator is a power of the quantic. This is Clebsch's theorem.

(2) For a system of quantics, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of the separate base forms of its several quantics and of any complete system of $(\mu - 1)$ independent Jacobians of the quantics taken in pairs, and whose denominator is a product of powers of the quantics of the system.

Also it will be observed that these theorems will continue to subsist when the base forms have for their roots in lieu of the basic differentiants, as above defined, any ascending scale of differentiants in which the letters enter successively one at a time and each letter on its first appearance figures only in the first degree and combined exclusively with powers of a.

On the theory of basic forms may be grounded a method for obtaining, *in propriâ personâ*, the fundamental in- and co-variants to a quantic or system of quantics in regular succession, by a process which continues so long as there are many more to be elicited and comes to a self-manifesting end as soon as the last irreducible form has been obtained, like an air pump that refuses to act as soon as the exhaustion has become complete. In a word, the cataloguing of the irreducible in- or co-variants is transferred to the province of, and becomes a problem in, ordinary algebra.

I have previously observed that any expression which represents a differentiant in regard to a quantic of a given degree necessarily does the same for quantics of all higher degrees. And I may take this occasion to remark, or to repeat, that a differentiant may be irreducible in respect to the quantic of minimum degree to which it can be referred, and yet not so for quantics of higher degrees. Thus, if we take the expression

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

this referred to a cubic is irreducible (as is well known), but regarded as a differentiant of a quartic or higher degreed quantic, is reducible, being in fact identical with

$$(ac-b^2)(ae-4bd+3c^2)-a\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

Let us suppose a linear function $yu-xv$ combined with a quantic into a system. Then it follows as a corollary from (2) at [p. 200], that if the quantic belongs to the form $(a, b, c, \ldots l \between u, v)^i$, or say more simply to the form $[a, b, c, \ldots l]$ any covariant of such quantic multiplied by a suitable power of a will be a function of y, $ax+by$ and of the differentiants, or in a word, every covariant of the quantic expressed as a function of x and $ax+by$ will have no coefficients but what are differentiants, or to use Professor Cayley's term, semi-invariants. Thus, for example, the Hessian of the cubic $(a, b, c, d \between x, y)^3$ may be put under the form

$$\frac{1}{a^2}\left\{(ac-b^2)(ax+by)^2+(a^2d-3abc+2b^3)(ax+by)\,y+(ac-b^2)^2\,y^2\right\}.$$

So it will be found that the Hessian of the quintic, namely

$$(ae-4bc+3c^2)\,x^2+(af-3be+2cd)\,xy+(bf-4cd+3d^2)\,y^2$$

on writing $ax+by=X$, becomes

$$\frac{1}{a^2}\Big\{(ae-4bc+3c^2)\,X^2+(a^2f-5abe+2acd+8b^2d-6bc^2)\,Xy$$
$$-\left[(ac-b^2)(ae-4bd+3c^2)+3a\,(ace+2bcd-ad^2-b^2e-c^3)\right]y^2\Big\},$$

where all the coefficients are semi-invariants-in-x, the second coefficient being one of the basic differentiants and the latter part of the third coefficient, the catalecticant

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

and so more generally, it may be shown to follow from (2), that if there be any number of binary quantics

$$[a, b, c \ldots],\ \ [a', b', c', \ldots],\ \ [a'', b'', c'', \ldots],$$

every covariant of such system, expressed as a function of y and of *any one* of the quantics

$$ax+by,\ a'x+b'y,\ \ldots$$

chosen at will, has differentiants-in-x exclusively for its coefficients.

It is easy to express the base-covariants in terms of the roots. Those of weight $2n$ and order 2 will be of the form

$$\Sigma F(a_1, a_2, a_3, \ldots a_{2n})(x-a_{2n+1})^2(x-a_{2n+2})^2 \ldots$$

where F may be expressed as

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \ldots (a_{2n-1} - a_{2n})^2,$$

or,
$$(a_1 - a_2)(a_2 - a_3)(a_3 - a_4) \ldots (a_{2n-1} - a_{2n})(a_{2n} - a_1),$$

or under a variety of other forms all equal to a numerical factor près; for the type $[2n: 2n, 2]$ and the more general one $[2n: 2n + \nu, 2]$ are monadelphic. And again those of the weight $2n + 1$ and order 3 may take, or at all events be replaced by, the form

$$\Sigma \{(a_1 - a_2)(a_2 - a_3) \ldots (a_{2n-1} - a_{2n})(a_{2n} - a_1)(a_1 - a_{2n+1})(x - a_1)(x - a_2) \ldots (x - a_{2n+1})(x - a_{2n+2})^3 (x - a_{2n+3})^3 \ldots\}.$$

It is proper to notice that the type $[2n + 1 : 2n + 1 + \nu\ ;\ 3]$ is only monadelphic so long as $2n + 1$ is less than 9, so that we cannot, without an investigation which might be tedious, determine whether the above representation coincides with the basic forms of the third order in the coefficients adopted in [p. 199]; but such investigation would be a work of supererogation, for the only *material* character for any of the base-covariants in question to possess is, that its root differentiant-in-x shall be not higher than of the third order in the coefficients and shall contain the element ϵ_{2n+1}. Any formula having this property (which is enjoyed by the root function above given) is just as good as any other for the purposes of this theory*.

It will be seen to follow from the theorem I have given for differentiants from which Clebsch's follows as an immediate consequence, that all the permutation-sums of any rational integer function of the differences of the roots of an algebraical equation of the nth degree are rational integer functions of $(n - 1)$ of them of the second and third order alternately; so, for example, all the coefficients in Lagrange's equations to the squares of the differences of the roots of an algebraical equation in its ordinary form are rational integer

* Writing the type under the form $[2n+1: 2n+1+\nu,\ 3]$, the degree of the corresponding covariant in the variables is $2n+1+3\nu$, which is the degree in x of the symmetrical function assumed in the text; also each letter in this function occurs 3 times agreeing with the order 3 of the type, and the number of factors in the coefficient of the highest power of x is $2n+1$, which is right for the weight. It is obvious also by inspection that the product $a_1 . a_2 \ldots a_{2n+1}$ will arise from each term of the assumed symbolical function affected always with the same sign, so that ϵ_{2n+1} will occur (as required) in its expression in terms of the coefficients. Of course all the same conclusions will apply if in the formula

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \ldots (a_{2n-1} - a_{2n})^2$$

is substituted in lieu of

$$(a_1 - a_2)(a_2 - a_3) \ldots (a_{2n-1} - a_{2n})(a_{2n} - a_1).$$

That the type to which Q_{2n+1} belongs is non-monadelphic from and after $2n+1=9$ is obvious from the fact that that type, when the degree of the parent quantic is made a minimum, is of the form $[2n+1: 2n+1,\ 3]$, the multiplicity of which is the same as that of $[2n+1: 3,\ 2n+1]$, or set out in full $[2n+1: 3,\ 2n+1: 2n+1]$; but cubics include covariants of orders and degrees $2: 2$ and $3: 3$ among their fundamental forms, and $9: 9$ can be formed either by taking a triplication of $3: 3$, or by combining $3: 3$ with a triplication of $2: 2$, so that when $2n+1=9$ the type is diadelphic, and *a fortiori*, it is non-monadelphic for values of $2n+1$ superior to 9.

functions of $(n-1)$ known quantities. Thus, for instance, the equation to the squares of the differences of a cubic equation will be

$$\rho^3 + 18(b^2 - ac)\rho^2 + 81(b^2 - ac)^2 + 27\Delta = 0,$$

where the coefficients are given in terms of two differentiants $(b^2 - ac)$ and Δ.

Throughout this paper the perspicuity of expression has been considerably marred by want of a complete nomenclature which the theory of graphs and types necessarily calls for and which I shall hereafter employ whenever I may have occasion to revert to the subject. It is as follows:

In the first place, w, the weight in respect to the selected variable, and j, the order in the coefficients, are terms well understood and need no change or further illustration; i, the degree of the parent quantic, I shall hereafter call the *rank* of the type, $ij - 2w$ which becomes the degree of a covariant got by expanding the differentiant of type $[w: i, j]$ may be called the *grade*. The order and rank may be termed collectively the *permutable indices*.

When a differentiant is given algebraically its weight and order are given but *not* its rank; in addition to the weight and order a third number which may be called the *range* (and which I shall denote by a Greek ϵ) is given, being the number less 1 of the letters which enter into it. The relation between *rank* and *range* is one of inequality. The former may be equal to, or greater than, but not less than the latter.

The multiplicity of the type to which a given differentiant belongs is a function of the *weight*, *order* and *rank* and is consequently not known until the *rank* is assigned. Thus, for example $(ac - b^2)^2$, considered as having the lowest possible rank, namely 2 (the *range*) is monadelphic; its type is then $[2: 2, 4]$, but if the rank 4 be assigned to it so that its type is $[2: 4, 4]$, it becomes diadelphic. We have then, in general, 6 characters (not all independent) appertaining to a differentiant, namely, *weight, rank, order, grade, range* and *multiplicity*. The theory of types has never hitherto formed the subject of distinct contemplation, and that is why the necessity for the use of some of the above terms has not been previously felt. But it will have been observed that throughout the preceding memoir it has forced itself upon our notice, and in particular, that it is impossible to go to the bottom of the so-called law of reciprocity or that of the radical representation of forms without keeping in view the question of type and multiplicity.

I have also to remark that since the preceding matter was completed I have been surprised to learn that recent chemical research favours the notion of simple elements (hydrogen atoms in special) being distinguishable from each other in chemical composition. If this view is confirmed, the discrepancy, which I have pointed to, between the known conditions for the existence of algebraical graphs and the unknown natural laws which govern the production of chemical substances may become partially or wholly

obliterated, so that, for example, the hydrogen molecule and the extended derivatives from marsh gas may exist in accordance with, and not in contradiction to, algebraical law, and thus it is possible to conceive that all the phenomena of chemistry and algebra may ultimately be shown to be identical.

Since the above matter was sent to press I have been led to study algebraically what may be termed the direct problem of isomerism, that is to say the determination of the number of combinations subject to given conditions that can be formed between the constituents of groups each containing a given number of equivalent chemical atoms, the valences of the several groups being either independent or given linear functions of a certain number of independent parameters. In this problem the numbers of atoms are given and the valences left indeterminate. In the inverse problem the valences are given and the numbers left indeterminate.

The problem of the enumeration of the saturated hydro-carbons, investigated by Professor Cayley, is a simple example of the inverse problem. The direct problem admits of a uniform and unfailing method of solution by generating functions, the exposition of which may probably form the subject of an additional Appendix in the following number*. This method is

* The principle employed in this method leads to the following theorem only a particular case of which comes into play in the general partition problem which covers the ground occupied by the allied invariantive and isomeric theories. Let there be given a product of a limited number of rational functions of

$$u_1^{a_1} . u_2^{a_2} \dots u_i^{a_i}; \quad u_1^{a_1'} . u_2^{a_2'} \dots u_i^{a_i'}; \text{ etc., etc.,}$$

where all the indices are *positive or negative* integers, and let $\mu_1, \mu_2, \dots \mu_i$ be given linear functions of $\nu_1, \nu_2, \dots \nu_j$ (j being not greater than i), then it is always possible to find a limited product of rational functions of

$$v_1^{\beta_1} . v_2^{\beta_2} \dots v_j^{\beta_j}; \quad v_1^{\beta_1'} . v_2^{\beta_2'} \dots v_j^{\beta_j'}; \text{ etc., etc.,}$$

where the indices are exclusively *positive*, such that the coefficient of $v_1^{\nu_1} . v_2^{\nu_2} \dots v_j^{\nu_j}$, in their product developed according to ascending powers of $v_1, v_2, \dots v_j$, shall be the same as the coefficient of $u_1^{\mu_1} u_2^{\mu_2} \dots u_i^{\mu_i}$ in the original product developed according to ascending powers of $u_1, u_2, \dots u_i$. Previous to the discovery of this principle the problem of isomerism, now completely solved potentially for the direct case, must have remained unattackable by any existing methods, such for example as were known to Euler, the inventor of the application of the method of generating functions to the theory of partitions. It renders supererogatory a large part of the methods devised by myself for the treatment of the problem of compound partitions contained in the printed notes of my lectures on Partitions, delivered at King's College, London, in the year 1859†. As an example of the direct problem of isomerism, suppose that three atoms of the same valence j are to combine with ϵ atoms of hydrogen which do not combine *inter se*; then the number of combinations which can be so formed is the coefficient of $a^j x^\epsilon$ in the development of the generating function

$$\frac{1 + ax + a^2x^2}{(1-a^2)(1-ax)^2(1-ax^3)}$$

if the three atoms are all unlike, and of the generating function

$$\frac{1}{(1-a^2)(1-ax)(1-a^2x^2)(1-ax^3)}$$

if they are all alike.

[† Volume II of this Reprint, p. 119.]

substantially the same as that which I have described* in general terms in the *Comptes Rendus* as applicable to the theory of ternary and other higher varieties of quantics but less difficult of application to the Isomeric Problem on account of the greater simplicity of the crude forms subject to reduction, which appear in it. Appendix 4 will contain the application of the theory of "Associirter Formen" to the algebraical deduction of the irreducible forms of the quintic and certain other cases which but for the press of matter awaiting publication in the *Journal* would have formed part (as announced) of the present Appendix.

As already stated in a previous footnote, the theory of irreducible forms reappears in the isomeric investigation, the general character of the reduced generating function to be interpreted in it being precisely the same as in the invariantive theory, which constitutes an additional and a closer and more real bond of connexion between the chemical and algebraical theories than any which I had in view when I commenced the subject of this memoir.

Note on the Ladenburg Carbon-Graph.

The reasoning by which I have† established, in the preceding number of the *Journal*, the validity of the Ladenburg graph (and the invalidity of the Kekulean one) as a representative of the root differentiant to a covariant of the 6th degree in the variables and of the 6th order in the coefficients to a quartic, is so peculiar and it may seem to some of my readers so far-fetched, that it appears highly desirable to confirm it by a direct demonstration founded on the principle, that the permutation-sum of the product of the bonds in a valid graph interpreted as differences between the letters which they connect, shall not vanish. Previous to applying this principle to Ladenburg's graph we must convert it into an invariant by attaching hydrogen atoms to the six apices. Let these apices be called a, b, c, d, e, f, and the hydrogen atoms $\alpha, \beta, \gamma, \delta, \epsilon, \phi$: then the permutation-sum under consideration is

$$\Sigma (a-b)(a-c)(b-c)(d-e)(d-f)(e-f)(a-d)(b-e)(e-f)(a-\alpha)(b-\beta)(c-\gamma)$$
$$(d-\delta)(e-\epsilon)(f-\phi)$$

where the 6 letters a, b, c, d, e, f are interpermutable, as are also the 6 letters $\alpha, \beta, \gamma, \delta, \epsilon, \phi$.

It may be well to observe at this point that if we struck off the hydrogen atoms and treated the graph as representing an invariant to a cubic form, the permutation-sum

$$\Sigma (a-b)(a-c)(b-c)(d-e)(d-f)(e-f)(a-d)(d-c)(c-f)$$

would be found to vanish, as may easily be shown and as it ought to do, because there exists no invariant of the 6th order in the coefficients to a cubic form. Let a and d be interchanged in the term given under the sign of summation in the permutation-sum formed from the Ladenburg graph; then the sum of this together with the original term becomes

$$(a-d)(b-e)(c-f)(b-c)(e-f)(b-\beta)(c-\gamma)(e-\epsilon)(f-\phi)$$

[* p. 100 above.] [† p. 155 above.]

multiplied by

$$(a\delta - da)\{a^2-(b+c)\,a+bc\}\{d^2-(e+f)\,d+ef\}-(d\delta - aa)\{d^2-(b+c)\,d+bc\}\{a^2-(e+f)\,a+ef\},$$

which last named multiplier will be found to contain the quantity $(a^3d^2-a^2d^3)(a+\delta)$. Again, in the multiplicand, let b and c be interchanged; then, since

$$(b-e)(c-f)-(c-e)(b-f)=(b-c)(e-f),$$

the sum of the original and permuted multiplicand will contain a term

$$(a-d)(b-c)^2(e-f)^2\,bc\,(e-\epsilon)(f-\phi),$$

and accordingly the entire permutation-sum will contain the terms

$$(a+\delta)(a-d)(a^3d^2-a^2d^3)(b-c)^2(e-f)^2\,bc\Sigma(e-\epsilon)(f-\phi).$$

The partial sum last written is

$$4ef+4\epsilon\phi-2(e+f)(\epsilon+\phi).$$

Hence we may readily see that the total permutation-sum will contain *inter alia* a positive multiple of the combination $a^4b^3e^3d^2cfa$ and will not vanish, and consequently the graph is valid and not illusory; I presume that the same method applied to Kekulé's graph regarded as a representation of the covariant to the type [9 : 4, 6 : 6], which is the same thing (except that the hydrogen atoms are suppressed) as the graph to the invariant [15 : 4, 6; 1, 6 : 0], would serve to show it to be illusory as previously inferred from other considerations.

25.

NOTE ON THE THEOREM CONTAINED IN PROFESSOR LIPSCHITZ'S PAPER.

[*American Journal of Mathematics*, I. (1878), pp. 341—343.]

I THINK it may be useful to state the principle to which the theorem demonstrated in the preceding paper leads, in the shape in which it has always presented itself to my mind, but which I found difficult to express when writing under the constraint of a foreign language.

It amounts simply to the statement that in a *prepared* form just as the variables $(x, y, z, \ldots)$ are contragredient to their symbolic inverses $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right)$ so the coefficients $(a, b, c, \ldots)$ are contragredient to theirs $\left(\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots\right)$; the latter statement in fact includes the former inasmuch as the so-called variables may be regarded as the coefficients of an auxiliary linear form.

In applying this principle it is expedient to enlarge our conception of invariants, covariants, etc., and to predicate invariance of functions not only of quantities ordinarily so termed, but of their symbolic inverses, or of functions in which quantities and operators enter conjointly*. To draw the

* It was through this idea that I was originally led to an intuitive perception of the theorems concerning the prepared form. For suppose $(a, b, c, \ldots l \between x, y, z)^n$ to be any prepared form; then if x', y', z'; x'', y'', z'' are cogredient with x, y, z, and we operate upon the given form with

$$(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between y'z'' - y''z', \;\; z'x'' - z''x', \;\; x'y'' - x''y')^n,$$

the result is the nth power of the determinant $\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$ which is a covariant. Hence we may conclude that the operator is a covariant; just as, if a covariant multiplied by any form is a covariant, we may conclude that the multiplier must be so too. Consequently

$$(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between x, y, z)^n$$

is a contravariant, and the same reasoning will apply whatever may be the number of variables. I originally used two forms, one the ordinary form for $(a, b, c, \ldots l \between x, y, z, \ldots)$ and the other the

conclusions which flow from this conception, we have only to add the rule that all *combinations* of invariants, or of covariants, or of contravariants, etc., are themselves invariants, or covariants, or contravariants, etc., respectively. We may then state that the effect of substituting in a covariant or contravariant (to a prepared form) in place of the variables, or in place of the coefficients, the symbolic inverses of the one or the other, is to reverse their character and convert covariants into contravariants and *vice versa*, leaving of course the character of invariants unaltered; and I may remark incidentally that we are thus provided with a means of making any two *invariants* operate on each other so as to produce a third, a mode of operation which was not possible previous to the introduction of the prepared form*.

Moreover, the word combination must be taken in its widest sense, as there is more than one *mode* of combination possible. For example, if F, G, H are covariantive and Φ a contravariantive function of $(a, b, c, \ldots; x, y, z, \ldots)$, where $a, b, c, \ldots$ are the coefficients and $x, y, z, \ldots$ the variables of a prepared form, we have of course $\Phi\left(\frac{d}{da}, \frac{d}{db}, \ldots; x, y, \ldots\right) F$ and

$$\Phi\left(a, b, \ldots; \frac{d}{dx}, \frac{d}{dy}, \ldots\right) F \text{ and } G\left(\frac{d}{da}, \frac{d}{db}, \ldots; \frac{d}{dx}, \frac{d}{dy}, \ldots\right) F$$

all of them covariants. This may be termed an external mode of combination, but we shall equally have covariants derived by an internal mode of combination, for example

$$\Phi\left(\frac{dF}{da}, \frac{dF}{db}, \ldots; x, y, \ldots\right), \quad \Phi\left(a, b, \ldots; \frac{dF}{dx}, \frac{dF}{dy}, \ldots\right),$$

$$H\left(\frac{dF}{da}, \frac{dF}{db}, \ldots; \frac{dG}{dx}, \frac{dG}{dy}, \ldots\right)$$

will also be covariants.

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ordinary form *divested* of its numerical coefficients for $(\dot{a}, \ldots \between y'z'' - y''z', \ldots)^n$, and of course with the same result.

The reasoning is perhaps not absolutely rigorous, but sufficiently so to bring conviction of the fact to be established. Of course when we have proved that $(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between x, y, z)^n$ is a contravariant, it follows more generally that if $(A, B, C, \ldots \between x, y, z, \ldots)^N$ is a covariant

$$(\dot{A}, \dot{B}, \dot{C}, \ldots \between x, y, z, \ldots)^N,$$

where $\dot{A}, \dot{B}, \dot{C}, \ldots$ are the same functions of $\dot{a}, \dot{b}, \dot{c}, \ldots$ as $A, B, C, \ldots$ are of $a, b, c, \ldots$, will be a contravariant and *vice versa*.

* The case may be stated thus: previous to the introduction of the *prepared form*, invariants of systems could be made to operate upon invariants solely through the instrumentality of the coefficients of the linear forms of the system; since its introduction the same operation may be made to take effect through the instrumentality of the coefficients of all the forms, linear or non-linear, indiscriminately. The first named mode of operation is equivalent to the hyperdeterminantive method, which includes that of *Ueberschiebung*; the latter transcends the sphere of hyperdeterminants.

So again it may be observed that these modes of combination admit of being applied in more than one way: thus, to confine ourselves for a moment to the case of two forms, their external operation on each other may be simple, or concurrent, or reciprocal: simple when in *one* of them one set of quantities are converted into operators, concurrent when both sets are so converted, but reciprocal when in one of the two forms the variables and in the other the coefficients undergo such conversion. As an example, suppose we take the prepared form $ax^3 + \ldots + dy^3$, and its skew covariant

$$(a^2d + \ldots)\, x^3 + \ldots - (ad^2 + \ldots)\, y^3.$$

We may combine the contravariant

$$\left(\frac{d}{da}x^3 + \ldots + \frac{d}{dd}y^3\right)^2$$

with the contravariant

$$\left(a^2d + \ldots\right)\left(\frac{d}{dx}\right)^3 + \ldots - \left(ad^2 + \ldots\right)\left(\frac{d}{dy}\right)^3,$$

and the result will be a numerical multiple of the contravariant $dx^3 + \ldots - ay^3$. If, in the above instance, we denote the square of the primitive and its skew covariant according to their degree and order by $2\cdot6$, $3\cdot3$ respectively, we may explain their mutual action stenographically by saying that $\dot{2}\cdot6$ and $3\cdot\dot{3}$ have acted reciprocally on each other, the *dot* signifying that the quantities typified by the number so marked have been replaced by their symbolic inverses; we cannot well represent this mutual action by writing $\dot{2}\cdot6 * 3\cdot\dot{3}$ or $3\cdot\dot{3} * \dot{2}\cdot6$, but may employ for the purpose $\overset{\dot{2}\cdot6}{\underset{3\cdot\dot{3}}{*}}$. So from the square of a quintic $2\cdot10$ and its linear covariant $5\cdot1$ we may derive by reciprocal action $\overset{\dot{2}\cdot10}{\underset{5\cdot\dot{1}}{*}}$, or the contravariant $3\cdot9$: or, again, we may take any even number of covariants and cause them to operate in various manners, the variables on the variables and the coefficients on the coefficients, so as to form a closed circuit, as, for example, with four, we may make the coefficients of the first operate on those of the second, the variables of the second on those of the third, the coefficients of the third on those of the fourth, and the variables of the fourth on those of the first. Thus we have passed from reciprocal to the more general notion of simultaneous or circulatory action between any even number of covariants. And it is not unlikely that further applications may be made of this fertile conception: when dealing with a principle (an intellectual force) as distinguished from a theorem (a mere law), we never can feel sure that its uses are exhausted, or its plastic power spent.

26.

A SYNOPTICAL TABLE OF THE IRREDUCIBLE INVARIANTS AND COVARIANTS TO A BINARY QUINTIC, WITH A SCHOLIUM ON A THEOREM IN CONDITIONAL HYPERDETERMINANTS.

[*American Journal of Mathematics*, I. (1878), pp. 370—378.]

It is well known that every binary quintic can be expressed, and in only one way, as the sum of three fifth powers of linear functions of its variables, or which is the same thing, as the sum of the fifth powers of three variables connected by a linear equation, or finally, under the form

$$ax^5 + by^5 + cz^5,$$

subject to the equation

$$x + y + z = 0.$$

If ϕ, ψ be any two covariants of a binary quintic in x, y, the most general expression of the covariant produced by their operation on each other through the variables is

$$\left(\dot{x}\frac{d}{dy} - \dot{y}\frac{d}{dx}\right)^i \phi\psi,$$

where i is any positive integer and $\dot{x}, \dot{y}$ $\left(\text{abbreviations for } \frac{\delta}{\delta x}, \frac{\delta}{\delta y}\right)$ operate on ϕ only whilst $\frac{d}{dx}, \frac{d}{dy}$ operate on ψ.

Suppose now that ϕ, ψ are expressed as functions, say Φ, Ψ, of x, y, z, between which there exists the linear relation $lx + my + nz = 0$; it may be shown that the preceding expression becomes identical with

$$\begin{vmatrix} l, & m, & n \\ \dot{x}, & \dot{y}, & \dot{z} \\ \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \end{vmatrix}^i \Phi\Psi,$$

where x, y, z are to be treated as independent variables. In the present case, therefore, writing

$$(\dot{y} - \dot{z})\frac{d}{dx} + (\dot{z} - \dot{x})\frac{d}{dy} + (\dot{x} - \dot{y})\frac{d}{dz} = \Lambda,$$

$\Lambda^i\Phi\Psi$, or (which will be more convenient for writing) $\Psi\Lambda^i\Phi$ will represent the covariant derived from the alliance of Φ and Ψ.

The twenty-three irreducibles of the quintic may be arranged in the following partially symmetrical order, which is that which I shall adopt as the order of their successive deduction: the first figure denotes the degree in the coefficients, the second the order in the variables*.

			2·2		4·0					
	1·5		3·3		5·1					
	2·6	3·5	4·4	5·3	6·2	7·1	8·0			
		4·6		6·4		8·2				
3·9		5·7		7·5		9·3		11·1	12·0	
									13·1	
										18·0

* Comparing this arrangement to the distribution of stars in a firmament, it will be observed that there is a tendency to concentration, or the formation of a sort of milky-way, in the zone situated towards the centre, consisting of three bands which comprise between them 15 out of 23, the total number of forms. This phenomenon becomes very much more distinctly marked in the distribution of the 124 irreducible forms appertaining to the septic, the *corrected* table of which I anticipate will have appeared, about simultaneously with the publication of this, in the *Comptes Rendus*†. The table previously given in that journal for the seventhic is affected with some inaccuracies chiefly arising from arithmetical errors of calculation, as I made the computation hurriedly and on the point of leaving England for this continent, and also, in part, from the existence of some errors in the table of the reduced generating function, which I accepted, without sufficient examination, as the basis of my work. It may perhaps be worthy of notice that, if we add a unit to the ordinarily received number of irreducible forms in each case (which it is proper to do, since an absolute number is an invariant of the order zero), the numbers of the irreducibles for the 1st, 3rd, 5th and 7th orders become 2, 5, 24, 125 respectively. As I am about to compute the irreducibles for the 9th order, we shall soon be in a position to ascertain whether the law indicated in this progression has any foundation in nature: if so, the number for that case should be 626, or thereabouts, but it is not unlikely that the fact of 9 being a composite number may have a tendency to affect the result, probably in the direction of decrease. For binary quantics of the even orders 0, 2, 4, 6, 8 the number of irreducible covariants is 1, 3, 6, 27, 70 respectively (for the last see *Comptes Rendus*‡, June 24, 1878), which appear to indicate a geometrical progression with the common ratio 3, subject to diminution for higher powers of 2 entering into the order of the quantic.

[† p. 146 above.] [‡ p. 114 above.]

There will be two sources of indeterminateness in the expressions obtained for these forms, one universal, arising from the arbitrary addition

$$(x + y + z)\, M,$$

the other special to those forms (such as 13·1) which can be obtained by the multiplication of lower forms (as 8·0, 5·1). Our object must be to seek in all cases the simplest expressions that can be obtained.

$2{\cdot}2 = 1{\cdot}5\Lambda^4 1{\cdot}5 \equiv \Sigma\,(abxy + acxz) \equiv \Sigma abxy.$

I use the sign of equivalence to signify that numerical common multipliers are to be rejected.

$$4{\cdot}0 = 2{\cdot}2\Lambda^2 2{\cdot}2$$
$$= \Sigma\,(\dot y - \dot z)\,(\dot z - \dot x)\,(abxy + acxz + bcyz)\,\frac{d}{dx}\cdot\frac{d}{dy}\,(abxy + acxz + bcyz)$$
$$= \Sigma\,(-ab + ac + bc)\,ab \equiv a^2b^2 + b^2c^2 + c^2a^2 - 2abc\,(a + b + c)$$
$$1{\cdot}5 = ax^5 + by^5 + cz^5$$
$$3{\cdot}3 = 2{\cdot}2\Lambda^2 1{\cdot}5 \equiv \Sigma ax^3\,(\dot y - \dot z)^2\,(abxy + bcyz + cazx) \equiv abc\Sigma x^3.$$

Since $x^3 + y^3 + z^3 = 3xyz + (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$ we have

(*bis*) $$3{\cdot}3 = abcxyz$$

$$5{\cdot}1 = 3{\cdot}3\Lambda^2 2{\cdot}2 \equiv abc\Sigma x\,(\dot y - \dot z)^2\,(abxy + bcyz + cazx) \equiv abc\Sigma bcx$$
$$2{\cdot}6 = 1{\cdot}5\Lambda^2 1{\cdot}5 \equiv \Sigma ax^3\,(\dot y - \dot z)^2\,(ax^5 + by^5 + cz^5) \equiv \Sigma ax^3\,(by^3 + cz^3) \equiv \Sigma abx^3y^3$$
$$3{\cdot}5 = 2{\cdot}2\Lambda\ 1{\cdot}5 = \Sigma\,(aby + acz)\,(\dot y - \dot z)\,(ax^5 + by^5 + cz^5)$$
$$\equiv \Sigma\,(aby + acz)\,(by^4 - cz^4) = \Sigma a\,(b^2y^5 - c^2z^5) + abc\Sigma\,(zy^4 - yz^4)$$
$$4{\cdot}4 = 3{\cdot}3\Lambda^2 1{\cdot}5 \equiv abc\Sigma x\,(\dot y - \dot z)^2\,(ax^5 + by^5 + cz^5) \equiv abc\Sigma\,(bxy^3 + cxz^3)$$
$$= abc\Sigma\,[(ax^3 + by^3 + cz^3)\,x - ax^4] \equiv abc\Sigma ax^4$$
$$5{\cdot}3 = 2{\cdot}2\Lambda\ 3{\cdot}3 = \Sigma\,(aby + acz)\,(\dot y - \dot z)\,abc\,(x^3 + y^3 + z^3)$$
$$\equiv abc\Sigma\,(y^2 - z^2)\,(aby + acz)^* \equiv abc\Sigma ax\,(by^2 - cz^2)$$
$$6{\cdot}2 = 3{\cdot}3\Lambda^2 3{\cdot}3 \equiv a^2b^2c^2\Sigma x\,(\dot y - \dot z)^2\,(x^3 + y^3 + z^3) \equiv a^2b^2c^2\Sigma\,(xy + xz)$$
$$\equiv a^2b^2c^2\,(xy + yz + zx) \equiv a^2b^2c^2\,(x^2 + y^2 + z^2)$$
$$7{\cdot}1 = 4{\cdot}4\Lambda^4 3{\cdot}5 \equiv abc\Sigma\,a\,(\dot y - \dot z)^4\Sigma\,\{(ab^2y^5 - ac^2z^5) + abc\,(zy^4 - yz^4)\}$$
$$\equiv a^2b^2c^2\Sigma\,a\,(\dot y - \dot z)^4\Sigma\,(zy^4 - yz^4) = a^2b^2c^2\Sigma a\,(y - z)$$
$$8{\cdot}0 = 4{\cdot}4\Lambda^4 4{\cdot}4 \equiv a^2b^2c^2\Sigma a\,(\dot y - \dot z)^4\,(ax^4 + by^4 + cz^4) \equiv a^2b^2c^2\,(ab + ac + bc)$$
$$4{\cdot}6 = 3{\cdot}3\Lambda\ 1{\cdot}5 \equiv abc\Sigma x^2\,(\dot y - \dot z)\,(ax^5 + by^5 + cz^5) \equiv abc\Sigma a\,(y^2 - z^2)\,x^4$$
$$6{\cdot}4 = 2{\cdot}2\Lambda\ 4{\cdot}4 = \Sigma\,(aby + acz)\,(\dot y - \dot z)\,abc\,(ax^4 + by^4 + cz^4)$$
$$= abc\,(aby + acz)\,(by^3 - cz^3)$$
$$= abc\Sigma\,(ab^2y^4 + ac^2z^4) + a^2b^2c^2\Sigma\,(zy^3 - yz^3),$$

which, since $\Sigma\,(zy^3 - yz^3)$ contains $x + y + z$,

$$= abc\Sigma\,(c - b)\,a^2x^4$$
$$8{\cdot}2 = 4{\cdot}4\Lambda^4 4{\cdot}6 = a^2b^2c^2\Sigma a\,(\dot y - \dot z)^4\,\{\Sigma a\,(y^2 - z^2)\,x^4\} = a^2b^2c^2\Sigma ab\,(x^2 - y^2)$$
$$3{\cdot}9 = 2{\cdot}6\Lambda\ 1{\cdot}5 \equiv \Sigma\,(abx^2y^3 + acx^2z^3)\,(\dot y - \dot z)\,(ax^5 + by^5 + cz^5)$$
$$\equiv \Sigma\,(abx^2y^3 + acx^2z^3)\,(by^4 - cz^4)$$
$$= \Sigma ax^2\,(b^2y^7 - c^2z^7) + abcx^2y^2z^2\Sigma\,(zy^2 - yz^2)\dagger$$

* For $y^2 - z^2$ I substitute $xz - xy$.

† Possibly this expression may be simplifiable by the addition of a suitable multiple of $x + y + z$.

$$5{\cdot}7 = 4{\cdot}4\Lambda\ 1{\cdot}5 \equiv abc\Sigma ax^3(\dot{y}-\dot{z})(ax^5+by^5+cz^5) \equiv abc\Sigma ab(x-y)x^3y^3$$
$$7{\cdot}5 = 4{\cdot}4\Lambda\ 3{\cdot}3 \equiv a^2b^2c^2\Sigma ax^3(\dot{y}-\dot{z})(x^3+y^3+z^3) \equiv a^2b^2c^2\Sigma ax^2(y^3-z^3)$$
$$11{\cdot}1 = 5{\cdot}1\Lambda\ 6{\cdot}2 \equiv a^3b^3c^3\Sigma bc(\dot{y}-\dot{z})(x^2+y^2+z^2) \equiv a^3b^3c^3\Sigma bc(y-z)$$
$$9{\cdot}3 = 6{\cdot}2\Lambda\ 3{\cdot}3 \equiv a^3b^3c^3\Sigma x(\dot{y}-\dot{z})(x^3+y^3+z^3) \equiv a^3b^3c^3(x-y)(y-z)(z-x)$$
$$12{\cdot}0 = 6{\cdot}2\Lambda^2 6{\cdot}2 \equiv a^4b^4c^4\Sigma(\dot{y}-\dot{z})^2(x^2+y^2+z^2) \equiv a^4b^4c^4$$
$$\begin{aligned}13{\cdot}1 = 7{\cdot}1\Lambda\ 6{\cdot}2 &= a^4b^4c^4\Sigma(b-c)(\dot{y}-\dot{z})(x^2+y^2+z^2) \equiv a^4b^4c^4\Sigma(b-c)(y-z)\\ &= a^4b^4c^4\{\Sigma(by+cz)-\Sigma(bz+cy)\}\\ &= a^4b^4c^4\{2(ax+by+cz)+(bx+cz+ay)\} \equiv a^4b^4c^4\Sigma ax\end{aligned}$$
$$\begin{aligned}18{\cdot}0 = 13{\cdot}1\Lambda\,5{\cdot}1 &= a^4b^4c^4\Sigma a(\dot{y}-\dot{z})(bcx+cay+abz) = a^4b^4c^4\Sigma a(c-b)\\ &= a^5b^5c^5(a-b)(b-c)(c-a).\end{aligned}$$

18·0 may also be obtained by the operation of 11·1 on 7·1, or instantaneously as the resultant of 1·5, $abcxyz$ and $x+y+z$. In the following table the preceding results are collected; for greater brevity instead of the sign of summation I employ the sign + or − to signify respectively the symmetrical or semi-symmetrical completion of the terms to which it is affixed; m is used to signify abc.

1—2	$abxy+ : (a^2b^2-2abc^2)+$
3—5	$ax^5+ : mx^3+$, or $mxyz : mbcx+$
6—12	$\begin{cases} abx^3y^3+ : a^2bx^5+myz^4- : max^4+ : \\ mabxy^2- : m^2x^2+ : m^2bx- : m^2ab+ \end{cases}$
13—15	$max^4y^2- : ma^2cx^4- : m^2abx^2-$
16—21	$\begin{cases} ab^2x^2y^7+mx^2y^4z^3- : mabx^4y^3- : m^2ax^2y^3- : \\ m^3bcy- : m^3x^2y- : m^4 \end{cases}$
22	m^4ax+
23	m^5a^2b-

I propose, at some future time, to apply a similar method to obtain an explicit representation of the irreducible forms appertinent to the binary seventhic, an arduous undertaking, but one that seems likely to lead to the apperception of new forms of complex symmetry. The primitive may, for that case, be represented by $x^7+y^7+z^7+t^7$, connected by the linear equations $(l, m, n, p \between x, y, z, t)=0$, $(\lambda, \mu, \nu, \pi \between x, y, z, t)=0$, and Λ, the symbol of alliance, will be represented by

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, & \frac{d}{dt} \\ \dot{x}, & \dot{y}, & \dot{z}, & \dot{t} \\ l, & m, & n, & p \\ \lambda, & \mu, & \nu, & \pi \end{vmatrix}.$$

Every in- and co-variant will then be a rational integer function of x, y, z, t and the six minor determinants, which are the parameters of the line represented by the above two linear equations.

It may be worth while to notice the representations of the irreducible derivatives of the quartic when put under the indeterminate form

$$ax^4 + by^4 + cz^4,$$

subject to the relation $x + y + z = 0$. We get

$$\begin{aligned} 2\cdot0 &= 1\cdot4\Lambda^4 1\cdot4 = \Sigma a\,(\dot{y} - \dot{z})^4\,(ax^4 + by^4 + cz^4) \equiv ab + bc + ca \\ 2\cdot4 &= 1\cdot4\Lambda^2 1\cdot4 = \Sigma ax^2\,(\dot{y} - \dot{z})^2\,(ax^4 + by^4 + cz^4) = abx^2y^2 + acx^2z^2 + bcy^2z^2 \\ 3\cdot0 &= 1\cdot4\Lambda^4 2\cdot4 = \Sigma a\,(\dot{y} - \dot{z})^4\,(abx^2y^2 + acx^2z^2 + bcy^2z^2) \equiv abc \\ 3\cdot6 &= 1\cdot4\Lambda\ 2\cdot4 = \Sigma ax^3\,(\dot{y} - \dot{z})\,(2\cdot4) \\ &= \Sigma\,(a^2bx^5y - a^2cx^5z) + abcxyz\Sigma\,(yz^2 - y^2z). \end{aligned}$$

As regards the sextic form, the first idea would be to regard it as the resultant, in respect to one of the variables (say z), of the canonical system discovered by me so long ago,

$$\left.\begin{matrix} ax^6 + by^6 + cz^6 + mxyz\,(x - y)\,(y - z)\,(z - x) \\ x + y + z \end{matrix}\right\},$$

but this will be found to give rise to expressions for the invariants and covariants of extreme complexity. The representations will, I think, be simplified by adopting the new canonical system

$$\left.\begin{matrix} x^3 + y^3 + z^3 + 3mxyz \\ ayz + bxz + cyx \end{matrix}\right\} \begin{matrix} (1) \\ (2) \end{matrix}$$

and considering the sextic as the resultant of (1) and (2). It will then be found that every covariant proper (calling its order, which is always an even number, 2ϵ) will still be a resultant of (2) and of some new form in x, y, z of order ϵ*. The fact of the lowering, by one-half, the order of the form in x, y, z, corresponding to a covariant of any given order in x, y, gives a great (though it may be not an unbalanced) advantage to the new canonical system over the old. On setting out the equation connecting the four completely symmetrical invariants with the square of the skew one of the sextic, and then making this latter equal to zero, we obtain an equation between three absolute invariants of the sextic which may be regarded as the equation to a surface, the analogue of my Bicorn, the *Nomen Triviale* for the bicuspidal unicursal quartic curve. This surface will divide space into two parts, one corresponding to equations of the sixth order with real, the other with conjugate coefficients, or by real linear substitutions transformable into such, the surface itself being the locus of equations of the recurrent form. The facultative part of space, that is, the part corresponding to the case of real coefficients will then separate into two pairs of regions, one pair belonging to the case of 0 and 4, the other to that of 2 and 6 imaginary roots. By this method, however laborious, the solution of the problem of determining the invariantive criteria of the quality of the roots of the sextic (to borrow a term

* For every quantic of an even order in x, y is a ternary quantic in x^2+xy, y^2+yx, $-xy$, which quantities are proportional to x, y, z connected by the equation $xy+xz+yz=0$.

from the chess table) becomes *forced*, and no other mode of attacking the question appears to me to be practicable; nor can it fail to bring into view a surface possessed of remarkable properties*.

SCHOLIUM.

The mode of representing the covariants to a sextic above employed made it imperative, or at least expedient, to discover a method by aid of which the process of alliance, or hyperdetermination, could be performed upon the representative forms themselves, without eliminating one of the variables by means of the equation of condition, and I have obtained the following very general theorem, which, it will presently be seen, contains a solution of the problem in question, and which, as the first example of conditional alliance, or hyperdetermination, it seems to me desirable to put on record.

Let $\phi, \psi, \ldots, \theta$ be i homogeneous functions of the orders $\alpha, \beta, \ldots, \lambda$ in $i+j$ variables, $x, y, \ldots, t$ being i of them and $u, v, \ldots, z$ the j others, and let the variables be connected by the j homogeneous equations

$$L=0, \quad M=0, \quad \ldots, \quad N=0.$$

Call the Jacobian $$\frac{d(L, M, \ldots, N)}{d(u, v, \ldots, z)}=\Omega.$$

Let $\Phi, \Psi, \ldots, \Theta$ be the values of $\phi, \psi, \ldots, \theta$ expressed in terms of $x, y, \ldots, t$ alone, and let

$$\begin{vmatrix} \delta_{x_1}, & \delta_{y_1}, & \ldots, & \delta_{t_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots, & \delta_{t_2} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{x_i}, & \delta_{y_i}, & \ldots, & \delta_{t_i} \end{vmatrix}^q (\Omega_1{}^\alpha \Omega_2{}^\beta \ldots \Omega_i{}^\lambda \Phi_1 \Psi_2 \ldots \Theta_i)$$

be called D, it being understood that the meaning of any subscript, say μ, is to cause the letters $x, y, \ldots, t$ to be changed into $x_\mu, y_\mu, \ldots, t_\mu$. Again let the operative determinant of the $(i+j)$th order written below

$$\begin{vmatrix} \delta_{x_1}, & \delta_{y_1}, & \ldots\ldots, & \delta_{z_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots\ldots, & \delta_{z_2} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{x_i}, & \delta_{y_i}, & \ldots\ldots, & \delta_{z_i} \\ (\delta_x L)_\epsilon, & (\delta_y L)_\epsilon, & \ldots\ldots, & (\delta_z L)_\epsilon \\ (\delta_x M)_\epsilon, & (\delta_y M)_\epsilon, & \ldots\ldots, & (\delta_z M)_\epsilon \\ \cdots & \cdots & \cdots & \cdots \\ (\delta_x N)_\epsilon, & (\delta_y N)_\epsilon, & \ldots\ldots, & (\delta_z N)_\epsilon \end{vmatrix}$$

* One may see at a glance that this surface cannot be of a higher order than 7, the integer part of 30 : 4. Possibly however, it may not be so high; there will be no difficulty in finding the actual order by means of the known expression for R^2 (Clebsch, *Binäre Formen*, p. 299), in terms of the invariants of even degrees.

be called J_ϵ (ϵ being any of the suffixes 1, 2, 3, ..., i) then it will be found that to a numerical factor près

$$(\Omega_1^{\alpha-q}\Omega_2^{\beta-q}\ldots\Omega_i^{\lambda-q})(J_1+J_2+\ldots+J_i)^q(\phi_1\psi_2\ldots,\theta_i)=D.$$

As a corollary, if the functions $L, M, \ldots, N$ are all linear in respect to $u, v, \ldots, z$, and if in respect to $1, u, v, \ldots, z$ the resultants of $\phi, L, M, \ldots, N$; $\psi, L, M, \ldots, N$; ... are $[\Phi], [\Psi], \ldots, [\Theta]$ (which is what we mean by saying that $\phi, \psi, \ldots, \theta$ represent $[\Phi], [\Psi], \ldots, [\Theta]$), it will be easily seen to follow from the above theorem that the qth alliance of these quantics will be itself *represented by*

$$(J_1+J_2+\ldots+J_i)^q(\phi_1\psi_2\ldots,\theta_i)^*.$$

Thus in the particular case where $x, y, \ldots, t$ becomes x, y and $u, \ldots, z$ becomes z and $L, M, \ldots, N$ becomes the single function $xy+yz+zx$, we see that the qth alliance of the quantics represented by ϕ, ψ will be itself represented by

$$\left\{\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_1+z_1, & z_1+x_1, & x_1+y_1\end{vmatrix}+\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_2+z_2, & z_2+x_2, & x_2+y_2\end{vmatrix}\right\}^q(\phi_1\psi_2)$$

on replacing x_1, y_1, z_1; x_2, y_2, z_2 by x, y, z after the differentiations have been executed. It will, of course, be understood that the factors in each cross product of the determinants above are to be taken in *their natural order*, that is,

$$\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_1+z_1, & z_1+x_1, & x_1+y_1\end{vmatrix}^\mu$$

is to be understood to mean, not

$$[\Sigma(x_1+y_1)(\delta_{x_1}\delta_{y_2}-\delta_{y_1}\delta_{x_2})]^\mu,$$

but

$$[\Sigma(\delta_{x_1}\delta_{y_2}-\delta_{y_1}\delta_{x_2})(x_1+y_1)]^\mu,$$

and so in general.

* This expression may be put under the more compact form J^q, J being a matrix in which the first i lines are the same as those common to $J_1, J_2, \ldots, J_i$, and the last j lines are the sums of the corresponding ones in $J_1, J_2, \ldots, J_i$. Although I had submitted it to a mental process of demonstration (or what seemed such) before sending it to the press, I am not without some little misgiving as to the exactitude of the theorem so far as it regards the higher alliances; for those of the first order it is easily verifiable, and, in that case, it should be noticed that each of the i terms in the expression given by it will reproduce separately (but under quite a distinct form) the value of the Jacobian of $\phi, \psi, \ldots, \theta$; $L, \ldots, N$. Some corresponding simplification in practice, it is not improbable, will apply in the general case, supposing my doubts as to the validity of the theorem to prove unfounded. It is important, and greatly enlarges the horizon of the subject, to remark that, inasmuch as any ternary quadric is linearly transformable into the form $xy+yz+zx$, it will follow that any binary quantic of an even order, with its train of covariants, may be represented by corresponding ternary forms of half their respective orders, combined with a perfectly general final conic, so that, for example, instead of the form $xy+yz+zx$, useful though it be as an intermediate step in the evolution of the theory, we may substitute the handier and more advantageous one $x^2+y^2+z^2$ as the auxiliary quadric.

The result of this investigation has been to open my eyes to the unquestionable fact that, as we know that the first "Ueberschiebung," or "transvectant," or "alliance," of two or more quantics (names significant and useful enough to indicate the particular modes under which they are considered to be generated) is the ordinary Jacobian, so the right general name for the Ueberschiebung or alliance of any order viewed *per se* (as a *Ding an sich*) and without reference to its mode of origination, which ought to supersede all others, is the *Jacobian of the corresponding order*; or, in other words, the theory of invariants falls into the theory of compound differentiation, and just as $\left(\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right)$ is called a Jacobian and designated by $\frac{d(u, v)}{d(x, y)}$, so $\frac{d^2u}{dx^2}\frac{d^2v}{dy^2} - 2\frac{d^2u}{dx\,dy}\frac{d^2v}{dx\,dy} + \frac{d^2u}{dy^2}\frac{d^2v}{dx^2}$ is entitled to be called the second Jacobian and to be designated by $\frac{d^2(u, v)}{d(x, y)^2}$, and more generally every hyperdeterminant may be designated as a compound differential coefficient (or derivative) of the type $\frac{d^\alpha d^\beta \ldots}{d(\ \ )^\alpha d(\ \ )^\beta \ldots}$, where the vacant spaces are to be filled up by the insertion of a certain number of letters, with liberty for any number of them in each parenthesis to be identical with the like number in any other. Since we are now in possession of a definite analogue to ordinary differential coefficients of all orders, I do not know whether I shall be considered too bold or fanciful in suggesting that there ought to exist, in the nature of things, some theorem of development for several sets of variables analogous to Taylor's for a single set: what such theorem is or could be I have at present no conception, but as little, be it remembered, could anyone, even Jacobi himself, before the creation of hyperdeterminants, have had the remotest conception in regard to a function of several variables bearing to $\left(\frac{d}{dx}\right)^i \phi$ the same relation of analogy as the ordinary functional determinant to $\frac{d\phi}{dx}$, whether such function could exist, and, if so, what it would be. I have always thought and felt that beyond all others the algebraist, in his researches, needs to be guided by the principle of faith, so well and philosophically defined as "the substance of things hoped for, the evidence of things not seen."

27.

SUR LES ACTIONS MUTUELLES DES FORMES INVARIANTIVES DÉRIVÉES.

[*Crelle's Journal*, LXXXV. (1878), pp. 89—114.]

JE comprends les invariants, les covariants, les contravariants et toutes les formes qui dérivent dans le même sens d'un système donné de Quantics sous le nom général de *dérivées invariantives*, et je vais établir un principe qui rend ces formes fécondes et donne à deux quelconques d'entre elles la faculté de produire, par l'action de l'une sur l'autre, de nouvelles formes invariantives. Si l'on se borne aux invariants d'un seul Quantic ou d'un système de Quantics, la manière de procéder pour cette génération est presque évidente d'elle-même. Car soient $F(a, b, c, \ldots)$, $G(a, b, c, \ldots)$ deux invariants du même Quantic, ou du même système de Quantics, et écrivons à la place de $a, b, c, \ldots$ dans l'une de ces deux fonctions $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$: si l'on opère avec la fonction ainsi modifiée sur l'autre, le résultat restera invariantif (sauf le cas dans lequel le résultat se réduit à zéro ou à une autre constante numérique). En effet, désignons par $\dot{a}, \dot{b}, \ldots$ les opérations $\frac{d}{da}, \frac{d}{db}, \ldots$ et admettons qu'une substitution quelconque appliquée aux variables des Quantics donnés dont F et G sont les dérivées, change en $a', b', c', \ldots$ les éléments donnés $a, b, c, \ldots$; alors on aura d'après un principe élémentaire du calcul différentiel

$$\dot{a}' = \dot{a}\frac{da}{da'} + \dot{b}\frac{db}{da'} + \ldots,$$

$$\dot{b}' = \dot{a}\frac{da}{db'} + \dot{b}\frac{db}{db'} + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

donc la substitution contraire à la substitution en question changera $\dot{a}, \dot{b}, \dot{c}, \ldots$ en $\dot{a}', \dot{b}', \dot{c}', \ldots$. Or $F(\dot{a}, \dot{b}, \dot{c}, \ldots)$ et $G(a, b, c, \ldots)$ ne subissant aucun changement, le résultat de l'opération de la première sur la seconde ne subira non plus de changement par une substitution quelconque opérée sur les variables.

Ce raisonnement reste bon dans le cas où l'on substitue à l'invariant G un *contravariant* quelconque. Mais dans le cas général, dans lequel les variables entrent en même temps dans F et dans G, on a besoin de s'appuyer sur des considérations additionnelles d'un genre nouveau.

Or je remarque que la formation d'un Quantic quelconque se compose de trois genres des quantités—des variables,—des parties littérales des coefficients et enfin—des multiplicateurs numériques qui les affectent et qui forment, pour ainsi dire, l'équipement arithmétique de la forme. Dans la vieille algèbre ces multiplicateurs numériques ont été réduits à l'unité, dans l'algèbre moderne on les égale aux nombres binômes ou polynômes.—Dans la théorie que je vais produire on aura besoin de se servir d'un équipement qui tient pour ainsi dire la moyenne entre les deux dont je viens de parler, c. à d. que le multiplicateur d'un élément quelconque sera la racine carrée du nombre binôme ou polynôme qui lui serait égalé dans la notation ordinaire des Quantics. Quand les multiplicateurs numériques sont mis sous cette forme, je dirai que le Quantic est un Quantic *préparé*.

Remarquons que, quel que soit l'équipement numérique d'un Quantic, une substitution quelconque opérée sur les variables *induit* une substitution corrélative opérée sur les éléments, c. à d. que si dans le Quantic $(\lambda a, \mu b, \nu c, \ldots \between x, y, z, \ldots)^i$ on écrit pour $x, y, z, \ldots$ des fonctions linéaires de $x, y, z, \ldots$, le Quantic se changera en un autre $(\lambda A, \mu B, \nu C, \ldots \between x, y, z, \ldots)^i$ où $A, B, C, \ldots$ seront des fonctions linéaires de $a, b, c, \ldots$. Qu'on sépare les coefficients qui entrent dans les fonctions linéaires données de $x, y, z, \ldots$: on obtient une matrice;—qu'on sépare les coefficients qui entrent dans les fonctions linéaires de $a, b, c, \ldots$: on aura une autre matrice,—et je dirai que cette seconde matrice est *induite* par la première. Puisque les changements peuvent être effectués par des altérations insensibles on pourrait même sans trop d'incorrection affirmer que le mouvement des variables *induit* ou entraîne avec lui un mouvement dans les éléments d'une forme donnée. L'ordre du déterminant inducteur ne dépend que du nombre des variables, celui du déterminant induit dépend en même temps du nombre des variables et du degré de la forme par rapport aux variables.

On comprend le sens qu'il faut attribuer à la désignation de matrices contraires. La matrice dont les termes servent à exprimer le système des variables $\xi, \eta, \zeta, \ldots$ en fonctions linéaires des variables $x, y, z, \ldots$ et celle dont les termes servent à exprimer les $x, y, z, \ldots$ en fonctions linéaires des $\xi, \eta, \zeta, \ldots$ seront appelées matrices contraires*, et de même les deux substitutions dont les coefficients sont données par deux matrices contraires seront appelées substitutions contraires.

Cela posé, je suis en état d'énoncer le suivant théorème fondamental. *Dans un Quantic préparé deux substitutions contraires opérées sur les variables induisent deux substitutions contraires opérées sur les éléments.* La préparation indiquée ci-dessus est la condition nécessaire pour la validité

* Mot plus précis que celui d'*inverse* ou de *réciproque* dont on se sert quelquefois dans un ens plus vague quant à la grandeur *absolue* des termes dont on parle.

du théorème: si le Quantic n'était pas dûment *préparé* de la manière indiquée, des substitutions contraires appliquées aux variables n'induiraient point des substitutions contraires opérées sur les éléments. Avant d'entrer dans la démonstration de ce théorème je rappellerai que la matrice produite par la multiplication de deux et conséquemment d'un nombre quelconque de matrices et la matrice produite par la multiplication des matrices respectivement contraires aux premières, sont encore des matrices contraires entre elles—cet énoncé suppose toutefois que les multiplications des matrices soient effectuées de manière que leurs éléments soient combinés ligne par colonne ou colonne par ligne, mais non ligne par ligne ou colonne par colonne.

Considérons pour un instant une matrice de la forme

$$\begin{array}{cccccccccc}
1 & r' \\
r & 1 & s' \\
 & s & 1 & t' \\
 & & t & 1 & \cdot \\
 & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & 1 & \tau' \\
 & & & & & & & \tau & 1 & \sigma' \\
 & & & & & & & & \sigma & 1 & \rho' \\
 & & & & & & & & & \rho & 1
\end{array}$$

où toutes les places vacantes dans le carré dont la ligne des unités est une des diagonales, doivent être remplies par des zéros.

En se bornant aux termes de la première dimension en r, s, t, ..., la valeur du déterminant est $1 - rr' - ss' - tt' \ldots$. Autant qu'il est permis de se servir de cette valeur approchée, les dérivées du déterminant par rapport à r, s, t, ... auront les valeurs $-r'$, $-s'$, $-t'$, ... et vice versâ, donc en attribuant des valeurs infiniment petites à r, s, t, ... et faisant évanouir chacune des lettres accentuées, la matrice

$$[\mathrm{A.}] \quad \left\{\begin{array}{cccccccccc}
1 \\
r & 1 \\
 & s & 1 \\
 & & t & 1 \\
 & & & \cdot & \cdot \\
 & & & & \cdot & \cdot \\
 & & & & & \cdot & \cdot \\
 & & & & & & \cdot & 1 \\
 & & & & & & & \tau & 1 \\
 & & & & & & & & \sigma & 1 \\
 & & & & & & & & & \rho & 1
\end{array}\right.$$

aura pour sa contraire la matrice

$$[\text{B.}]\quad \left\{\begin{matrix} 1 & \bar{r} & & & & & & & & \\ & 1 & \bar{s} & & & & & & & \\ & & 1 & \bar{t} & & & & & & \\ & & & 1 & \cdot & & & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & \cdot & \cdot & & & \\ & & & & & & \cdot & \cdot & & \\ & & & & & & 1 & \bar{\tau} & & \\ & & & & & & & 1 & \bar{\sigma} & \\ & & & & & & & & 1 & \bar{\rho} \\ & & & & & & & & & 1 \end{matrix}\right.$$

où $\bar{r}$, $\bar{s}$, $\bar{t}$, ... doivent être remplacées par $-r$, $-s$, $-t$, ...

J'établirai d'abord la loi de l'induction des contraires dans le cas d'un Quantic binôme dûment préparé, x et y étant les variables et a, b, c, ... les éléments.

Si la loi des contraires est vraie pour des substitutions dont la matrice a l'unité pour valeur de son déterminant, elle sera vraie pour des substitutions quelconques. De plus on démontre aisément que toute substitution au déterminant $=1$ opérée sur x, y peut être effectuée par trois substitutions *simples* successives, c. à d. par les substitutions successives de $x+hy$ pour x, de $y+kx$ pour y et de $x+ly$ pour x. Donc en vertu d'une remarque faite précédemment, la démonstration cherchée se réduit à la démonstration dans le cas d'une substitution *simple*, c. à d. de la substitution de $x+hy$ pour x. Mais cette substitution même peut être effectuée par une succession infinie de substitutions de la forme $x+\epsilon y$, où ϵ est une quantité infiniment petite: donc en vertu de la même remarque il ne nous reste qu'à établir la loi des contraires dans le cas où l'on substitue $x+\epsilon y$ au lieu de x dans la forme préparée

$$ax^i + \sqrt{(i)}\, bx^{i-1}y + \sqrt{\left\{\frac{i(i-1)}{2}\right\}}\, cx^{i-2}y^2 + \dots$$
$$+ \sqrt{\left\{\frac{i(i-1)}{2}\right\}}\, hx^2y^{i-2} + \sqrt{(i)}\, kxy^{i-1} + ly^i.$$

Soit $$a'x^i + \sqrt{(i)}\, b'x^{i-1}y + \dots + \sqrt{(i)}\, k'xy^{i-1} + l'y^i$$

ce que devient la forme après cette substitution, de sorte que

$$a'=a,\ b'=b+\sqrt{(i)}\,\epsilon a,\ c'=c+\sqrt{\{2(i-1)\}}\,\epsilon b,\ \dots,$$
$$k'=k+\sqrt{\{2(i-1)\}}\,\epsilon h,\ l'=l+\sqrt{(i)}\,\epsilon k,$$

et posons

$$\sqrt{(i)}\,\epsilon=r,\ \sqrt{\{2(i-1)\}}\,\epsilon=s,\ \sqrt{\{3(i-2)\}}\,\epsilon=t,\ \dots,\ \sqrt{\{2(i-1)\}}\,\epsilon=s',\ \sqrt{(i)}\,\epsilon=r',$$

la matrice de la substitution par laquelle $a, b, c, \ldots$ se transforment en $a', b', c', \ldots$ formera le cas particulier contenu dans la matrice [A.] lorsque le système des quantités $r, s, t, \ldots, \tau, \sigma, \rho$, que je désigne ici par $r, s, t, \ldots, t', s', r'$, est réversible, c. à d. que l'on a

$$r=r',\ s=s',\ t=t',\ \ldots.$$

Or au lieu de la substitution $\begin{matrix} 1 & \epsilon \\ 0 & 1 \end{matrix}$ opérée sur x, y prenons la substitution contraire $\begin{matrix} 1 & 0 \\ -\epsilon & 1 \end{matrix}$, c. à d. la substitution simple de $y-\epsilon x$ pour y: il est évident que la forme de substitution induite sera donnée par la matrice [B.] *avec les mêmes valeurs* de $r, s, t, \ldots, t', s', r'$ qu'auparavant. Mais ces deux matrices sont contraires. Donc le théorème est démontré dans le cas des formes binaires. On voit la nécessité de la condition que le Quantic soit *préparé* quant à son équipement numérique. Car sans cela les deux matrices induites qui se trouvent toujours sous les deux formes [A.] et [B.] ne seraient plus contraires, car le système des quantités $r, s, t, \ldots, t', s', r'$ étant renversé dans ces deux formes et les lettres accentuées et non-accentuées ne conservant plus des valeurs identiques, les deux matrices [A.] et [B.] cesseraient d'être corrélatives.

Comme exemple du théorème qui vient d'être établi, considérons le cas très-simple de la forme préparée $ax^2+\sqrt{(2)}\,bxy+cy^2$.

Opérons sur x, y la substitution $fx+gy$ pour x et $hx+ky$ pour y (où pour plus de simplicité je supposerai que $fk-gh=1$); les valeurs induites en a, b, c répondront à la matrice

$$\begin{matrix} f^2, & \sqrt{(2)}fh, & h^2, \\ \sqrt{(2)}fg, & fk+gh, & \sqrt{(2)}\,hk, \\ g^2, & \sqrt{(2)}\,gk, & k^2, \end{matrix}$$

dont l'inverse, en négligeant le facteur commun $fk-gh$, sera

$$\begin{matrix} k^2, & -\sqrt{(2)}\,gk, & g^2, \\ -\sqrt{(2)}\,kh, & gh+fk, & -\sqrt{(2)}fg, \\ h^2, & -\sqrt{(2)}fh, & f^2, \end{matrix}$$

qui est évidemment la matrice d'induction qui répond à la substitution de $kx-hy$ pour x et de $-gx+fy$ pour y: c. à d. que les deux substitutions contraires $\begin{matrix} f & g \\ h & k \end{matrix}$, $\begin{matrix} k & -h \\ -g & f \end{matrix}$, opérées sur les variables induisent des substitutions contraires opérées sur les éléments a, b, c.

J'ajouterai deux observations dont la première trouvera son application dans la démonstration générale et dont la seconde facilitera l'application du principe que je vais fonder sur la loi des contraires.

1°. Il est évident que pour *préparer* un Quantic, il n'est pas nécessaire que les multiplicateurs numériques soient les nombres binômes eux-mêmes;

il suffit que les rapports entre ces multiplicateurs soient les mêmes qu'entre les nombres binômes.

2°. Si l'on applique aux variables deux substitutions contraires dans deux Quantics ayant les mêmes éléments mais des multiplicateurs numériques distincts, les substitutions induites seront contraires pourvu que les produits des multiplicateurs qui affectent le même élément dans les deux Quantics, suivent la loi des multiplicateurs dans un Quantic préparé: ainsi comme cas particulier, si l'on introduit deux substitutions contraires, l'une dans un Quantic de la forme normale $(a, b, c, \ldots \between x, y)^i$, l'autre dans un Quantic où les éléments sont les mêmes mais dépourvus de tout multiplicateur numérique (c. à d. dans deux Quantics avec les mêmes éléments, l'une écrite selon la méthode ancienne, l'autre selon la méthode moderne) les deux substitutions induites sur les éléments seront contraires. A l'aide de cette remarque on évite l'inconvénient d'introduire des racines carrées qui doivent nécessairement disparaître dans les résultats.

Passons à l'application du théorème sur les substitutions contraires. Soit $F(a, b, c, \ldots : x, y)$ un covariant d'un Quantic ou d'un système de Quantics: écrivons comme auparavant $\dot{a}, \dot{b}, \dot{c}, \ldots$ pour $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$; je dis que $F(\dot{a}, \dot{b}, \dot{c}, \ldots : x, y)$ possédera toutes les propriétés d'un contravariant, c. à d. que si $G(a, b, c, \ldots : x, y)$ est un contravariant quelconque du même Quantic ou du même système, $F(\dot{a}, \dot{b}, \dot{c}, \ldots : x, y)$ appliqué comme opérateur à la forme $G(a, b, c, \ldots : x, y)$ conduira à un contravariant. Dans cet énoncé on suppose toutefois que les Quantics soient exprimés chacun dans leurs formes préparées ou bien (ce qui revient au même) que des deux formes F et G l'une appartienne à un système de Quantics pleins (c. à d. à éléments affectés de nombres binômes) et l'autre à un système de Quantics vides (c. à d. à éléments dépourvus de multiplicateurs binômes). Sous cette condition la forme $F \divideontimes G$, c. à d. le résultat de l'opération du covariant F sur le contravariant G, sera un contravariant du système auquel G appartient. Si au contraire F est un contravariant et G un covariant le résultat $F \divideontimes G$ de l'opération de F sur G sera un covariant. Dans ce qui suit je supposerai pour plus de simplicité que les formes dont il est question soient présentées dans leur forme préparée.

Je vais passer maintenant à des générations de formes dérivées que l'on obtient, si dans la forme dérivée $F(a, b, c, \ldots : x, y)$, qui pourra être covariant ou contravariant, on remplace non seulement les éléments $a, b, c, \ldots$ par leurs inverses symboliques, c. à d. par $\dot{a} = \frac{d}{da}, \dot{b} = \frac{d}{db}, \dot{c} = \frac{d}{dc}, \ldots$, mais en même temps les variables x, y par leurs inverses symboliques, c. à d. par

$$\dot{x} = \frac{d}{dx}, \quad \dot{y} = \frac{d}{dy}.$$

Soit Φ ce que devient F après ce remplacement, de sorte que

$$\Phi = F(\dot{a}, \dot{b}, \dot{c}, \ldots : \dot{x}, \dot{y}),$$

et soit $G(a, b, c, \ldots : x, y)$ une seconde forme dérivée du même système, qui pourra être covariant ou contravariant; cela posé, suivant que le produit $F . G$ (c. à d. $F(a, b, c, \ldots : x, y)$ multiplié par $G(a, b, c, \ldots : x, y)$) est un covariant ou un contravariant, le résultat $\Phi \divideontimes G$ de l'opération de Φ sur G sera également un covariant ou un contravariant.

Considérons encore l'opération Ψ qui résulte d'une forme dérivée F lorsque, sans altérer les éléments, on remplace seulement les variables x, y par leurs inverses symboliques $\dot{x} = \frac{d}{dx}$, $\dot{y} = \frac{d}{dy}$. Dans ce cas comme dans celui que nous avons considéré en premier lieu et dans lequel on remplaçait seulement les éléments (et non les variables) par leurs inverses symboliques, le caractère de F est renversé, de covariant il devient contravariant et vice versâ. En un mot: une seule inversion symbolique renverse, deux inversions simultanées reproduisent le caractère de F. Ces propositions n'ont pas besoin d'être démontrées formellement, elles découlent comme conséquences des deux principes:

1°. que la marche du mouvement d'un système quelconque de lettres et de leurs inverses symboliques est contraire,

2°. que les mouvements induits dans les éléments* d'un Quantic *préparé* par deux mouvements contraires des variables sont eux-mêmes contraires.

Donnons le nom de différentiant-en-x à une fonction D' des éléments d'un Quantic binaire ou d'un système de plusieurs Quantics binaires, qui ait la propriété de rester la même après la substitution de $x + hy$ au lieu de x, c. à d. qui dans la notation pleine d'éléments affectés de multiplicateurs binômes satisfasse à l'identité

$$\Sigma (a\dot{b} + 2b\dot{c} + 3c\dot{d} + \ldots + ik\dot{l})\, D' = 0.$$

De même soit $'D$ un différentiant-en-y, c. à d. une fonction des éléments qui satisfasse à l'identité

$$\Sigma \{ib\dot{a} + (i-1)\, c\dot{b} + \ldots + l\dot{k}\}\, 'D = 0.$$

Pour les Quantics préparés ces équations prennent la forme

$$(1) \quad \Sigma [\sqrt{(n)}\, a\dot{b} + \sqrt{\{2\,(n-1)\}}\, b\dot{c} + \ldots + \sqrt{\{2\,(n-1)\}}\, h\dot{k} + \sqrt{(n)}\, k\dot{l}]\, D' = 0,$$

$$(2) \quad \Sigma [\sqrt{(n)}\, b\dot{a} + \sqrt{\{2\,(n-1)\}}\, c\dot{b} + \ldots + \sqrt{\{2\,(n-1)\}}\, k\dot{h} + \sqrt{(n)}\, l\dot{k}]\, 'D = 0.$$

On sait que D' sera toujours le coefficient de la plus haute puissance de x dans quelque covariant du système et $'D$ celui de la plus haute puissance de y

* De la combinaison de ces deux principes il résulte que le second principe peut être énoncé non seulement pour les éléments mais également pour leurs inverses symboliques.

dans quelque contravariant; et puisque en substituant au lieu des éléments leurs inverses symboliques le résultat de son action sur le covariant sera en vertu de notre dernier théorème un covariant et le coefficient de la plus haute puissance en x un différentiant-en-x, on en tire la conséquence que l'action d'un différentiant-en-y *rendu opératif* (par inversion symbolique) sur un différentiant-en-x donnera naissance à un différentiant-en-x: ce qui revient à dire que si $\dot{\Phi}$ est une fonction (des systèmes de $\dot{a}, \dot{b}, \dot{c}, \ldots$) qui satisfait à l'équation (2) quand on y remplace $a, b, c, \ldots$ par $\dot{a}, \dot{b}, \dot{c}, \ldots$ et $\dot{a}, \dot{b}, \dot{c}, \ldots$ par $\frac{d}{d\dot{a}}, \frac{d}{d\dot{b}}, \frac{d}{d\dot{c}}, \ldots$ c. à d. par $\ddot{a}, \ddot{b}, \ddot{c}, \ldots$ et que si D' satisfait à l'équation (1), alors $\dot{\Phi}D'$ doit satisfaire à la même équation (1).

Pour donner une démonstration indépendante de cette conclusion, je nommerai Ω' l'opérateur qui reduit D' à zéro, $'\Omega$ celui qui réduit $'D$ à zéro. La démonstration restant essentiellement la même dans le cas d'un système et dans celui d'un seul Quantic, on se bornera pour plus de simplicité à ce dernier cas, ce qui permet de supprimer les signes de sommation (Σ). Evidemment la proposition qu'on veut établir sera vraie si les deux opérations $\dot{\Phi}\Omega'$ et $\Omega'\dot{\Phi}$ que l'on obtient en appliquant l'opération $\dot{\Phi}$ et l'opération Ω' l'une après l'autre dans un ordre différent, ne diffèrent pas entre elles, ou ce qui est la même chose, si $\dot{\Phi}\Omega'$ et $\Omega'\dot{\Phi}$ ne diffèrent pas en puissance opérative.

Or bornons-nous pour le moment à un seul terme quelconque, p. e. au terme $\lambda p\dot{q}$ contenu dans Ω' (λ étant un nombre), et considérons la différence entre l'opération de $p\dot{q}\dot{\Phi}$ et de $\dot{\Phi}p\dot{q}$. Comme ce n'est que l'existence de $\dot{p}$ en $\dot{\Phi}$ qui produit cette différence, étudions l'effet de chaque terme $M\dot{p}^i$ séparément où M ne contient pas p. D'après le théorème de Leibnitz la différence entre l'effet de $\dot{p}^i(p\dot{q})$ et de $p\dot{q}(\dot{p}^i)$ sera $\dot{q}i\dot{p}^{i-1}$, c. à d. $\dot{q}\frac{d}{d\dot{p}}(\dot{p}^i)$ ou bien $\dot{q}\ddot{p}(\dot{p}^i)$.

Donc la valeur de la différence opérative entre $p\dot{q}\dot{\Phi}$ et $\dot{\Phi}p\dot{q}$ sera $\dot{q}\ddot{p}\dot{\Phi}$, et conséquemment la valeur totale de la différence entre $\Omega'\dot{\Phi}$ et $\dot{\Phi}\Omega'$ sera $\Sigma(\lambda\dot{q}\ddot{p}\dot{\Phi})$, c. à d. elle sera ce que Ω' devient quand après avoir renversé l'ordre des lettres dans chaque conjonction $a\dot{b}, b\dot{c}, c\dot{d}$ qui s'y trouve, on remplace les lettres non-accentuées par les lettres une fois accentuées et ces dernières par les lettres deux fois accentuées, ce qui fait voir que la différence opérative entre $\dot{\Phi}\Omega'$ et $\Omega'\dot{\Phi}$ sera nulle, vu que par hypothèse $\Phi(\dot{a}, \dot{b}, \dot{c}, \ldots)$ est un différentiant-en-y de l'expression dans laquelle se change le Quantic donné (ou bien les Quantics simultanés donnés) quand on y remplace les éléments $a, b, c, \ldots$ par leurs inverses $\dot{a}, \dot{b}, \dot{c}, \ldots$, et que Ω' se change en $'\Omega$ quand on renverse l'ordre des éléments. J'ajouterai un seul exemple pour illustrer ce résultat, et pour éviter l'emploi des racines carrées je me servirai de la forme pleine pour les opérandes et de la forme vide pour les opérateurs.

Soit donné le Quantic $(x, y)^3$ et choisissons-en le discriminant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.$$

Differentiant par rapport à a cet invariant que je regarde pour l'instant comme un différentiant-en-y, j'obtiens le nouveau différentiant-en-y

$$ad^2 - 3bcd + 2c^3.$$

Pour obtenir l'opérateur qui y répond par rapport à la forme vide, il faut écrire $\frac{b}{3}, \frac{c}{3}$ au lieu de b, c, ce qui donne l'opérateur

$$27\dot{a}\dot{d}^2 - 9\dot{b}\dot{c}\dot{d} + 2\dot{c}^3.$$

Appliquons cet opérateur au différentiant-en-x, que l'on obtient en multipliant le discriminant par $ac - b^2$ et qui est

$$4a^2c^4 - 7ab^2c^3 - (6a^2bd + 3b^4)\, c^2 + (a^3d^2 + 10ab^3d)\, c - a^2b^2d^2 - 4b^5d.$$

Le résultat des différentiations indiquées sera

$$192a^2c - 84ab^2 - 270ab^2 + 108a^2c - 108ab^2 + 162a^2c = 462a\,(ac - b^2),$$

ce qui est évidemment un différentiant-en-x, comme il doit être. Passons rapidement à l'établissement des théorèmes analogues relatifs aux formes dérivées d'un nombre quelconque de variables.

La loi des mouvements contraires étant vraie pour les Quantics binaires dûment préparés, sera également vraie pour les Quantics ternaires pareillement préparés. Car soit i le degré d'un Quantic dans ses variables x, y, z; qu'on le range suivant les puissances ascendantes de z, évidemment chacun des Quantics binaires qui multiplient ces puissances sera dûment préparé. Le premier aura pour son équipement numérique les racines carrées des nombres binômes de l'ordre i, le second les racines carrées des nombres binômes de l'ordre $i-1$ multipliés chacun par $\sqrt{i}$, le troisième les racines carrées des nombres binômes de l'ordre $i-2$ multipliés chacun par $\sqrt{\tfrac{1}{2}i\,(i-1)}$, et ainsi de suite. Or il est facile de voir comme auparavant que le théorème sera vrai pour des substitutions quelconques s'il est vrai pour les substitutions pour lesquelles le déterminant est l'unité, et chaque substitution de ce dernier genre peut être effectuée par une succession de substitutions simples de la forme $x + hy$; $y + kz$, etc. Donc on n'a besoin que de démontrer le théorème pour une seule substitution de ce genre comme $x + hy$: mais pour cette substitution, tous les Quantics en x, y dont j'ai parlé étant dûment préparés, on a déjà démontré que le théorème est vrai. Donc le théorème est vrai pour chaque Quantic ternaire. De la même façon on passe des Quantics ternaires aux Quantics quaternaires et de même progressivement aux Quantics d'un nombre quelconque de variables. De plus il est facile de voir que la démonstration peut être étendue sans difficulté à des Quantics multipartites, c. à d. contenant un nombre quelconque de systèmes de

variables: car chacun de ces systèmes étant assujetti à une substitution à part, le système des éléments subira une substitution composée des substitutions induites par chacune des substitutions partielles relatives à un système isolé de variables. De plus deux substitutions contraires appliquées à un quelconque des systèmes de variables induira deux substitutions contraires appliquées aux éléments.

Si pour donner plus de simplicité aux énoncés, on se borne au cas de Quantics unipartites, on peut résumer les conséquences qui découlent des principes établis en affirmant qu'une dérivée invariantive d'un système quelconque de Quantics unipartites préparés reste une dérivée invariantive, quand on substitue pour les variables ou pour les éléments ou pour les unes et les autres simultanément, leurs inverses symboliques avec la distinction que sous la première supposition le caractère est changé dans son opposé et sous la dernière il reste le même.—Dans mes premiers mémoires sur ce sujet dans le *Quarterly Journal of Mathematics* j'ai déjà donné substantiellement cette loi en me servant de la forme pleine pour les opérandes et de la forme vide pour les opérateurs—mais je crois que personne n'en a jamais donné la preuve.—C'est l'idée lumineuse et très-inattendue de la loi des mouvements contraires relative aux Quantics *préparés* qui simplifie la théorie et en rend la démonstration presque intuitive.

Cependant ce n'est que par exception qu'on doit se servir de la forme *préparée* pour désigner les Quantics.—Parmi les autres avantages de la notation ordinaire on peut citer la *permanence* de chaque expression d'un différentiant, c. à d. qu'un différentiant qui appartient à un Quantic d'un degré quelconque restera un différentiant de tout Quantic contenant le même nombre de variables d'un degré supérieur. Car soit

$$\Omega = a\dot{b} + 2b\dot{c} + \ldots + ik\dot{l} \quad \text{et} \quad \Omega F(a, b, c, \ldots, l) = 0,$$

il est évident que si l'on augmente Ω par des termes additionnels

$$(i+1)\, l\dot{m} + \text{etc.}$$

et que l'on désigne par Ω_1 l'opération Ω augmenté, on aura

$$\Omega_1 F(a, b, c, \ldots, l) = 0.$$

Dans cette notation un covariant ou contravariant qui appartient à un Quantic quelconque donné, appartiendra donc également à tout autre Quantic composé du même nombre de variables et qui, en dépendant des mêmes éléments, s'élève pourtant à un degré supérieur*.

* Il peut arriver qu'un différentiant qui est irréductible pour un degré donné de son Quantic cesse de l'être pour un degré supérieur. Cela a lieu, par exemple, dans le cas du discriminant de la forme binaire du troisième ou du cinquième degré (il va sans dire qu'en élevant le degré, on augmente en même temps le nombre des éléments). Il y a donc des différentiants qui sont absolument irréductibles et d'autres qui ne le sont que conditionellement. Ainsi $a^2d - 3abc + 2b^3$

De plus il y a dans cette notation des moyens qui permettent de donner à un différentiant unique la faculté de propager, pour ainsi dire, son espèce, sans agir sur une autre forme du même genre et sans en subir l'action.— Voici un exemple de ce genre de propagation. Soit $F(a, b, c, \ldots, l)$ un différentiant-en-x d'un Quantic binaire donné $(a, b, c, \ldots, l \between x, y)^j$, de l'ordre j dans les éléments; remplaçons les éléments $a, b, c, \ldots, l$ de F par $s_0, s_1, s_2, \ldots, s_j$, où s_q signifie la somme des puissances $q^{\text{èmes}}$ des racines $\frac{x}{y}$ du Quantic donné; alors $a^\mu F(s_0, s_1, s_2, \ldots, s_j)$ restera encore un différentiant-en-x du même Quantic, μ étant un nombre égal à l'ordre du différentiant transformé. En effet les formules connues du calcul différentiel pour passer d'un système donné de variables indépendantes à un autre, étant appliquées à la transformation de l'expression $\frac{d}{d\alpha_1} + \frac{d}{d\alpha_2} + \ldots + \frac{d}{d\alpha_j}$, où $\alpha_1, \alpha_2, \ldots, \alpha_j$ désignent les valeurs de $\frac{x}{y}$ qui annulent le Quantic donné, cette expression se transformera dans l'une et l'autre des deux expressions

$$-\frac{1}{a}[a\dot{b} + 2b\dot{c} + 3c\dot{d} + \ldots], \quad [s_0\dot{s}_1 + 2s_1\dot{s}_2 + 3s_2\dot{s}_3 + \ldots].$$

Par conséquent l'identité $(a\dot{b} + 2b\dot{c} + \ldots)D = 0$ étant satisfaite, l'identité corrélative $(s_0\dot{s}_1 + 2s_1\dot{s}_2 + \ldots)\Delta = 0$ le sera également, Δ désignant la transformée de D selon la règle donnée; et comme à chaque différentiant appartient un seul covariant dont il constitue un coefficient principal, on a le moyen de passer par une substitution facile d'un invariant ou covariant à un autre covariant qui sera en général d'un degré différent par rapport aux variables. Ainsi par exemple si l'on regarde l'invariant $ae - 4bd + 3c^2$ appartenant au Quantic $(a, b, c, d, e \between x, y)^4$ comme un différentiant-en-x, on voit que $\alpha, \beta, \gamma, \delta$ étant les quatre valeurs de $\frac{x}{y}$ qui annulent le Quantic donné,

$$a^4\{(\alpha^4+\beta^4+\gamma^4+\delta^4) - 4(\alpha+\beta+\gamma+\delta)(\alpha^3+\beta^3+\gamma^3+\delta^3) + 3(\alpha^2+\beta^2+\gamma^2+\delta^2)^2\}$$

sera aussi un différentiant-en-x du Quantic donné. On vérifie aisément que cette expression est:

$$a^4[\tfrac{1}{3}\{\Sigma(\alpha-\beta)^2\}^2 - \tfrac{2}{3}\Sigma(\alpha\gamma-\beta\delta)^2] = -b(ac-b^2)^2 - \tfrac{3}{2}a^2(ae-4bd+3c^2);$$

ainsi l'invariant $ae - 4bd + 3c^2$ donne naissance au différentiant $ac - b^2$ dont le poids est égal à $\frac{2.4}{2} - 2$ et qui par conséquent sert à déterminer un

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appartient à la première catégorie, $a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$ à la seconde; car en l'attribuant au Quantic $(a, b, c, d, e \between x, y)^4$, il peut être exprimé sous la forme

$$(ac-b^2)(ae-4bd+3c^2) - a(ace-ad^2+2bcd-c^3-b^2e),$$

c. à d. il devient une fonction entière des quatre différentiants

$$a, \quad ac-b^2, \quad ae-4bd+3c^2, \quad ace-ad^2+2bcd-c^3-b^2e.$$

covariant quadratique de l'ordre 2 dans les éléments. Ainsi un invariant a servi à produire un covariant. La double représentation des différentiants au moyen des éléments et au moyen des racines, fournit une démonstration d'un théorème assez important dans la théorie des partitions qu'il serait peut-être difficile d'établir par une démonstration directe. Voici en quoi consiste ce théorème. Désignons le nombre de manières de composer un nombre n avec j des nombres 0, 1, 2, ..., i par $(n:i,j)$: on sait que $(n:i,j)=(n:j,i)$ et l'on voit sans peine que si

$$m+\mu=ij, \quad (m:i,j)=(\mu:i,j).$$

Cela posé, le théorème en question affirme que pour les valeurs de m qui n'excèdent pas $\frac{1}{2}ij$, $(m:i,j)$ peut être égal à $(m-1:i,j)$ ou plus grand que ce nombre, mais non plus petit que ce nombre, ou, ce qui revient au même, que si m est plus grand que $\frac{1}{2}ij$, $(m:i,j)$ ne peut pas être plus grand que $(m-1:i,j)$. L'une de ces propositions implique l'autre en vertu de l'égalité $(m:i,j)=(ij-m:i,j)$. C'est la première que je veux établir et je l'établirai au moyen de la seconde. Cette démonstration étant accomplie je ferai une généralisation facile et du théorème et de la démonstration qui y conduit, pour en faire l'extension aux systèmes de couples i, j.

En vertu de l'équation $\Omega D=0$, on sait selon l'observation précieuse que M. Cayley a fait le premier, que le nombre de différentiants linéairement indépendants de l'ordre j dans les éléments, qui appartiennent à un Quantic binaire donné du degré i, dont le poids par rapport à x est w, doit être égal à $(w:i,j)-(w-1:i,j)$.

Sans même se servir de cette proposition qui est certainement vraie mais qui exige la vérification de l'indépendance des équations qu'on obtient en satisfaisant à l'identité $\Omega D=0$, on peut affirmer avec une certitude absolue que le nombre des différentiants dont il s'agit *ne peut pas être inférieur à* $(w:i,j)-(w-1:i,j)$, ce qui suffit pour la démonstration proposée. Or je dis qu'il ne peut pas exister de différentiants pour lesquels w est plus grand que $\frac{1}{2}ij$. Car en vertu de l'identité $\Omega=a\Sigma\frac{d}{d\alpha}$ un différentiant quelconque doit être de la forme $a^j\Sigma(\alpha-\alpha')(\alpha''-\alpha''')(\alpha^{\text{IV}}-\alpha^{\text{V}})\ldots$ où le nombre des facteurs est w et chaque α une des i valeurs de $\frac{x}{y}$ qui annulent le Quantic donné du degré i, bien entendu qu'il n'y a nulle restriction sur la répétition des mêmes racines. L'ordre de cette fonction symétrique relatif aux coefficients étant j, on en conclura d'après un théorème connu de l'algèbre ordinaire qu'aucune racine α ne peut se présenter plus de j fois, mais dans chacun des w facteurs il paraîtra deux lettres; donc le poids w est la moitié du nombre total de ces apparitions. Or puisque nulle lettre ne paraît plus de j ois, le nombre total de ces apparitions aura ij pour son maximum et conséquemment la valeur maximum de w est $\frac{1}{2}ij$, c. à d. qu'il n'existe pas de

différentiants pour lesquels w excède $\frac{1}{2}ij$; donc comme il existe toujours $(w:i,j)-(w-1:i,j)$ *au moins* (je dis *au moins* pour ne pas m'appuyer sur la vérification de l'indépendance citée plus haut), il s'ensuit que, pour $w>\frac{1}{2}ij$, $(w:i,j)$ ne peut pas excéder $(w-1;i,j)$ et par conséquent que pour $w=\frac{1}{2}ij$ ou pour $w<\frac{1}{2}ij$, $(w:i,j)$ ne peut pas être plus petit que $(w-1:i,j)$, ce qu'il fallait démontrer. On peut étendre ce raisonnement en se fondant sur la proposition que pour un nombre quelconque de Quantics, p. e. pour deux Quantics $(a, b, c, \ldots, l \between x, y)^i$, $(a', b', \ldots, f' \between x, y)^{i'}$, le nombre des différentiants du poids w en x, de l'ordre j dans $a, b, c, \ldots$ et de l'ordre j' dans $a', b', \ldots$ a pour expression ou au moins pour valeur maximum* la différence entre deux dénumérants dont l'un est le nombre de solutions en nombres positifs entiers du système

$$x_0+x_1+\ldots+x_i=j, \quad y_0+y_1+\ldots+y_{i'}=j',$$
$$x_1+2x_2+\ldots+ix_i+y_1+2y_2+\ldots+i'y_{i'}=w,$$

et l'autre le dénumérant du système qui en résulte lorsqu'on y remplace w par $w-1$. En suivant cette voie et après avoir démontré par la même méthode dont on s'est servi ci-dessus et à l'aide des fonctions symétriques des racines des deux Quantics donnés que la valeur maximum de w est $\frac{ij+i'j'}{2}$, on arrivera à cette conclusion analogue que la valeur de la différence entre ces deux dénumérants ne peut jamais être négative, conclusion qui reste vraie en général. A ce résultat on peut donner l'énoncé remarquable : Que l'on développe suivant les puissances de $a, b, c, \ldots$ le produit d'un nombre quelconque de fonctions

$$\begin{aligned}&[(1-a)(1-at)(1-at^2)\ldots(1-at^i)]^{-1}\times\\&[(1-b)(1-bt)(1-bt^2)\ldots(1-bt^k)]^{-1}\times\\&[(1-c)(1-ct)(1-ct^2)\ldots(1-ct^l)]^{-1}\times\\&\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}$$

que l'on cherche dans ce développement la fonction de t qui multiplie un produit quelconque donné de puissances de $a, b, c, \ldots$; cette fonction ordonné suivant les puissances ascendantes de t présentera une série de coefficients numériques distribués symétriquement autour de son milieu et ayant des valeurs non décroissantes depuis l'une des extrémités jusqu'au terme unique ou jusqu'aux deux termes qui forment le milieu de la série.

L'importance de cette proposition pour la théorie des invariants consiste dans le fait qu'elle énonce et d'après lequel *l'expression analytique* du nombre des différentiants linéairement indépendants d'un système de

* Abstraction faite de l'indépendance non démontrée des équations données par l'application de l'opérateur $\left(a\frac{d}{db}+2b\frac{d}{dc}+\ldots\right)+\left(a'\frac{d}{db'}+2b'\frac{d}{dc'}+\ldots\right)$.

Quantics binaires donné est toujours un nombre positif ou nul, pourvu que ces différentiants soient d'une forme concevable, c. à d. que le poids donné w n'excède pas le maximum dont il est susceptible. Au contraire, pour les différentiants inconcevables, c. à d. dont le poids donné excède le maximum, l'expression analytique du nombre est toujours négatif ou zéro.

La loi de l'accroissement et du décroissement citée ci-dessus peut être exprimée au moyen d'une transformation facile à vérifier. En se bornant au cas d'un seul Quantic on a l'énoncé qu'en représentant par $\chi(\theta)$ un des deux produits

$$(1-2a\cos\theta+a^2)(1-2a\cos 3\theta+a^2)\ldots\{1-2a\cos(2q+1)\theta+a^2\},$$

$$(1-a)(1-2a\cos 2\theta+a^2)(1-2a\cos 4\theta+a^2)\ldots(1-2a\cos 2q\theta+a^2),$$

la fraction $\dfrac{\sin\theta}{\chi(\theta)}$ développée selon les puissances positives de a et les sinus des multiples de θ sera *omnipositive*, c. à d. ne contiendra que des coefficients numériques positifs.—La même conclusion aura lieu quand on multiplie ensemble plusieurs fonctions de l'une ou l'autre forme de χ, savoir

$$\chi(a,q,\theta)\,\chi(a',q',\theta)\ldots.$$

En désignant par $\Pi\chi(\theta)$ ce produit, le développement de $\dfrac{\sin\theta}{\Pi\chi(\theta)}$ suivant les puissances de a, a', a'', ... et leurs combinaisons et suivant les sinus des multiples de θ sera omnipositif.

Il paraît que ce théorème reste vrai quand on considère la fonction entière $\sin\theta\Pi\chi(\theta)$ au lieu de la fonction fractionnaire, mais je n'en possède point de preuve. Dans le cas simple de $\sin\theta\chi(\theta)$ cela reviendrait à l'énoncé que les coefficients des puissances de t dans le développement de

$$(1+a)(1+at)(1+at^2)\ldots(1+at^i),$$

qui forme évidemment une série symétrique, jouissent de la même propriété que les coefficients du développement de la fonction réciproque, c. à d. que les valeurs des coefficients peuvent augmenter ou rester stationnaires en passent de l'une ou l'autre extrémité de la série vers le milieu, mais qu'elles ne peuvent jamais décroître.—Il paraît qu'une proposition analogue peut être avancée pour le produit $\phi(1)\,\phi(t)\,\phi(t^2)\ldots\phi(t^i)$, où $\phi(x)$ signifie

$$1+ax+a^2x^2+\ldots+a^jx^j,$$

et pour des formes encore plus générales.

Dans les recherches précédents je suis tombé sur une démonstration exacte du théorème fondamental de la théorie des invariants, théorème qui a été accepté comme vrai par son illustre auteur M. Cayley sur la foi d'une induction *à posteriori* purement empirique et dont l'exactitude a été révoquée en doute par un écrivain distingué sur les formes binaires, apparemment en conséquence d'une méprise relative à l'explication donnée par M. Cayley sur la source de la conclusion erronée qu'il avait énoncée sur le nombre des invariants fondamentaux pour les degrés supérieurs.

On démontre facilement que si $D(w:i,j)$ est le nombre des différentiants linéairement indépendants* de l'ordre j, du poids w, et qui appartiennent à un Quantic binaire du degré i,

$$D(w:i,j) = \text{ou} > (w:i,j) - (w-1:i,j),$$

différence que je dénoterai désormais par $\Delta(w:i,j)$. Cette conclusion est une conséquence immédiate de l'identité $\Omega D = 0$, D étant un différentiant quelconque du Quantic $(a, b, c, \ldots, l \between x, y)^i$ et Ω l'opérateur

$$a\delta_b + 2b\delta_c + 3c\delta_d + \ldots.$$

Mais pour établir le théorème en question, c. à d. l'équation

$$D(w:i,j) = \Delta(w:i,j),$$

il faudrait avoir prouvé l'indépendance de toutes les équations entre les constantes indéterminées, que l'identité $\Omega D = 0$ fournit (en regardant D comme une fonction composée des combinaisons des $a, b, c, \ldots$ multipliées chacune par une telle constante)—ce qui n'a jamais été fait et offre des difficultés presque insurmontables si l'on se propose de résoudre la question par escalade.—Je suivrai une marche différente—commençant par l'alternative d'égalité ou de supériorité entre D et Δ, je démontre que la dernière est inadmissible—l'indépendance dont j'ai parlé est donc une conséquence et non la clef de la démonstration.—Lorsqu'un opérateur quelconque Φ satisfait à l'équation $\Phi F = G$, je dirai dans ce qui suit que Φ transforme F en G, et lorsqu'on a identiquement $\Phi F = 0$, je dirai que Φ annule F.

Je remarque que $D(0:i,j) = 1$ parce que dans tous les cas il y a un seul différentiant-en-x libre du poids zéro, à savoir une puissance de a: d'autre part $(0:i,j)$, c. à d. le nombre de manières de composer zéro avec $0, 1, 2, 3, \ldots, i$ prises j à j, est aussi $=1$, par conséquent la relation $D(w:i,j) \geqq \Delta(w:i,j)$ fournit

$$D(w:i,j) + D(w-1:i,j) + D(w-2:i,j) + \ldots + D(0:i,j) \geqq (w:i,j),$$

où le symbole $\geqq$ signifie "est égal à ou plus grand que."

* Pour plus de commodité je dirai différentiants *libres* au lieu de différentiants linéairement indépendants.

De plus on aura

$$D(w:i,j) + 2D(w-1:i,j) + 3D(w-2:i,j) + \ldots$$
$$\mathrel{\substack{=\\>}} (w:i,j) + (w-1:i,j) + (w-2:i,j) + \ldots,$$

ce qui est vrai pour toutes les valeurs de w. Je supposerai à présent que w ait la valeur $\frac{1}{2}ij$ pour ij pair et la valeur $\frac{1}{2}(ij-1)$ pour ij impair. Dans ce cas la somme qui forme la seconde partie de la dernière relation devient évidemment égale au nombre des combinaisons j à j (avec répétitions) formées des $(i+1)$ chiffres $0, 1, 2, 3, \ldots, i$, et assujetties à la restriction que la somme des chiffres d'une combinaison n'excède pas $\frac{1}{2}ij$: nombre de combinaisons que je dénoterai par $P(i,j)$.

Remplaçons chaque différentiant qui fait partie du groupe dont le nombre est $D(w:i,j)$, de même du groupe dont le nombre est $D(w-1:i,j)$, etc. par le covariant qui y correspond.—Le degré de ces covariants par rapport aux variables étant, pour une valeur quelconque de w, $ij-2w$, les degrés des covariants dans les groupes successifs seront $1, 3, 5, \ldots$ dans le cas de ij impair, et $0, 2, 4, \ldots$ dans le cas de ij pair. Imaginons que chaque coefficient de chacun de ces covariants soit représenté par une dame d'un damier, qu'on se borne à prendre le premier coefficient des covariants du premier groupe, les deux premiers coefficients des covariants du second groupe, les trois premiers du troisième groupe, etc., on peut alors former un triangle rectangulaire de piles des dames. La pile au sommet contiendra $D(w:i,j)$, les deux piles qui suivent $D(w-1:i,j)$, les trois piles qui suivent $D(w-2:i,j)$ chacune, et ainsi de suite. Le nombre total des dames sera la fonction qui est $\mathrel{\substack{=\\>}} P(i,j)$.

Pour donner plus de précision à cette image je remarque que les dames dans la première colonne verticale représentent des différentiants et que chaque pile à la base se réduit nécessairement à une seul dame, dont la partie essentielle (abstraction faite de la partie numérique qui n'a pas d'influence sur le raisonnement et que je négligerai dans tout ce qui suit) n'est autre chose qu'un coefficient du Quantic du degré i élevé à la puissance j.

Remarquons que, d'après une propriété bien connue des covariants, chaque quantité dans la première colonne sera annulée par Ω, dans la seconde par $(\Omega)^2$, en général dans la $q^{\text{ème}}$ par $(\Omega)^q$ et que l'opérateur $(\Omega)^{q-1}$ appliqué à un terme de la $q^{\text{ème}}$ colonne produit le différentiant qui se trouve à la première place de la même horizontale avec ce terme.

Remarquons encore qu'en avançant de gauche à droite dans la même horizontale, les poids des quantités augmentent d'une unité de l'une à l'autre, qu'au contraire, en avançant de haut en bas dans la même verticale, les poids des quantités diminuent d'une unité de l'une à l'autre, de sorte

que dans une ligne diagonale descendante (de gauche à droite), ou ce qui est la même chose, parallèle à l'hypoténuse, tous les termes sont de poids égal.

Or j'affirme que nulle liaison linéaire ne peut exister entre les quantités du triangle en question. Evidemment ce n'est qu'entre les quantités isobariques qu'une telle liaison serait imaginable. Prenons une ligne quelconque parallèle à l'hypoténuse ou bien l'hypoténuse elle-même.

1°. Je dis que nulle équation linéaire ne peut lier des quantités qui se trouvent exclusivement dans une seule pile. Car si cette pile se trouvait dans la $q^{\text{ème}}$ colonne, en vertu du fait que l'opérateur $(\Omega)^{q-1}$ fait naître de chacune d'elle le différentiant qui se trouve à la première place de la même ligne horizontale, la liaison supposée subsisterait encore entre des différentiants d'une même pile, ce qui est contraire à l'hypothèse de la construction.

2°. Je dis que nulle équation linéaire ne peut lier les quantités qui se trouvent dans des piles distinctes. En effet, supposons donnée une relation de ce genre, d'après la condition du poids égal il ne peut y avoir dans chaque colonne qu'une seule pile comprenant des quantités qui entrent dans l'équation supposée. Soit q le rang de la colonne la plus avancée qui renferme une pile comprenant des quantités liées entre elles par l'équation linéaire. L'opérateur Ω, appliqué au premier membre de l'équation qui exprime cette liaison un nombre de fois inférieur à $q-1$, produira une équation d'une forme analogue. Mais lorsqu'on applique l'opérateur $(\Omega)^{q-1}$, toutes les quantités comprises dans des colonnes d'un rang inférieur à q seront annulées, tandis que celles qui sont comprises dans la pile de la $q^{\text{ème}}$ colonne seront transformées en des différentiants appartenant à la même ligne, c. à d. qu'il y aurait une liaison linéaire entre les différentiants d'une même pile, ce qui est contraire à l'hypothèse de la construction.

On a donc démontré que nulle équation linéaire ne subsiste entre les quantités du triangle.—De plus il est évident que le poids d'un coefficient quelconque qui se trouve dans le triangle ne peut excéder $\frac{1}{2}ij$. Donc les quantités comprises dans le triangle sont des fonctions linéaires et homogènes sans liaison linéaire entre elles de $P(i, j)$ quantités. Donc le nombre de ces quantités ne peut pas excéder $P(i, j)$, c. à d. que

$$D(w : i, j) + 2D(w-1 : i, j) + 3D(w-2 : i, j) + \dots$$

ne peut pas excéder $P(i, j)$.

Mais si dans une seule des relations $D(w : i, j) \geqq \Delta(w : i, j)$, pour des valeurs de w quelconques, le signe applicable était $>$ et non $=$, la somme en question serait $> P(i, j)$.

Donc on a toujours $D(w:i,j) = \Delta(w:i,j)$, ce qu'il fallait démontrer. Comme corollaire il s'ensuit que l'indépendance des équations données par l'identité $\Omega D = 0$ est établie. Précisément la même méthode peut être suivie pour démontrer l'égalité

$$D(w:i,j:i',j':\text{etc.}) = \Delta(w:i,j:i',j':\text{etc.})$$

où D dénote le nombre des différentiants libres d'un système de Quantics binaires, i, i', ... désignant les degrés des Quantics, j, j', ... l'ordre des différentiants par rapport aux coefficients de chacun des Quantics, et où Δ dénote la différence entre deux dénumérants, l'un désignant le nombre des solutions en nombres entiers et positifs du système des équations simultanées

$$x_0 + x_1 + x_2 + \ldots + x_i = j, \quad x'_0 + x'_1 + \ldots + x'_{i'} = j', \text{ etc.}$$
$$x_1 + 2x_2 + \ldots + ix_i + x'_1 + 2x'_2 + \ldots + i'x'_{i'} + \ldots = w,$$

et l'autre le dénumérant du système qui en résulte lorsqu'on y remplace w par $w-1$.

Un autre corollaire que le théorème contient comme cas particulier est la proposition déjà démontrée, que $\Delta(w:i,j)$ ne peut jamais devenir négatif pour des valeurs de w qui n'excèdent pas $\frac{1}{2}ij$. En effet si cette assertion n'était pas vraie, il devrait exister une valeur de w qui n'excède pas $\frac{1}{2}ij$ et pour laquelle $D(w:i,j) > \Delta(w:i,j)$, ce qui a été prouvé impossible.

En dernier lieu je remarque qu'en démontrant inadmissible le signe de supériorité, on a établi pour $w = \frac{1}{2}ij$ quand ij est pair et pour $w = \frac{1}{2}(ij-1)$ quand ij est impair, l'équation

$$D(w:i,j) + 2D(w-1:i,j) + 3D(w-2:i,j) + \ldots = P(i,j).$$

Soit ij impair, en vertu de l'équation $(x:i,j) = (ij-x:i,j)$ le nombre $P(i,j)$ sera évidemment la moitié du nombre total des combinaisons j à j des $i+1$ éléments 0, 1, 2, 3, ..., i. Donc pour ij impair $P(i,j) = \frac{1}{2}\dfrac{\Pi(i+j)}{\Pi i \Pi j}$.

Soit au contraire ij pair, on aura

$$\begin{aligned} P(i,j) &= \tfrac{1}{2}\left\{\frac{\Pi(i+j)}{\Pi i \Pi j} + (w:i,j)\right\} \\ &= \tfrac{1}{2}\frac{\Pi(i+j)}{\Pi i \Pi j} + \tfrac{1}{2}\{D(w:i,j) + D(w-1:i,j) + D(w-2:i,j) + \ldots\}. \end{aligned}$$

Le degré des covariants qui correspondent un à un aux différentiants dont le nombre est $D(x:i,j)$ étant $ij-2x$, on peut substituer pour $D(x:i,j)$ le nombre $K(i,j:ij-2x)$ où i est le degré du Quantic donné, j l'ordre par rapport aux coefficients, $ij-2x$ le degré relatif aux variables des covariants dont K exprime le nombre total.

Par conséquent quand ij est impair, on aura

$$2K(i, j:1) + 4K(i, j:3) + 6K(i, j:5) + \ldots = \frac{\Pi(i+j)}{\Pi i \Pi j}$$

et quand ij est pair

$$K(i, j:0) + 3K(i, j:2) + 5K(i, j:4) + \ldots = \frac{\Pi(i+j)}{\Pi i \Pi j}.$$

En remarquant que pour ij impair il n'existe pas de covariants de degré pair, et pour ij pair il n'en existe pas de degré impair, on peut réunir ces deux formules dans une seule formule remarquable, qui assujettit les quantités transcendantes K à une loi algébrique et qui pourrait même être très-utile dans certains cas comme formule de vérification :

$$K(i, j:0) + 2K(i, j:1) + 3K(i, j:2) + \ldots = \frac{\Pi(i+j)}{\Pi i \Pi j}.$$

J'en donnerai quelques exemples.

Soit $i = 4, \quad j = 2.$

On trouve

$$K(4, 2:0) = 1 ; \; K(4, 2:2) = 0 ; \; K(4, 2:4) = 1 ; \; K(4, 2:6) = 0 ;$$
$$K(4, 2:8) = 1$$

et de là $1 + 5 + 9 = 15 = \dfrac{\Pi 6}{\Pi 2 \, \Pi 4}.$

Soit $i = 3, \quad j = 3.$

En se rappelant l'échelle fondamentale pour les cubiques

$$3.1, \quad 4.0, \quad 2.2, \quad 3.3$$

on trouve

$$K(3, 3:1) = 0, \; K(3, 3:3) = 1, \; K(3, 3:5) = 1, \; K(3, 3:7) = 0,$$
$$K(3, 3:9) = 1$$

et de là $4 + 6 + 10 = 20 = \dfrac{\Pi 6}{\Pi 3 \, \Pi 3}.$

Soit $i = 3, \quad j = 4.$

On trouve

$$K(3, 4:0) = 1, \; K(3, 4:2) = 0, \; K(3, 4:4) = 1, \; K(3, 4:6) = 1,$$
$$K(3, 4:8) = 1, \; K(3, 4:10) = 0, \; K(3, 4:12) = 1$$

et de là $1 + 5 + 7 + 9 + 13 = 35 = \dfrac{\Pi 7}{\Pi 3 \, \Pi 4}.$

Le théorème que j'ai vérifié par ces exemples peut être résumé dans les termes suivants. Chaque covariant d'un ordre donné j par rapport aux coefficients d'un Quantic binaire de degré donné i, étant répété autant de fois qu'il y a de chiffres dans la série qui commence par zéro et se termine

par le degré du covariant, relatif aux variables qui y entrent, le nombre total de ces expressions, chacune comptée autant de fois qu'elle est répétée, est égal au nombre binôme symétrique par rapport aux nombres i et j, c. à d. égal à $\frac{\Pi(i+j)}{\Pi i \Pi j}$.

La règle des nombres binômes s'applique avec une modification légère au cas de plusieurs Quantics binaires de degrés donnés et de covariants d'ordres donnés relatifs aux coefficients de ces Quantics. Dans ce cas général on substituera au nombre binôme unique qui se présente dans le cas d'un seul Quantic, le produit de plusieurs nombres binômes dont chacun est symétrique par rapport au degré i de l'un des Quantics et à l'ordre j du covariant relatif aux coefficients du même Quantic.

Considérons comme exemple le cas de deux quadratiques binaires. Dans ce cas qui correspond à $i=2$, $i'=2$ il y a trois covariants de l'ordre $j=1$ par rapport aux coefficients de chacune, savoir:

1° le produit des deux Quantics,

2° leur *Hessien*,

3° leur *Connectif*.

Les degrés de ces trois expressions relatifs aux variables étant respectivement 4, 2, 0, on aura

$$5+3+1=\left(\frac{\Pi 3}{\Pi 1 \Pi 2}\right)^2=9,$$

ce qui s'accorde avec la règle énoncée ci-dessus.

A l'énumération que j'ai faite des propriétés essentielles du triangle de piles, j'ajoute la remarque que le poids maximum d'une quelconque des quantités qui s'y trouvent, est évidemment celui de la quantité qui appartient à l'hypoténuse et se trouve au sommet du triangle. Ce poids maximum est $\frac{1}{2}ij$ ou $\frac{1}{2}(ij-1)$ et par conséquent n'excède jamais $\frac{1}{2}ij$. C'est ainsi qu'on voit que les quantités comprises dans le triangle ne sont autre chose que des fonctions linéaires des combinaisons de l'ordre j par rapport aux coefficients du Quantic proposé, combinaisons dont le nombre est $P(i, j)$.

POSTSCRIPTUM 1. La démonstration donnée du théorème fondamental $D(w:i, j)=\Delta(w:i, j)$ peut être abrégée et simplifiée comme il suit.

Au lieu de se servir de la condition

$$D(w:i,j)+2D(w-1:i,j)+\ldots \gtreqless P(i,j)$$

il suffit de considérer l'équation préalable

$$\Sigma D(w:i,j)=(w:i,j).$$

Pour un différentiant quelconque que je désignerai par $[w-\delta]$ et dont le poids soit $w-\delta$ substituons l'expression $('\Omega)^\delta[w-\delta]$, expression qui résulte

de $[w-\delta]$ en y appliquant l'opérateur* $('\Omega)^\delta$ et qui jouit de la propriété qu'en opérant sur elle avec $(\Omega')^\delta$, le résultat est un multiple numérique de $[w-\delta]$. Toutes les expressions ainsi obtenues seront du même poids w et par conséquent des fonctions linéaires des $(w:i,j)$ combinaisons qui sont du poids w et de l'ordre j dans les $i+1$ coefficients du Quantic donné.

On démontre comme auparavant que ces expressions sont linéairement indépendantes entre elles et que par conséquent leur nombre ne peut pas excéder $(w:i,j)$; donc leur nombre est égal à $(w:i,j)$, et la proposition est établie.

Pour démontrer que $(\Omega')^\delta('\Omega)^\delta[w-\delta]$ est, à un facteur numérique près, égal à $[w-\delta]$, on n'a pas besoin de sortir de la sphère des différentiants et de faire appel aux propriétés des covariants. On établit aisément que pour une quantité quelconque D du poids w et de l'ordre j dans les coefficients d'un Quantic binaire du degré i, on aura

$$(\Omega'\,'\Omega-'\Omega\,\Omega')\,D=(ij-2w)\,D.$$

En partant de là et supposant que $\Omega'D=0$, on trouve par une induction algébrique facile que

$$(\Omega')^k('\Omega)^k D=1\,.\,2\ldots k(ij-w)(ij-w-1)\ldots(ij-w-k+1)\,.\,D,$$

où le facteur numérique ne s'évanouit que lorsque k est plus grand que $ij-w$.

Considérons le système complet des expressions

$$[w-\delta],\quad '\Omega[w-\delta],\quad ('\Omega)^2[w-\delta],\quad \ldots,\quad ('\Omega)^{ij-2w}[w-\delta],$$

dont la dernière, qui résulte de l'opération $'\Omega$ répétée $ij-2w$ fois, se réduit à un différentiant-en-y, tandis que les suivantes produites par la même opération répétée $ij-2w+1$ ou un plus grand nombre de fois s'évanouissent identiquement.

En représentant toujours par des piles l'ensemble de toutes les expressions $('\Omega)^m[w-\delta]$ pour les mêmes valeurs de m et de δ, distinguant les deux cas de ij pair ou impair, et commençant par la plus grande valeur de w qui est $\frac{1}{2}ij$ pour ij pair et $\frac{1}{2}(ij-1)$ pour ij impair, on arrivera aux deux tableaux† suivants de points, qui donnent une image du système en question de piles

pour ij impair | pour ij pair

* Le signe $('\Omega)^\delta$ exprime l'opération $'\Omega$ répétée δ fois.

† Le point au sommet du premier tableau correspond à des invariants, les points à gauche se rapportent aux différentiants-en-x, ceux à droite aux différentiants-en-y, ceux de la base aux coefficients successifs du Quantic élevé à la puissance j (pour l'un et l'autre des deux tableaux).

Prenons l'ensemble de toutes les combinaisons des coefficients du Quantic proposé du degré i, qui sont de l'ordre j dans les coefficients et des poids $0, 1, 2, \ldots, ij$ quant à x, alors, dans l'un cas comme dans l'autre, les quantités qui se trouvent dans chaque colonne verticale, seront du même nombre que l'ensemble correspondant des combinaisons des coefficients, elles seront en même temps des fonctions homogènes et linéaires des combinaisons qui y appartiennent.

Les piles qui se trouvent dans une ligne horizontale quelconque peuvent se réduire à une seule quantité, cas qui se présente toujours pour la dernière ligne horizontale: elles peuvent même s'évanouir identiquement, ce qui arrive pour certaines valeurs de w, i, j pour lesquelles il n'existe point de différentiant.

POSTSCRIPTUM 2. Je suppléerai dans ce qui suit à une lacune qui se trouve dans les recherches précédentes, en donnant la démonstration de la proposition suivante :

Dans un Quantic préparé les inverses symboliques des éléments subissent par une substitution quelconque des variables une substitution induite qui est identique avec celle que les éléments eux-mêmes subiraient par la substitution contraire des variables.

Les mêmes raisonnements dont on s'est déjà servi plusieurs fois, font voir que pour la démonstration générale de cette proposition il suffit de la vérifier dans le cas spécial dans lequel le Quantic est binaire et $x+\epsilon y$ la valeur que l'on substitue pour x, ϵ étant infiniment petit. Soit i le degré du Quantic donné, soient $a, b, c, \ldots, h, k, l$ ses éléments et

$$1, \quad \sqrt{i}, \quad \sqrt{\left\{\frac{i(i-1)}{2}\right\}}, \quad \ldots, \quad \sqrt{i}, \quad 1,$$

les multiplicateurs numériques des éléments, soient $a', b', c', \ldots$ les valeurs dans lesquelles se changent les éléments donnés après la substitution de $x+\epsilon y$ au lieu de x, ϵ étant infiniment petit; cela posé et en faisant

$$\lambda=\sqrt{i}, \quad \mu=\sqrt{\{2(i-1)\}}, \quad \nu=\sqrt{\{3(i-2)\}}, \quad \ldots$$

les nouveaux éléments et les éléments primitifs s'exprimeront les uns par les autres au moyen des relations linéaires

$$a'=a, \quad b'=b+\lambda\epsilon a, \quad c'=c+\mu\epsilon b, \quad \ldots, \quad k'=k+\mu\epsilon h, \quad l'=l+\lambda\epsilon k,$$
$$a=a', \quad b=b'-\lambda\epsilon a', \quad c=c'-\mu\epsilon b', \quad \ldots, \quad k=k'-\mu\epsilon h', \quad l=l'-\lambda\epsilon k'.$$

D'autre part les inverses symboliques de ces deux systèmes d'éléments

$$\dot{a}=\frac{d}{da}, \quad \dot{b}=\frac{d}{db}, \quad \ldots,$$
$$\dot{a}'=\frac{d}{da'}, \quad \dot{b}'=\frac{d}{db'}, \quad \ldots$$

satisfont aux équations

$$\begin{aligned}
\dot{a}' &= \dot{a}\,\frac{da}{da'} + \dot{b}\,\frac{db}{da'} = \dot{a} - \lambda\epsilon\dot{b},\\
\dot{b}' &= \dot{b}\,\frac{db}{db'} + \dot{c}\,\frac{dc}{db'} = \dot{b} - \mu\epsilon\dot{c},\\
&\vdots \qquad\qquad\qquad\qquad \vdots\\
\dot{k}' &= \dot{k}\,\frac{dk}{dk'} + \dot{l}\,\frac{dl}{dk'} = \dot{k} - \lambda\epsilon\dot{l},\\
\dot{l}' &= \dot{l}\,\frac{dl}{dl'} \qquad\qquad = \dot{l},
\end{aligned}$$

ce qui fait voir que la substitution des inverses symboliques $\dot{a}, \dot{b}, \ldots, \dot{l}$ induite par la substitution de $x + \epsilon y$ au lieu de x, y restant inaltéré, est précisément la même que la substitution contraire de $y - \epsilon x$ au lieu de y, x restant inaltéré, induirait dans les éléments mêmes $a, b, \ldots, l$.

Je terminerai ces additions par l'énoncé d'un théorème général sur les formes invariantives dérivées qui montre d'une manière frappante le parti avantageux que l'on tire de la forme préparée sous laquelle je présente les Quantics.

Soit $F(a, b, c, \ldots : x, y, \ldots)$ un contravariant et $\Phi(a, b, c, \ldots : x, y, \ldots)$ un covariant du même Quantic donné; on connaît depuis longtemps le théorème que la nouvelle forme

$$F\left(a, b, c, \ldots : \frac{d\Phi}{dx}, \frac{d\Phi}{dy}, \ldots\right)$$

est un covariant du même Quantic. Or j'ajoute que si le Quantic proposé est présenté dans la forme préparée, alors la nouvelle forme

$$F\left(\frac{d\Phi}{da}, \frac{d\Phi}{db}, \frac{d\Phi}{dc}, \ldots : x, y, \ldots\right)$$

sera également un covariant du même Quantic. Si le Quantic proposé est présenté dans la forme ordinaire (pleine), cette dernière expression se change en

$$F\left(\frac{1}{m}\frac{d\Phi}{da}, \frac{1}{n}\frac{d\Phi}{db}, \frac{1}{p}\frac{d\Phi}{dc}, \ldots : x, y, \ldots\right),$$

$m, n, p, \ldots$ désignant les nombres binômes ou polynômes qui multiplient les éléments $a, b, c, \ldots$, elle se change au contraire en

$$F\left(m\frac{d\Phi}{da}, n\frac{d\Phi}{db}, p\frac{d\Phi}{dc}, \ldots : x, y, \ldots\right),$$

si le Quantic est présenté dans la forme vide. La démonstration de ce théorème se fait immédiatement à l'aide des principes exposés dans ce mémoire.

28.

ON A RULE FOR ABBREVIATING THE CALCULATION OF THE NUMBER OF IN- OR CO-VARIANTS OF A GIVEN ORDER AND WEIGHT IN THE COEFFICIENTS OF A BINARY QUANTIC OF A GIVEN DEGREE.

[*Messenger of Mathematics,* VIII. (1879), pp. 1—8.]

IF i is the degree of a quantic we know now by *apodictic* reasoning that the number of its in- or co-variants of order j and of weight w in the coefficients is $(w: i, j) - \{(w-1): i, j\}$, where in general $(x: i, j)$ denotes the number of modes of composing x with j numbers each having any value from 0 to i (both inclusive) or (what is the same thing) with i numbers each having any value from 0 to j. The object of this note is to show how to calculate the *difference* between the two denumerants above given without calculating each of them separately, whereby the actual amount of calculation required will be reduced to a small fraction of what it would otherwise be. I shall not stop to draw theoretical consequences from this theorem, but present it to the readers of the *Messenger* in the way it has occurred to me as a rule for abbreviating labour.

It is founded on the exhaustive method of representing partition systems by following a dictionary order of sequence, and it will be best understood by beginning with an example.

Suppose then that $w=7$, $i=5$, $j=4$, we may find $(7: 5, 4)$ by setting out and counting the arrangements where 4 is the number of parts and 5 the limit to each part, namely, 5 . 2, 5 . 1 . 1, 4 . 3, 4 . 2 . 1, 4 . 1 . 1 . 1, 3 . 3 . 1, 3 . 2 . 2, 3 . 2 . 1 . 1, 2 . 2 . 2 . 1.

For brevity the zeros required to fill up the number of parts to 4 are omitted in this table.

To find $(6:\ 5,\ 4)$ we may consider

(1) Those arrangements which begin with 5.

(2) Those arrangements which begin with a number less than 5.

To obtain the latter also arranged in dictionary order of sequence, we may (subject to an exception to be stated immediately below) proceed by diminishing each initial number in the above table by unity.

The exception to be made is where 2 initial numbers are alike, as in $3\,.\,3\,.\,1$; $2\,.\,2\,.\,2\,.\,1$. These arrangements must not be counted in, as the arrangements $2\,.\,3\,.\,1$; $1\,.\,2\,.\,2\,.\,1$ will already have been obtained from $4\,.\,2\,.\,1$; $3\,.\,2\,.\,1\,.\,1$ respectively.

Hence the number of arrangements in the above table to be preserved is less by 2 than the total number.

On the other hand we shall have the arrangement $5\,.\,1$, to which there is nothing corresponding in the table for $(7:\ 5,\ 4)$. Hence the difference required is

$$2-1, \text{ that is, } (7:\ 5,\ 4)-(6:\ 5,\ 4)=1.$$

Let us take as a second example w (the weight) 12, i (the limit to each part) 6, and j (the number of parts) 4.

Let A be the table for $(12:\ 6,\ 4)$ in dictionary order, and let A' be the part of the table for $(11:\ 6,\ 4)$, also arranged in dictionary order, for which 6 is nowhere the initial term. Let A_1 be what A becomes when each initial number is diminished by unity.

Then, by the same reasoning as above, we must have $A'-A_1=6\,.\,6$, $5\,.\,5\,.\,2$, $5\,.\,5\,.\,1\,.\,1$, $4\,.\,4\,.\,4$, $4\,.\,4\,.\,3\,.\,1$, $4\,.\,4\,.\,2\,.\,2$, $3\,.\,3\,.\,3\,.\,3$, 7 in number.

Also calling B the part of the table for $(11:\ 6,\ 4)$, beginning with 6 we have $B=6\,.\,5$, $6\,.\,4\,.\,1$, $6\,.\,3\,.\,2$, $6\,.\,3\,.\,1\,.\,1$, $6\,.\,2\,.\,2\,.\,1$, 5 in number.

Hence $(12:\ 6,\ 4)-(11:\ 6,\ 4)=7-5, =2$.

To verify this, let us interchange the values 6 and 4, this by a well-known theorem leaves the value of each denumerant unaltered.

We have now $A'-A_1=4\,.\,4\,.\,4$, $4\,.\,4\,.\,3\,.\,1$, $4\,.\,4\,.\,2\,.\,2$, $4\,.\,4\,.\,2\,.\,1\,.\,1$, $4.4.1.1.1.1$, $3.3.3.3$, $3.3.3.2.1$, $3.3.3.1.1.1$, $3.3.2.2.2$, $3.3.2.2.1.1$, $2.2.2.2.2.2$, number is 11.

Also $B=4\,.\,4\,.\,3$, $4\,.\,4\,.\,2\,.\,1$, $4\,.\,4\,.\,1\,.\,1\,.\,1$, $4.3.3.1$, $4.3.2.2$, $4.3.2.1.1$, $4.3.1.1.1.1$, $4.2.2.2.1$, $4.2.2.1.1.1$, number is 9, and thus

$$(12:\ 4,\ 6)-(11:\ 4,\ 6)=11-9=2$$

as before. Evidently this identity between the two forms of

$$(w:\ i,\ 5)-\{(w-1):\ i,\ 5\},$$

given by this method, and also the incapability of this difference becoming negative when w is not greater than $\frac{1}{2}ij$, which I have elsewhere demonstrated, may be made to yield arithmetical properties of a new kind, and not unlikely to prove very valuable in certain parts of the theory of numbers; but what has impressed itself on my mind is the *enormous saving* of labour in the actual business of calculating invariantive formulæ, which this method confers. The existence of a perfectly definite table exhibiting an exhaustive arrangement of *ruled partitions* (as I call partitions subject to the two indices i, j) in itself constitutes a theorem (however simple), and the method above given is a further and more recondite theorem deduced from it, combined of course with other *intuitional* propositions.

Let us take as another example $w = 20$, $i = 13$, $j = 3$.

Here $A' - A_1 = 10.10,\ 9.9.2,\ 8.8.4,\ 7.7.6.$ $B = 13.6,\ 13.5.1,\ 13.4.2,\ 13.3.3.$ Therefore $(20:13, 3) - (19:13, 3) = 0$.

Again let us calculate $(40:20, 4) - (39:20, 4)$.

Here $A' - A_1 = 20.20,\ 19.19.2,\ 19.19.1.1,\ 18.18.4,\ 18.18.3.1,\ 18.18.2.2,\ 17.17.6,\ 17.17.5.1,\ 17.17.4.2,\ 17.17.3.3$, and similarly 16.16 with 5 duads, 15.15 with 6 duads, 14.14 with 7 duads. Also 13.13 with $13.1,\ 12.2,\ 11.3,\ 10.4,\ 9.5,\ 8.6,\ 7.7$, 12.12 with $12.4,\ 11.5,\ 10.6,\ 9.7,\ 8.8$, 11.11 with $11.7,\ 10.8,\ 9.9$, $10.10.10.10$. Thus the number of terms in $A' - A_1$ is

$$(1 + 2 + 3 + 4 + 5 + 6 + 7) + (7 + 5 + 3 + 1) = 44.$$

And B is composed of arrangements containing 20, together with the number of triads into which $39 - 20$, that is, 19 can be decomposed, none greater than 20, that is, the number of terms in B is $19:20, 3$, which is the same as the absolute number of modes of resolving 19 into 3 parts or fewer, which is

$$1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + (10:9, 2) + (9:10, 2) + (8:11, 2) + (7:12, 2) = 25 + 5 + 5 + 3 + 2 = 40.$$

Thus
$$(40:20, 4) - (39:20, 4) = 44 - 40 = 4,$$

which is easily verified, for the difference between the above two denumerants is the number of linearly independent invariants of the 20th order to a quartic, that is, is the number of ways of composing 20 with 2 and 3 (the orders of the fundamental invariants) which is 4 as found above.

The method thus simply and almost intuitively deduced, may be expressed in the form of a theorem as follows:

$$\Sigma_{q=0}^{q=i}(w - 2q : q, j - 2) - (w - i - 1 : i, j - 1) = (w : i, j) - (w - 1 : i, j)$$
$$= \Sigma_{q=0}^{q=j}(w - 2q : q, i - 2) - (w - j - 1 : j, i - 1).$$

The inferior unit is taken zero for the purpose of theoretical simplicity. Let the effective value of this limit be called $[q]$, and consider the first of the above three equals.

The value of $[q]$ is given by the condition that

$$w - 2[q] \text{ shall be not greater than } (j-2)[q],$$

that is, $$[q] \text{ not less than } \frac{w}{j},$$

that is, $$[q] \text{ is } \frac{w}{j} \text{ if } \frac{w}{j} \text{ is an integer, } \frac{w}{j}+1 \text{ if } \frac{w}{j} \text{ is fractional,}$$

that is, $$[q] = E\,\frac{w+j-1}{j};$$

(E standing as usual for the integer part of the quantity which it precedes).

The number of actual terms differing from zero under the sign of summation is therefore

$$i+1-E\,\frac{w+j-1}{j}, \text{ that is } 1+E\,\frac{ij-w}{j},$$

similarly the number of terms under the sign of summation in the conjugate form will be $1+E\,\dfrac{ij-w}{i}$.

Thus the first or second expression will be the best to employ, according as j is greater or less than i.

Again, since $$(w : i, j) = (ij - w : i, j),$$
we may in place of
$$(w : i, j) - (w-1 : i, j),$$
employ $$(w' : i, j) - (w'+1 : i, j),$$
which is $$-[(w'+1 : i, j) - (w' : i, j)].$$

Hence, we may always secure in the application of this method, that the numerator in $E\,\dfrac{ij-w}{i}$ or in $E\,\dfrac{ij-w}{j}$ shall not be greater than $\frac{1}{2}ij$. Supposing j to be greater or not less than i, so that the first formula is applied, it will be found most convenient, so long as q is less or not greater than $j-2$, to consider q the number of the parts in any of the quantities

$$(w-2q : q, j-2),$$

and $j-2$ the limit to the magnitude of each part, and until q becomes equal to $i-1$, this hypothesis will always be the case. When $q=i$ or when $q=i$ and $q=i-1$ in the respective cases of j being only one unit greater than i or equal to i, the two indices $q : j-2$ may with advantage be reversed. For any other values of $j-i$, the order of the indices need not be disturbed. It may be worth while to call attention to the *two* independent theorems

of reciprocity made use of in the preceding discussion, indicated by the equations

$$\begin{aligned}&(w:i,\,j)\\&=(w:j,\,i)\\&=(ij-w:i,\,j)\\&=(ij-w:j,\,i),\end{aligned}$$

both of them of importance in the theory of invariants after the English method.

ADDITION.

Notwithstanding what has been stated above as to the choice between the two formulæ representing $\Delta(w:\,i,\,j)$, the advantage of diminishing the smaller of the two indices $i,\,j$, will simplify the calculations to a degree that far more than outweighs the disadvantage of increasing the number of terms under the sign of summation. Let us suppose then that j is less than w, and that $\Delta(w:\,i,\,j)$ is positive, representing in fact indifferently the number of linearly independent covariants of order i to a quantic of degree j, or of order j to a quantic of degree i. Then, unless these covariants are invariants, we must have $w<\frac{1}{2}ij$.

Consequently, the best formula to apply in such case will be obtained by writing

$$\begin{aligned}\Delta(w:\,i,\,j)&=(ij-w:\,i,\,j)-(ij-w+1:\,i,\,j)\\&=-\Delta(ij-w+1:\,i,\,j)\\&=(ij-i-w:\;i,\,j-1)-\Sigma_{q=0}^{q=i}(ij-w+1-2q:\,q,\,j-2).\end{aligned}$$

The number of terms other than zero under the sign of summation will then be $1+E\,\dfrac{w}{j}$.

For the case of invariants we may with at least equal advantage use the formula

$$\Sigma_{q=0}^{q=i}(\tfrac{1}{2}ij-2q:\,q,\,j-2)-(\tfrac{1}{2}ij-1-i:\,i,\,j-1).$$

Let us apply this to the case of finding

$$\Delta\left(\frac{18\,.\,5}{2}:\;18,\;5\right),\text{ that is }(45:\;18,\;5).$$

In the work below I use, whenever useful, the formula of transformation

$$(x:i,\,j)=(ij-x:\,i,\,j),$$

and employ $\dfrac{\mu}{3}$ to denote the number of ways of breaking up μ into three or

fewer parts, which we know is the nearest integer to $\frac{(\mu+3)^2}{12}$; and in like manner $\frac{\nu}{2}$ for the number of ways of breaking up ν into two parts: also in place of $(x : k, 3)$, whenever k is at least as great as x, I use the obviously equivalent value $\frac{x}{3}$.

Let us then first calculate

$$\Sigma_{q=9}^{q=18}\,[45 - 2q : q,\ 3],\ \text{say } S.$$

The values of q inferior to 9 will give quantities in which $3q < 45 - 2q$, and which will therefore be zero.

We have thus

$$\begin{aligned} S = {} & (9:18, 3) + (11:17, 3) + (13:16, 3) + (15:15, 3) \\ & + (17:14, 3) + (19:13, 3) + (21:12, 3) + (23:11, 3) \\ & + (25:10, 3) + (27:9, 3) \\ = {} & \tfrac{9}{3} + \tfrac{11}{3} + \tfrac{13}{3} + \tfrac{15}{3} + (17:14, 3) + (19:13, 3) + (15:12, 3) \\ & + (10:11, 3) + (5:10, 3) + (0:9, 3). \end{aligned}$$

Also

$$\begin{aligned} (17:14, 3) &= (17:17, 3) - \tfrac{1}{2} - \tfrac{2}{2} - \tfrac{3}{2} = \tfrac{17}{3} - 1 - 2 - 2 = \tfrac{17}{3} - 5, \\ (19:13, 3) &= (19:19, 3) - \tfrac{1}{2} - \tfrac{2}{2} - \tfrac{3}{2} - \tfrac{4}{2} - \tfrac{5}{2} - \tfrac{6}{2} = (19:19, 3) - 15, \\ (15:12, 3) &= (15:15, 3) - \tfrac{1}{2} - \tfrac{2}{2} - \tfrac{3}{2} = \tfrac{15}{3} - 5. \end{aligned}$$

Thus

$$\begin{aligned} S &= \tfrac{9}{3} + \tfrac{11}{3} + \tfrac{13}{3} + \tfrac{15}{3} + \tfrac{17}{3} + \tfrac{19}{3} + \tfrac{15}{3} + \tfrac{10}{3} + \tfrac{5}{3} + \tfrac{0}{3} - 25 \\ &= \tfrac{0}{3} + \tfrac{5}{3} + \tfrac{9}{3} + \tfrac{10}{3} + \tfrac{11}{3} + \tfrac{13}{3} + 2\,.\,\tfrac{15}{3} + \tfrac{17}{3} + \tfrac{19}{3} - 25. \end{aligned}$$

Again let $S' = (44 - 18 : 18, 4) = (26 : 18, 4)$.

Then

$$\begin{aligned} S' = {} & (8:18, 3) + (9:17, 3) + (10:16, 3) + (11:15, 3) \\ & + (12:14, 3) + (13:13, 3) + (14:12, 3) + (15:11, 3) \\ & + (16:10, 3) + (17:9, 3) + (18:8, 3) + (19:7, 3) \\ = {} & \tfrac{8}{3} + \tfrac{9}{3} + \tfrac{10}{3} + \tfrac{11}{3} + \tfrac{12}{3} + \tfrac{13}{3} + (\tfrac{14}{3} - 3) + (\tfrac{15}{3} - 8) \\ & + (\tfrac{14}{3} - 8) + (\tfrac{10}{3} - 1) + \tfrac{6}{3} + \tfrac{2}{3} - 20 \\ = {} & \tfrac{2}{3} + \tfrac{6}{3} + \tfrac{8}{3} + \tfrac{9}{3} + 2\,.\,\tfrac{10}{3} + \tfrac{11}{3} + \tfrac{12}{3} + \tfrac{13}{3} + 2\,.\,\tfrac{14}{3} + \tfrac{15}{3} - 20; \end{aligned}$$

therefore

$$\begin{aligned} S - S' &= \tfrac{0}{3} - \tfrac{2}{3} + \tfrac{5}{3} - \tfrac{6}{3} - \tfrac{8}{3} - \tfrac{10}{3} - \tfrac{12}{3} - 2\,.\,\tfrac{14}{3} + \tfrac{15}{3} + \tfrac{17}{3} + \tfrac{19}{3} - 5 \\ &= 1 - 2 + 5 - 7 - 10 - 14 - 19 - 48 + 27 + 33 + 40 - 5 \\ &= 106 - 105 = 1, \end{aligned}$$

which is right, there being just one invariant to the quantic of the eighteenth order in the coefficients, so that $\Delta(45 : 18, 5) = 1$.

It appears from the tables given in M. Faà de Bruno's valuable *Théorie des Formes Binaires,* Turin, 1877, that this invariant contains 848 terms. Therefore the value of $(18: 8, 5)$ is very considerably greater* than 848.

Thus, by the direct method of calculating $\Delta(45: 18, 5)$, many more than 1695 terms would have required setting out.

There is one case which deserves special consideration, namely, when one of the indices i or j becomes infinite.

The function $\Delta(w: \mu, \infty)$ then represents the total number of in- and co-variants of weight w of any given order not less than w to a quantic of the μth degree.

The two formulæ for this case become respectively

$$\Sigma_{q=0}^{q=\infty} [w - 2q: q, \mu],$$

and

$$\Sigma_{q=0}^{q=\mu} [w - 2q: q, \infty] - [w - \mu - 1: \mu, \infty],$$

or if we agree to understand in all cases by $\frac{n}{m}$ the number of ways of making up n with the integers $0, 1, 2, 3 \ldots m$, or, what is the same, the number of ways of breaking up n into m or fewer parts, the second formula becomes

$$\Sigma_{q=0}^{q=\mu} \frac{w-2q}{q} - \frac{w-\mu-1}{\mu};$$

of these two the first is by far the most expeditious.

Let us take as an example $\Delta(20: 6, \infty)$, that is $\frac{20}{6} - \frac{19}{6}$.

The first formula {neglecting the values of q which make $w - 2q$ negative and those which make $4q < (w - 2q)$}, will give for the value of Δ

$$\begin{array}{lll}
(0: 10, 4) & = & (0) \\
+(2: 9, 4) & & +(2) \\
+(4: 8, 4) & & +(4) \\
+(6: 7, 4) & & +\frac{6}{4} \\
+(8: 6, 4) & & +(8: 6, 4) \\
+(10: 5, 4) & & +(10: 5, 4) \\
+(12: 4, 4) & & +(4: 4, 4), \text{ that is } (4),
\end{array}$$

* I say very considerably greater than, because only a certain number of the terms which satisfy the required conditions of order and weight actually appear in the octodecimal invariant in question. Thus, for example, there is no f^9, no f^8, and of the $(10: 11, 5)$ that is $\frac{10}{5}$ terms which might contain f^7, only six, namely the terms contained in $a(ac - b^2)^5$ actually make their appearance in it.

where in general (m) means *all* the modes of breaking up m into parts. The value of $(10:5,\ 4)$ will be easily found to be 9, of $(8:6,\ 4)$ 12 and of $\frac{6}{4}$, 9, also of (4) is 5. The value of $\frac{20}{6}-\frac{19}{6}$ thus becomes

$$1+2+5+9+12+9+5=43.$$

By the second formula the value of the same quantity would be

$$\tfrac{8}{6}+\tfrac{10}{5}+\tfrac{12}{4}+\tfrac{14}{3}+\tfrac{16}{2}+\tfrac{18}{1}-\tfrac{13}{5},$$

which would be exceedingly tedious to calculate.

In like manner if w is odd we shall have a series of denumerants of the form

$$\left(1:\frac{w-1}{2},\ \mu\right),\quad \left(3:\frac{w-3}{2},\ \mu\right),\quad \left(5:\frac{w-5}{2},\ \mu\right),\ \text{\&c.}$$

Thus, for example, $\frac{11}{6}-\frac{10}{6}$ (that is, the number of in- and co-variants to a sextic of weight 11 and of any given order not inferior to 11, or, if we please to vary the expression, the number of in- and co-variants of weight 11 and the sixth order to any quantic of a degree not inferior to 11) will be

$$\left.\begin{array}{r}(1:\ 5,\ 4)\\ +(3:\ 4,\ 4)\\ +(5:\ 3,\ 4)\\ +(7:\ 2,\ 4)\end{array}\right\}=\left\{\begin{array}{l}(1)\\ +(3)\\ +(5:\ 3,\ 4)\\ +(1:2,\ 4)\ \text{that is}\ (1)\end{array}\right.$$

$$=1+3+4+1=9.$$

29.

NOTE ON CONTINUANTS.

[*Messenger of Mathematics*, VIII. (1879), pp. 187—189.]

To find the number of terms in the cumulant or continuant $(a_1, a_2, \ldots a_n)$, we may proceed as follows:

(1) There is the term $a_1 a_2 a_3 \ldots a_n$.

(2) The number of terms of the first order of degradation, that is, obtained by leaving out any pair of consecutive elements, is $n-1$, say $u_{n,\,1}$.

(3) The number of terms of the second order of degradation obtained by leaving out any two pairs of such, that is, by leaving out the first and second and some other pair of those that follow the second, the second and third and a pair of those that follow the third, the third and fourth and a pair of those that follow the fourth and so on, is

$$u_{n-2,\,1} + u_{n-3,\,1} + u_{n-4,\,1} + \ldots,$$

and, consequently,

$$(n-3) + (n-4) + (n-5) + \ldots$$

$$= \frac{(n-2)(n-3)}{2} =, \text{ say, } u_{n,\,2}.$$

(4) The number of the third order of degradation is in like manner

$$u_{n-2,\,2} + u_{n-3,\,2} + u_{n-4,\,2} + \ldots,$$

that is

$$= \frac{(n-4)(n-5)}{1\,.\,2} + \frac{(n-5)(n-6)}{1\,.\,2} + \ldots$$

$$= \frac{(n-3)(n-4)(n-5)}{1\,.\,2\,.\,3},$$

and so in general

$$u_{n,r} = u_{n-2,r-1} + u_{n-3,r-1} + u_{n-4,r-1} + \dots$$
$$= \frac{(n-r)(n-r-1)\dots(n-2r-1)}{1.2\dots r}.$$

Hence, the total number is

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots.$$

Verification. In general

$$(i \sin\theta + \cos\theta)^n - (i\sin\theta - \cos\theta)^n$$
$$= 2\cos\theta\,\{(2i\sin\theta)^{n-1} + (n-2)(2i\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(2i\sin\theta)^{n-5} + \dots\},$$

for we know that

$$\cos\theta\,\{(2\sin\theta)^{n-1} - (n-2)(2\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(2\sin\theta)^{n-5} - \dots\}$$
$$= (-1)^{\frac{1}{2}(n-1)}\cos n\theta, \text{ or } (-1)^{\frac{1}{2}(n-2)}\sin n\theta, \text{ according as } n \text{ is odd or even.}$$

Hence, putting

$$i\sin\theta + \cos\theta = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5},$$

so that

$$i\sin\theta - \cos\theta = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5},$$
$$2i\sin\theta = 1,$$

and

$$2\cos\theta = \sqrt{5},$$

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots$$
$$= \frac{(\frac{1}{2} + \frac{1}{2}\sqrt{5})^{n+1} - (\frac{1}{2} - \frac{1}{2}\sqrt{5})^{n+1}}{\sqrt{5}}.$$

But because

$$(a_1, a_2, \dots a_n) = a_n(a_1, a_2, \dots a_{n-1}) + (a_1, a_2, \dots a_{n-2}),$$

if u_n is the number of terms in $(a_1, a_2, \dots a_n)$,

$$u_n = u_{n-1} + u_{n-2},$$

with the initial conditions

$$u_0 = 1, \quad u_1 = 1.$$

Solving this difference-equation, we shall obtain

$$u_n = \frac{1}{\sqrt{5}}\{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^{n+1} - (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n+1}\},$$

agreeing with the preceding result.

Corollary 1. The value of the continuant of the nth order $(x, x, \dots x)$, is

$$x^n + (n-1)\,x^{n-2} + \frac{(n-2)(n-3)}{1\,.\,2}\,x^{n-4} + \dots,$$

which admits also of the clumsy representation

$$[\{\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1} - \{\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1}] \div \sqrt{(x^2+4)}.$$

Corollary 2. The value of the pro-continuant of the nth order

$$(2\cos\theta,\ 2\cos\theta, \dots 2\cos\theta),$$

is

$$\frac{\sin(n+1)\,\theta}{\sin\theta}.$$

By the pro-continuant is to be understood what a continuant becomes, when in its representative determinant, the oblique lines of negative units are all changed into positive units so that the matrix has two precisely similar bands of units one above and one below the diagonal line and in opposition with it.

Corollary 3. The integral of the partial-difference equation

$$u_{x+1,\,y} - u_{x,\,y} - u_{x-1,\,y-1} = 0,$$

limited by the conditions

$$u_{x,\,0} = 1,\quad u_{x+1,\,x+1} = 0,$$

is

$$u_{x,\,y} = \frac{\Pi\,(x-y)}{\Pi\,(x-2y)\,\Pi y}.$$

30.

SUR UNE PROPRIÉTÉ ARITHMÉTIQUE D'UNE CERTAINE SÉRIE DE NOMBRES ENTIERS.

[*Comptes Rendus*, LXXXVIII. (1879), pp. 1297, 1298.]

NOMMONS le nombre de termes distincts qui figurent dans le développement d'un déterminant gauche son *dénumérant*. Soit

$$[1\,.\,3\,.\,5\,\ldots\,(2n-1)]\,u_n$$

le dénumérant d'un déterminant gauche de l'ordre $2n$. On aura pour u_1, u_2, u_3, u_4, u_5, u_6, ... les valeurs successives

$$1,\ 2,\ 8,\ 50,\ 418,\ 4348,\ \ldots$$

et en général

$$u_x = (2x-1)\,u_{x-1} - (x-1)\,u_{x-2}.$$

Soit $\theta\left(\frac{2x+1}{8}\right)$ l'entier le plus proche (en excès ou en défaut) de $\frac{2x+1}{8}$. Alors je dis que *le plus grand diviseur commun à u_x, u_{x+1} est égal au nombre 2 élevé à la puissance* $\theta\left(\frac{2x+1}{8}\right)$.

Ce théorème se déduit des deux propositions suivantes:

1°. On démontre que u_x et x ne peuvent avoir un facteur commun impair pour aucune valeur de x; c'est une conséquence immédiate de cette loi que deux u consécutifs ne peuvent avoir non plus un facteur commun impair.

2°. On démontre que $\frac{u_{4x-2}}{2^x}$, $\frac{u_{4x-1}}{2^x}$, $\frac{u_{4x}}{2^x}$, $\frac{u_{4x+2}}{2^x}$ sont tous les quatre des nombres entiers, dont le premier et le troisième sont des nombres impairs; cela suffit pour établir le théorème. Mais j'ajoute, comme corollaire, que la quatrième de ces quantités est aussi un nombre impair et la seconde un nombre pair, qui est toujours divisible par 4.

Le fondement du raisonnement au moyen duquel on établit cette proposition remarquable est l'identité que j'ai donnée dans* l'*American Journal of Mathematics*

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{(1-t)}} = 1 + u_1\frac{t}{2} + u_2\frac{t^2}{2\,.\,4} + u_3\frac{t^3}{2\,.\,4\,.\,6} + \ldots.$$

[* p. 272 below.]

31.

SUR LA VALEUR MOYENNE DES COEFFICIENTS DANS LE DÉVELOPPEMENT D'UN DÉTERMINANT GAUCHE OU SYMÉTRIQUE D'UN ORDRE INFINIMENT GRAND ET SUR LES DÉTERMINANTS DOUBLEMENT GAUCHES*.

[*Comptes Rendus*, LXXXIX. (1879), pp. 24—26.]

DANS un déterminant ou gauche ou symétrique, j'ai fait voir ailleurs que tous les coefficients qui ne sont pas des unités seront des puissances de 2. J'ajoute que, dans le dernier cas, si n est l'ordre du déterminant, la plus haute puissance de 2 qui entre comme coefficient sera la partie entière de $\frac{n}{3}$ et dans le premier cas $\frac{n}{4}$ (n dans ce cas étant un nombre pair).

M. Cayley a le premier démontré que, si le nombre des termes distincts dans le développement d'un déterminant symétrique de l'ordre x est $(1.2.3\ldots x)\,\Omega_x$, Ω_x aura pour sa fonction génératrice $\dfrac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$; et, de ma part, j'ai démontré que, si le nombre des termes distincts dans un déterminant gauche de l'ordre $2x$ est $1.3.5\ldots(2x-1)\,\omega_x$, ω_x aura pour sa fonction génératrice $\sqrt[4]{\left(\dfrac{e^t}{1-t}\right)}$.

Ces deux formules suffisent pour la solution du problème proposé. Commençons par le déterminant gauche. En vertu de la formule donnée, on aura

$$\omega_x = \left[1 + x + 1.5x + 1.5.9\frac{x(x-1)}{2}\right.$$
$$\left. + 1.5.9.13\frac{x(x-1)(x-2)}{2.3} + \ldots + 1.5.9\ldots(4x-3)\right]\frac{1}{2^x},$$

nombre qui est toujours entier, car ω_x est assujetti à satisfaire à l'équation $\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2}$; de sorte que ω_0, ω_1 étant 1, 1, tous les ω

[* See below, p. 257.]

seront des nombres entiers. En posant $1.3.5\ldots(2x-1)\,\omega_x = u_{2x}$, on trouve facilement, à l'aide de cette expression, que, pour $x=\infty$,

$$\frac{u_{2x}}{1.2.3\ldots 2x} = e^{\frac{1}{4}}\frac{1.5.9\ldots(4x-3)}{4.8.12\ldots 4x}.$$

De plus, par une méthode bien connue, on trouve

$$\log\{1.5.9\ldots(4x-3)\} = C - x + \frac{3}{4} + \frac{4x-1}{4}\log(4x-3)$$
$$+\frac{1}{12}\frac{d}{dx}\log(4x-3) - \frac{1}{720}\frac{d^3}{dx^3}\log(4x-3) + \ldots$$
$$=\left(C - \frac{\log 4}{4}\right) - x + \log 4x + x\log x + \frac{A}{x}\ldots.$$

On a aussi

$$\log(4.8\ldots 4x) = x\log 4 + \log\sqrt{(2\pi)} + x\log x - x + \frac{1}{2}\log x + \frac{A'}{x}\ldots.$$

On aura donc

$$\frac{1.5\ldots(4x-3)}{4.8\ldots 4x} = \frac{e^C}{2\sqrt{(\pi)}}\frac{1}{x^{\frac{3}{4}}},$$

et, puisque la somme des coefficients pris tous positivement en u_{2x} est égale à $\{1.3.5\ldots(2x-1)\}^2$ et $\dfrac{\{1.3.5\ldots(2x-1)\}^2}{1.2\ldots 2x} = \dfrac{1}{\sqrt{(\pi x)}}$, on a finalement la valeur moyenne des coefficients, c'est-à-dire

$$\frac{\{1.3.5\ldots(2x-1)\}^2}{u_{2x}} = \frac{2}{e^{\frac{1}{4}+C}}x^{\frac{1}{4}}.$$

Pour trouver C je me sers de la formule

$$C = \log\{1.5.9\ldots(4x-3)\} - \frac{3}{4}$$
$$+ x - \frac{4x-1}{4}\log(4x-3) - \frac{1}{3}\frac{1}{4x-3} + \frac{8}{45}\frac{1}{(4x-3)^3}\ldots$$

et, en mettant $4x-3=125$, on trouve, à l'aide des Tables ordinaires de logarithmes,

$$C = -\,0{,}022508\ldots,$$

ce qui donne pour la valeur moyenne cherchée $(1{,}593\ldots)\,x^{\frac{1}{4}}$.

Comme vérification, j'ai fait calculer u_4, u_8, u_{12}, u_{16}, par le moyen des formules

$$u_{2x} = 1.3.5\ldots(2x-1)\,\omega_x,$$
$$\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2},$$

et, en posant

$$\frac{1.3.5\ldots(2x-1)}{\omega_x} = \rho_x x^{\frac{1}{4}},$$

j'ai trouvé

$$\rho_4 = 1{,}262\ldots,\quad \rho_8 = 1{,}485\ldots,\quad \rho_{12} = 1{,}528\ldots,\quad \rho_{16} = 1{,}551\ldots,$$

ce qui s'accorde très bien avec la valeur $\rho_\infty = 1{,}593\ldots$.

Pour le déterminant symétrique, en vertu de la formule de M. Cayley, on sait que la valeur moyenne cherchée est le coefficient de t^x dans $\dfrac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$, qui sera le même, quand $x = \infty$, que dans $\dfrac{e^{\frac{3t}{4}}}{\sqrt{(1-t)}}$, et l'on trouve facilement que cette valeur est égale à $e^{-\frac{3}{4}}\sqrt{(\pi x)}$.

J'ajoute quelques mots sur les déterminants *doublement* gauches, c'est-à-dire gauches par rapport à l'une et à l'autre des deux diagonales.

1°. Je trouve que, pour que ces déterminants ne s'évanouissent pas, l'ordre doit être divisible par 4.

2°. Considérons la *racine carrée* d'un déterminant doublement gauche de l'ordre $4x$. Je trouve que la somme de ses coefficients pris tous positivement est égale à

$$1 \,.\, 2 \,.\, 5 \,.\, 6 \,.\, 9 \,.\, 10 \ldots (4x-3)(4x-2).$$

3°. Soit ϕ_x le nombre des termes *distincts* dans cette racine carrée. Je trouve qu'en posant $\phi_x = 2 \,.\, 4 \,.\, 6 \ldots (4x-2)\,\psi_x$, ψ_x sera toujours un nombre entier défini par l'équation

$$\psi_x = (4x-3)\,\psi_{x-1} - 2x\psi_{x-2}, \quad \psi_0 = 1, \quad \psi_1 = 1,$$

et que la fonction génératrice de ψ_x sera $\sqrt[8]{\left(\dfrac{e^t}{1-t}\right)}$, de sorte que

$$\psi_x = \left[1 + x + 1 \,.\, 9\,\frac{x(x-1)}{2} + 1 \,.\, 9 \,.\, 17\,\frac{x(x-1)(x-2)}{1 \,.\, 2 \,.\, 3} \ldots \right.$$
$$\left. + 1 \,.\, 9 \,.\, 17 \ldots (8x-7)\right] \div 2^x.$$

4°. On démontre facilement que deux des ψ consécutifs quelconques seront toujours premiers entre eux et que tous les coefficients dans la racine carrée du déterminant doublement gauche de l'ordre $4x$ sont des puissances de 2, dont la plus haute sera désignée par la partie entière de $\dfrac{4x}{8}$, c'est-à-dire de $\dfrac{x}{2}$.

32.

TABLE DES NOMBRES DE DÉRIVÉES INVARIANTIVES D'ORDRE ET DE DEGRÉ DONNÉS, APPARTENANT À LA FORME BINAIRE DU DIXIÈME ORDRE.

[*Comptes Rendus*, LXXXIX. (1879), pp. 395, 396.]

Degré dans les coefficients.	Ordre dans les variables.													
	0	2	4	6	8	10	12	14	16	18	20	22	24	26
1.........						1								
2.........	1		1		1		1		1					
3.........		1		2	1	1	2	1	1	1	1		1	
4.........	1		3	1	3	3	2	3	1	2	1	1		1
5.........		3	3	4	5	4	5	2	4		1			
6.........	4	2	5	8	6	8	2	3						
7.........		7	10	8	12	2	3							
8.........	5	8	11	15	4	5								
9.........	5	13	19	8	4									
10.........	8	20	12	10										
11.........	8	18	21											
12.........	12	30												
13.........	15	16												
14.........	13	17												
15.........	19													
16.........	5													
17.........	3													

Pour trouver par cette Table le nombre d'invariants ou covariants fondamentaux de l'ordre ω et du degré δ, on cherche dans la colonne numérotée ω et dans la ligne numérotée δ; le chiffre qui se trouve au point de concours de cette colonne et de cette ligne est le nombre en question. S'il n'existe aucune combinaison de colonne et de ligne numérotées ω et δ respectivement, il n'y aura aucun covariant (ou invariant) du degré δ et de l'ordre ω.

Cette Table a été construite sous ma direction par M. Franklin, de Baltimore, avec l'aide des fonds que l'Association britannique pour l'avancement de la Science, dans sa dernière session à Dublin, a eu la bonté de mettre à ma disposition pour effectuer des calculs de ce genre.

Les Tables analogues pour la forme binaire de l'ordre 7 et de l'ordre 8 ont déjà paru* dans ces *Comptes rendus*, et celle pour l'ordre† 9 dans l'*American Journal of Mathematics* de cette année, de sorte qu'aujourd'hui on connaît toutes les dérivées invariantives fondamentales ayant rapport à des formes uniques binaires de chaque ordre, depuis 2 jusqu'à 10 inclusivement.

[* pp. 146, 115 above.] [† p. 281 below.]

33.

SUR LA VALEUR MOYENNE DES COEFFICIENTS NUMÉRIQUES DANS UN DÉTERMINANT GAUCHE D'UN ORDRE INFINIMENT GRAND.

[*Comptes Rendus*, LXXXIX. (1879), pp. 497, 498.]

PAR une inadvertance regrettable, j'ai omis* de donner la valeur moyenne des coefficients numériques dans un déterminant gauche d'un ordre infini sous sa forme exacte. Pour cela, on n'a besoin que de se servir de la formule

$$\frac{a(a+\delta)(a+2\delta)\dots(a+x\delta)}{b(b+\delta)(b+2\delta)\dots(b+x\delta)}=\frac{\Gamma\frac{b}{\delta}}{\Gamma\frac{a}{\delta}}x^{\frac{a-b}{\delta}},$$

où l'on suppose que x est infiniment grand.

Or la somme des coefficients, tous pris positivement dans le déterminant gauche de l'ordre x, est

$$[1.3.5\dots(x-1)]^2,$$

et le nombre des termes distincts (x étant supposé infiniment grand) est

$$e^{\frac{1}{4}}(1.2.3\dots 2x)\frac{1.5.9\dots(4x-3)}{4.8.12\dots 4x};$$

en conséquence, la valeur moyenne cherchée sera

$$\frac{1}{e^{\frac{1}{4}}}\frac{1.3.5\dots(2x-1)}{2.4.6\dots 2x}\frac{4.8.12\dots 4x}{1.5.9\dots(4x-3)}=\frac{1}{e^{\frac{1}{4}}}\frac{\Gamma 1}{\Gamma\frac{1}{2}}\frac{\Gamma\frac{1}{4}}{\Gamma 1}x^{\frac{3}{4}-\frac{1}{2}}=\frac{\Gamma\frac{1}{4}}{\Gamma\frac{1}{2}}\left(\frac{x}{e}\right)^{\frac{1}{4}}.$$

Si l'on écrit cette valeur sous la forme $Cx^{\frac{1}{4}}$, on aura

$$\begin{aligned}\log C&=\log\Gamma\tfrac{5}{4}+\log 2-\log\Gamma\tfrac{3}{2}-\tfrac{1}{4}\log e\\&=9573211+3010300-9475449-1085711=2022351.\end{aligned}$$

On a donc

$$C=1{,}59307,$$

expression dont les quatre premiers chiffres avaient été précédemment trouvés; mais l'expression exacte $\dfrac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{(\pi)}}x^{\frac{1}{4}}$, qui me paraît remarquable, est ici donnée pour la première fois.

[* above, p. 253.]

34.

SUR LE VRAI NOMBRE DES COVARIANTS FONDAMENTAUX D'UN SYSTÈME DE DEUX CUBIQUES.

[*Comptes Rendus*, LXXXIX. (1879), pp. 828—832.]

L'ÉNUMÉRATION des invariants et covariants pour un système de deux cubiques binaires, donnée par M. Salmon (*Modern Higher Algebra*, p. 186) et attribuée par lui à MM. Clebsch et Gordan, comprend huit covariants linéaires, dont deux sont du degré 3 par rapport aux coefficients de l'une des cubiques, et l'autre du degré 4. Par ma méthode, j'avais trouvé précisément les mêmes invariants et covariants fondamentaux que MM. Clebsch et Gordan; mais tout récemment, en refaisant mes calculs, M. Franklin, de Baltimore, a découvert qu'il y avait une faute d'arithmétique commise dans mon tamisage, et que les deux covariants linéaires dont j'ai parlé plus haut ne doivent pas figurer dans ma Table. Je vais donc démontrer qu'en effet ces covariants, supposés fondamentaux également par MM. Clebsch et Gordan et moi-même, ne le sont pas; de sorte que le nombre total des *Grundformen*, pour un système de deux cubiques, est 26 et non pas 28, comme on avait pensé jusqu'à ce jour.

En démontrant une chose pareille dans le cas d'un système de deux biquadratiques, je me suis servi de la méthode pour ainsi dire positive, c'est-à-dire j'ai donné la décomposition de deux des formes supposées fondamentales par M. Gordan. Dans le cas beaucoup plus difficile du système traité par M. Gundelfinger d'une cubique et une biquadratique, je me suis servi de la méthode négative en prouvant *à priori* l'impossibilité de l'existence de formes fondamentales ayant le type (c'est-à-dire les degrés et l'ordre) qu'avaient trois des *Grundformen* imaginées par cet auteur distingué.

Je vais me servir de cette dernière méthode comme étant la plus courte dans le cas actuel, en démontrant qu'un covariant linéaire du type 3, 4 ou du type gémeau 4, 3 appartenant à un système de deux cubiques ne peut pas être indécomposable.

Je commence avec la détermination du nombre des covariants du type 4, 3 : 1 (ou bien, ce qui est absolument le même, du type 3, 4 : 1), linéairement indépendants, appartenant à un système de deux cubiques. Pour cela, par le théorème que j'ai démontré avec le dernier degré de rigueur dans le *Journal de M. Borchardt** et dans le *Philosophical Magazine*†, on sait, puisque $\frac{4.3+3.3-1}{2} = 10$, que le nombre cherché sera

$$(10:3,4:3,3)-(9:3,4:3,3),$$

en se servant, en général, de la notation $(w: i, j: i', j')$ pour signifier le nombre des représentations de w par la somme bifide

$$x_1 + 2x_2 + 3x_3 + \ldots + ix_i + y_1 + 2y_2 + 3y_3 + \ldots + i'y_{i'},$$

où les x peuvent être chacun 0, 1, 2, 3, ... ou j, et les y, 0, 1, 2, 3, ... ou j'. Le nombre de partitions, sans exclusion des zéros, en trois parties, dont aucune n'excède 4, est respectivement pour les chiffres

0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	4	5	4	4	3	2
1	1	2	3	3	3	3	2	1	1	0

quand, le nombre des parties restant 3, la limite supérieure de chaque partie, au lieu de 4, devient 3. Conséquemment on aura

$$\begin{aligned}(10:3,4:3,3) &= 1.1+2.1+3.2+4.3+4.3+5.3+4.3+4.2+3.1+2.1\\ &= 1+2+6+12+12+15+12+8+3+2=73,\\ (9:3,4:3,3) &= 1.1+1.1+2.2+3.3+4.3+4.3+5.3+4.2+4.1+3.1\\ &= 1+1+4+9+12+12+15+8+4+3=69;\end{aligned}$$

c'est-à-dire que le nombre des covariants des degrés 3, 4 pour les coefficients et de l'ordre 1 pour les variables linéairement indépendants sera 73 − 69 ou 4.

Je vais démontrer qu'il y a, en effet, exactement quatre covariants de ce type non irréductibles, mais linéairement indépendants ; de sorte qu'il n'y aura pas place dans la nature des choses pour des covariants irréductibles, c'est-à-dire non composés ou fondamentaux, de ce même type.

Prenons les deux formes $(a, b, c, d \between x, y)^3$, $(\alpha, \beta, \gamma, \delta \between x, y)^3$. Je me servirai de la notation $p.q.i$ qui signifiera un covariant du degré p pour les coefficients a, b, c, d ; q pour $\alpha, \beta, \gamma, \delta$; et i pour les variables. On connaît les invariants fondamentaux $1.1.0$, $2.2.0$, $3.1.0$, disons A, B, C, et les covariants linéaires $2.1.1$, $1.2.1$, $3.2.1$, disons U, V, W, avec l'aide desquels on peut former les quatre covariants décomposables A^2U, BU, CV, AW, du type $4.3.1$.

[* p. 232 above.] [† p. 117 above.]

3 . 1 . 0 et 2 . 2 . 0 seront les valeurs des deux émanants, $E\Delta$, $E^2\Delta$, où

$$E = \alpha \frac{d}{da} + \beta \frac{d}{db} + \gamma \frac{d}{dc} + \delta \frac{d}{dd}$$

et
$$\Delta = a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd.$$

1 . 1 . 0 sera le combinant $a\delta - 3b\gamma + 3c\beta - d\alpha$; 2 . 1 . 1 sera*

$$\begin{vmatrix} a & b & c \\ b & c & d \\ \alpha x + \beta y & \beta x + \gamma y & \gamma x + \delta y \end{vmatrix}$$

et 3 . 2 . 1 sera le produit de l'opération du hessien de $(\alpha, \beta, \gamma, \delta)\left(\frac{d}{dy}, -\frac{d}{dx}\right)^3$ sur le covariant cubique de $(a, b, c, d)(x, y)^3$. Pour plus de facilité, faisons $b=0$, $d=0$, $\alpha=0$, $\gamma=0$; alors on voit que 3 . 1 . 0 s'évanouit et que 2 . 2 . 0 et 1 . 1 . 0 deviennent (en omettant dans le premier le coefficient numérique 2) $a^2\delta^2 - 6ac\beta\delta - 3c^2\beta^2$ et $a\delta + 3c\beta$ respectivement.

Bornons-nous aux coefficients de y dans 2 . 1 . 1 et 3 . 2 . 1; le dernier devient $ac\delta - c^2\beta$, et, puisque le hessien écrit plus haut devient

$$\beta\delta\left(\frac{d}{dx}\right)^2 - \beta^2\left(\frac{d}{dy}\right)^2,$$

si l'on nomme le covariant cubique dont j'ai parlé

$$Lx^3 + Mx^2y + Nxy^2 + Py^3,$$

le coefficient de y dans 3 . 2 . 1 deviendra $2\beta\delta M - 6\beta^2 P$, ou

$$M = 3abd - 6ac^2 + 3b^2c = -6ac^2,$$
$$P = -ad^2 + 3bcd - 2c^3 = -2c^3,$$

de sorte que ce coefficient, en omettant le coefficient numérique -12, devient $ac^2\beta\delta - c^3\beta^2$.

* Cela est une conséquence immédiate du fait connu qu'aux deux formes $(a, b, c, d)(x, y)^3$, $(\lambda, \mu, \nu)(x, y)^2$ appartient un déterminant invariantif

$$\begin{vmatrix} a & b & c \\ b & c & d \\ \lambda & \mu & \nu \end{vmatrix};$$

de même, pour deux biquadratiques, il y aura un déterminant invariantif

$$\begin{vmatrix} a & b & c & d \\ b & c & d & e \\ \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \epsilon \end{vmatrix};$$

et, en général, à un système de i formes binaires des degrés $n_1, n_2, \ldots, n_i$, en faisant $\frac{\Sigma(n) - 2}{i+1} = \mu$, pourvu que μ soit entier et moindre qu'un quelconque des n, on peut toujours former avec les coefficients des i formes un déterminant de l'ordre $\mu + 2$, analogue à ceux que j'ai écrits plus haut, qui sera un invariant du système. Cet invariant est, en effet, l'analogue pour un système de l'invariant bien connu nommé *catalecticant* dans le cas d'une seule forme.

Si donc une équation linéaire telle que $\lambda A^2 U + \mu BU + \nu CV + \rho AW = 0$ lie ensemble les quatre covariants composés dans leur forme générale, on aura

$$\lambda (a\delta + 3c\beta)^2 (ac\delta - c^2\beta) + \mu (a^2\delta^2 - 6ac\beta\delta - 3c^2\beta^2)(ac\delta - c^2\beta) \\ + \rho (a\delta + 3c\beta)(ac^2\beta\delta - c^3\beta^2)$$

identiquement égal à zéro; c'est-à-dire

$$\lambda (a\delta + 3c\beta)^2 + \mu (a^2\delta^2 - 6ac\beta\delta - 3c^2\beta^2) + \rho c\beta (a\delta + 3c\beta) = 0.$$

En égalant à zéro les coefficients de $a^2\delta^2$, $ac\beta\delta$, $c^2\beta^2$, dans cette identité, on obtient trois équations linéaires et homogènes en λ, μ, ρ auxquelles (vu que leur déterminant

$$\begin{vmatrix} 1 & 1 & 0 \\ 6 & -6 & 1 \\ 9 & -3 & 3 \end{vmatrix}$$

n'est pas zéro) on ne peut pas satisfaire simultanément sans poser

$$\lambda = 0,\ \mu = 0,\ \rho = 0.$$

Conséquemment nulle liaison linéaire ne peut exister entre les quatre covariants composés qu'on a formés du type 4 . 3 . 1 ; en sorte que ces quatre covariants étant linéairement indépendants, en dehors d'eux ne peut exister nul covariant indécomposable de ce même type : ce qui était à démontrer.

Ainsi, pour la troisième fois, l'exactitude de mon *postulatum* fondamental s'est trouvée en contradiction avec les résultats obtenus par les géomètres allemands, et pour la troisième fois elle est sortie victorieuse du conflit. C'est à la précision, qu'on ne peut trop louer, de M. Franklin comme calculateur et à sa passion pour ne laisser échapper aucune erreur, que la Science est redevable de cette troisième correction, bien remarquable et tout à fait inattendue.

Tous mes autres résultats, qui, avec ces trois exceptions, sont en conformité avec ceux de MM. Clebsch, Gordan et Gundelfinger, et y ajoutent un caractère de certitude qu'auparavant ils étaient très loin de posséder, ont été pleinement confirmés par les calculs indépendants exécutés par M. Franklin. Quelques erreurs typographiques, dont il est bon d'avertir, existent dans les Tables que j'ai publiées; elles seront corrigées dans la collection complète de Tables qui va prochainement* paraître dans l'*American Journal of Mathematics.*

[* p. 283 below.]

35.

NOTE ON AN EQUATION IN FINITE DIFFERENCES.

[*Philosophical Magazine*, VIII. (1879), pp. 120, 121.]

I GAVE* a great many years ago in this Magazine the integral of the equation in differences

$$u_x = \frac{u_{x-1}}{x} + u_{x-2},$$

which I obtained by observing that the equation could be solved by supposing each u of an odd order to be equal to the u of the order immediately superior, and also by supposing it to be equal to the u of the order immediately inferior. The upshot of the investigation expressed in the simplest language was to furnish two particular integrals of which one gives rise to the series

$$u_0 = 1, \quad u_1 = 1, \quad u_2 = \tfrac{1}{2}, \quad u_3 = \tfrac{1}{2}, \quad u_4 = \frac{1\,.\,3}{2\,.\,4}, \quad u_5 = \frac{1\,.\,3}{2\,.\,4} \ldots,$$

the other

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 2, \quad u_3 = \frac{2\,.\,4}{1\,.\,3}, \quad u_4 = \frac{2\,.\,4}{1\,.\,3}, \quad u_5 = \frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5} \ldots.$$

See also Boole's *Finite Differences*, 2nd Edition (edited by Mr Moulton), p. 235.

Now let ϕ, a function of any letter t, be the generating function of u_x. Then, since

$$xu_x - (x-2)\,u_{x-2} - u_{x-1} - 2u_{x-2} = 0,$$

we shall have

$$(1-t^2)\frac{d\phi}{dt} + (-1-2t)\,\phi = C;$$

and integrating we find

$$(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}\,\phi = C\int dt\,\sqrt{\left(\frac{1-t}{1+t}\right)},$$

or

$$\phi = C'\frac{1+t}{(1-t^2)^{\frac{3}{2}}} + C\frac{\sin^{-1}t + \sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}.$$

[* Vol. II. of this Reprint, p. 690.]

$\dfrac{1+t}{(1-t^2)^{\frac{3}{2}}}$ we see at a glance gives the values of u_x corresponding to the first particular integral; and since the two first terms of the function multiplied by C are $1+2t$, it follows that this function is the generatrix of the second particular integral—in other words, that

$$\frac{\sin^{-1}t+\sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}=1+2t+2t^2+\frac{2\,.\,4}{1\,.\,3}t^3+\frac{2\,.\,4}{1\,.\,3}t^4+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}t^5+\ldots.$$

Hence

$$\frac{t\sin^{-1}t+\sqrt{(1-t^2)}}{(1-t^2)^{\frac{3}{2}}}=\frac{1}{1+t}\left\{t\left(\frac{\sin^{-1}t+\sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}\right)+1\right\}$$

$$=1+\frac{2}{1}t^2+\frac{2\,.\,4}{1\,.\,3}t^4+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}t^6+\ldots;$$

and integrating

$$\frac{\sin^{-1}t}{\sqrt{(1-t^2)}}=t+\frac{2}{1}\cdot\frac{t^3}{3}+\frac{2\,.\,4}{1\,.\,3}\cdot\frac{t^5}{5}+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}\frac{t^7}{7}+\ldots.$$

Thus we have the remarkable identity

$$\left(1+\frac{1}{2}\tau+\frac{1\,.\,3}{2\,.\,4}\tau^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\tau^3\ldots\right)$$

$$\times\left(1+\frac{1}{2}\frac{\tau}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{\tau^2}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\tau^3}{7}\ldots\right)$$

$$=1+\frac{2}{1}\frac{\tau}{3}+\frac{2\,.\,4}{1\,.\,3}\frac{\tau^2}{5}+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}\frac{\tau^3}{7}\ldots.$$

I do not recollect ever having met with these remarkable series before I discovered them by the preceding method; but on showing them to Dr Story of this University, he ascertained that they had been stated not long ago by Mr Glaisher in a paper in the *Mathematical Messenger*, and made the foundation there of various summations for calculating π; but where Mr Glaisher found these series, which are not given in the ordinary books on the Calculus, or (if new) how he lighted upon them, he has not stated, and it is desirable that he should do so.

36.

NOTE ON DETERMINANTS AND DUADIC DISYNTHEMES.

[*American Journal of Mathematics*, II. (1879), pp. 89—96, 214—222.]

A GENERAL algebraical determinant in its developed form (viewed in relation to any one arbitrarily selected term) may be likened to a mixture of liquids seemingly homogeneous, but which being of differing boiling points, admit of being separated by the process of fractional distillation. Thus, for example, suppose a general determinant of the 6th order. The 720 terms which make it up will fall, in relation to the leading diagonal product, into as many classes (most of which comprise several similarly constituted families) as there are unlimited partitions of 6. These, 11 in number, are

6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 3, 1, 1, 1; 2, 2, 2; 2, 2, 1, 1; 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1.

Let the determinant be represented, in the umbral notation, by

$$\begin{matrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{matrix}\,.*$$

Let us, by way of illustration, consider the class corresponding to 6; this will consist of the 1 . 2 . 3 . 4 . 5 (120) terms obtained by forming the 120 distinct circular arrangements that belong to $a\ b\ c\ d\ e\ f$. Thus:

$$\begin{array}{cccc} & \multicolumn{2}{c}{\longrightarrow} & \\ & a & c & \\ b & & & e \\ & f & d & \\ & \multicolumn{2}{c}{\longleftarrow} & \end{array}$$

* The cyclical method of the text shows what was not previously apparent, that the umbral notation $\begin{matrix} ab \dots l \\ ab \dots l \end{matrix}$ possesses an essential advantage over $\begin{matrix} ab \dots l \\ \alpha\beta \dots \lambda \end{matrix}$ even for unsymmetrical determinants. This mode of notation of course implies some ground of preference for one diagonal group over all others and thus virtually regards a general determinant as related to a lineo-linear as a symmetrical one is to a quadratic form. For instance the general determinant of the second order is to be regarded as appurtenant to the lineo-linear form $aaxx' + abxy' + bayx' + bbyy'$.

will signify $ac \times ce \times ed \times df \times fb \times ba$, which will be one of the 120 in question. So, again, 3, 3 will denote, in the first place, the 10 sets of double triads of the general form $abc : def$, and, as each triad will give two cyclical orders, there will in all be 10×2^2, that is, 40, terms of the form $ab . bc . ca . de . ef . fd$. So, again, there will be $15 . 1^3$, that is, 15, corresponding to 2, 2, 2. So 3, 2, 1 will give 10 groupings of the form $abc : de : f$, and each of these will give rise to two terms, namely,

$$ab . bc . ca . de . ed . ff, \quad ac . cb . ba . de . ed . ff,$$

the number of cycles corresponding to two elements de being 1, and to one element f also 1.

This simple theory affords us a direct means of calculating the number of distinct terms in a symmetrical determinant, that is, one in which $i . j$ and $j . i$ are identical. It enables us to see at once that the coefficient of every term is unity or a power of 2; the rule being that plus or minus terms* of the class corresponding to $m_1, m_2, m_3, \ldots$ will take the coefficient 2^ν, ν being the number of the quantities m which are neither 1 nor 2, for, in every other case, the total number of cycles in each partial group will arrange themselves in pairs which give the same result, thus, for example,

$$\begin{array}{ccc} & a & \\ d & & b \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ b & & d \\ & c & \end{array}$$

will give the equal products $ab . bc . cd . da$ and $ad . dc . cb . ba$.

As an example of the direct method of computation, take a symmetrical determinant of the 5th order. Write

$$5 \quad 4.1 \quad 3.2 \quad 3.1.1 \quad 2.2.1 \quad 2.1.1.1 \quad 1.1.1.1.1.$$

To these 7 classes there will belong respectively

1 . 12	with the	coefficient	2
5 . 3	„	„	2
10 . 1	„	„	2
10 . 1	„	„	2
15	„	„	1
10	„	„	1
1	„	„	1.

Thus the number of distinct terms will be

$$12 + 15 + 10 + 10 + 15 + 10 + 1 = 73,$$

and the sum of the coefficients

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 120,$$

both of which are right.

* The complete value of the coefficient is $(-)^\mu 2^\nu$, ν being the number of elements in the partition other than 1 or 2, and μ the number of even elements.

Again, if we have a skew determinant of an even order, it will easily be seen that any partition embracing one or more odd numbers will give rise to pairs of terms that mutually cancel, but when all the parts into which the exponent of the order is divided are even, the coefficient will be given by the same rule as for symmetrical determinants, that is, its arithmetical value will be 2^ν, where ν is the number of parts exceeding 2. Thus, for example, for a skew determinant of the order 6 we have

$$6 \quad 4.2 \quad 2.2.2.$$

The number of terms corresponding to these partitions being 60 with coefficient 2, 15×3 also with coefficient 2, and 15 with coefficient 1, making 120 distinct terms in all, the sum of the coefficients will be

$$120 + 90 + 15 = (1.3.5)^2,$$

which is right, because the result is the square of the sum of 15 synthemes of the form $1.2 \times 3.4 \times 5.6$. It may be observed that 120 is $\frac{15.16}{2}$, as it ought to be, because, until we reach the order 8, the same *double duadic syntheme* can only be made up in one way of two simple ones, but this ceases to be the case from and after 8. Thus, for example, the pair of synthemes

$$1.2 \quad 3.4 \quad 5.6 \quad 7.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.7 \quad 6.8$$

combined will produce the same double syntheme as the pair

$$1.2 \quad 3.4 \quad 5.7 \quad 6.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.6 \quad 7.8,$$

and accordingly for 8 we have the partitions

$$8 \quad 6.2 \quad 4.4 \quad 4.2.2 \quad 2.2.2.2,$$

giving rise to

2520	with	coefficient	2
28.60	,,	,,	2
35.3^2	,,	,,	4
210.3	,,	,,	2
105	,,	,,	1,

making in all $2520 + 1680 + 315 + 630 + 105$, that is, 5250, distinct terms, whereas

$$\frac{(1.3.5.7)^2 + (1.3.5.7)}{2} = 5565,$$

the difference, 315, being due to the fact that there are that number of double synthemes which admit of a twofold resolution into two single synthemes.

I will not stop to prove, but any person conversant with the subject will see at once that this method gives an intuitive and direct proof of the theorem that a pure skew determinant for an even order is a perfect square*. Having

* That a skew determinant of an odd order vanishes is apparent from the fact that an odd number cannot be made up of a set of even ones. I use the term skew determinant in its strict sense as referring to a matrix for which $ij = -ji$ and $ii = 0$.

only a limited space at my command, I will pass on at once to forming the equation in differences for the case of a symmetrical, a skew, and one or two other special forms of determinants.

For a symmetrical determinant, taking as a diagram, to fix the ideas, the matrix of the 6th order

$$\begin{array}{cccccc} a & b & c & d & e & f \\ b & g & h & k & l & m \\ c & h & n & p & q & r \\ d & k & p & s & t & u \\ e & l & q & t & v & w \\ f & m & r & u & w & \omega \end{array},$$

calling u_m the number of distinct terms in a symmetrical matrix of the mth order, and, resolving the entire determinant into a sum of determinants of the order $(m-1)$ multiplied by the letters in the top line, we shall obviously get u_{m-1} together with $(m-1)$ quantities, positive or negative (and we know, by what precedes, that there can be no cancelling, so that the sign, for the object in view, may be entirely neglected) of the form

$$b \times \begin{array}{ccccc} b & h & k & l & m \\ c & n & p & q & r \\ d & p & s & t & u \\ e & q & t & v & w \\ f & r & u & w & \omega \end{array}.$$

Among these $(m-1)$ quantities all the terms containing *bc, bd, be, bf* will occur twice over, but those containing b^2 do not recur. Hence, to find the number of distinct terms we may reckon each of such distinct terms as contain *bc, bd, be, bf* worth only $\frac{1}{2}$, the others counting as 1. But if, instead of the column (which I write as a line) *bcdef*, we had the column *bhklm*, the rule for calculating the number of distinct terms might be calculated by this very same rule, except that the terms multiplied by *hc, kd, le, mf* ought to count as *units* instead of *halves*. Hence obviously

$$u_m + (m-1)(m-2)\,u_{m-3} \times \tfrac{1}{2} = u_{m-1} + (m-1)\,u_{m-1} = m u_{m-1},$$

or
$$u_m = m u_{m-1} - \tfrac{1}{2}(m-1)(m-2)\,u_{m-3},$$

which is Mr Cayley's equation, but obtained by a much more expeditious process (see Salmon's *Higher Algebra*, 3rd edition, pp. 40—42); writing $u_m = (1 \,.\, 2 \,\ldots\, m)\,v_m$ we obtain the equation in differences, linear in regard to the independent variable,

$$m v_m - m v_{m-1} + \tfrac{1}{2} v_{m-3} = 0,$$

and this, treated by the general method applicable to all such, gives rise to a linear differential equation in which, on account of the particular initial

values of u_0, u_1, u_2, the third term is wanting, and finally v_m is found to be the coefficient of t^m in

$$\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}.$$

If we apply a similar method to the case of a symmetrical determinant in which the diagonal of symmetry is filled out with zeros (an invertebrate symmetrical or symmetrical bialar determinant, as we may call it) we shall easily obtain the equation in differences

$$u_m = (m-1)[u_{m-1} + u_{m-2}] - \tfrac{1}{2}(m-1)(m-2)\, u_{m-3},$$

and, making $u_m = 1 \cdot 2 \ldots m v_m$,

$$m v_m - (m-1) v_{m-1} - v_{m-2} + \tfrac{1}{2} v_{m-3} = 0,$$

from which, calling $y = v_0 + v_1 t + v_2 t^2 + \ldots$ and having regard to the initial values v_0, v_1, v_2, we obtain

$$2\frac{dy}{y} = \frac{2t - t^2}{1-t} dt,$$

and
$$y = \frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}.$$

By way of distinction, using u' and v' for this case, and u, v for the preceding one, the slightest consideration shows that

$$u_m = u'_m + m u'_{m-1} + \frac{m(m-1)}{2} u'_{m-2} + \frac{m(m-1)(m-2)}{2 \cdot 3} u'_{m-3} + \ldots,$$

or
$$v_m = v'_m + v'_{m-1} + \frac{v'_{m-2}}{1 \cdot 2} + \frac{v'_{m-3}}{1 \cdot 2 \cdot 3} + \ldots.$$

Hence the generating function for v_m ought to be that for u_m multiplied by e^t, as we see is the case.

So, in like manner, the generating function for v_m, that is, $\frac{u_m}{1 \cdot 2 \ldots m}$, in the case of a general determinant being $\frac{1}{1-t}$, that of v_m for an invertebrate or zero-axial but otherwise general determinant we see must be $\frac{e^{-t}}{1-t}$*, that is,

$$v_m = 1 - 1 + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \ldots \pm \frac{1}{1 \cdot 2 \ldots m},$$

* It may easily be proved that the difference between the numbers of positive and negative combinations in the development of an invertebrate determinant of the mth order is $(-)^{m-1}(m-1)$ in favour of the former. From this it is easy to prove that the generating function for $\frac{\text{number of positive terms in such determinant}}{1 \cdot 2 \cdot 3 \ldots m}$ is

$$\frac{1}{2}\left\{\frac{e^{-t}}{1-t} - (1+t)e^{-t}\right\}, \text{ or } \frac{t^2 e^{-t}}{2(1-t)}.$$

the well known value $\left(\text{ultimately equal to } \frac{1}{e}\right)$, as it ought obviously to be, of the chance of two cards of the same name not coming together when one pack of m distinct cards is laid card for card under another precisely similar pack.

Returning to the case of the invertebrate symmetrical determinant, it will readily be seen, by virtue of the prolegomena, that the number of terms (the u_m) for such a determinant of the mth order is the same thing as the total number of duadic disynthemes that can be formed with m things, meaning by a duadic disyntheme any combination of duads with or without repetition, in which each element occurs twice and no oftener. Thus, when $m=6$, 1.2 2.3 1.3 4.5 4.6 5.6 and 1.2 2.3 3.4 5.6 6.1 and 1.2 2.3 3.4 1.4 5.6 5.6 are all three of them disynthemes. But the two latter ones are each resoluble into single synthemes, whereas the first one is not. It is clear that, when a disyntheme is formed by means of cycles all of an even order, it will be resoluble into a pair of single synthemes, and in no other case. The problem, then, of finding the number of distinct double synthemes with m elements is one and the same as that of finding the number of distinct terms in a *proper* (that is, invertebrate) skew determinant, which I proceed to consider.

Following a method (not identical with but) analogous to that adopted for the symmetrical cases, we shall find, by a process which the terms below written will sufficiently suggest

$$u_m + \frac{(m-1)(m-2)(m-3)}{2} u_{m-4} = (m-1) u_{m-2} + (m-1)(m-2) u_{m-2},$$

or
$$u_m = (m-1)^2 u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2} u_{m-4}.$$

Of course, when m is odd $u_m = 0$. From this it is readily seen that $\frac{u_{2m}}{1.3.5 \dots 2m-1}$, say ω_m, is an integer; for we shall have

$$\omega_m = (2m-1)\omega_{m-1} - (m-1)\omega_{m-2},$$

also,
$$\omega_1 = 1, \quad \omega_2 = 2,$$

Whence it follows that the number of positive terms in a general invertebrate determinant of the mth order is $m\frac{m-1}{2}$ times the total number of the terms in one of the $(m-2)$th order. The equation of differences for U_m, the total number, is of course

$$U_m = (m-1)(U_{m-1} + U_{m-2}),$$

and the successive values of

$$\begin{array}{lllllllll} U_m \text{ for } & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, \dots, \\ \text{are} & 0, & 1, & 2, & 9, & 44, & 265, & 1854, & 14833, \dots . \end{array}$$

so that
$$\begin{aligned}\omega_3 &= 5\,.\,2 \;\; - 2\,.\,1 \;=\; 8,\\ \omega_4 &= 7\,.\,8 \;\; - 3\,.\,2 \;=\; 50,\\ \omega_5 &= 9\,.\,50 \; - 4\,.\,8 \;=\; 418,\\ \omega_6 &= 11\,.\,418 - 5\,.\,50 = 4348,\end{aligned}$$
and the conventional $\omega_0 = 3\omega_1 - \omega_2 = 1$.

By the above formula u_m can be calculated with prodigious rapidity. If, however, we wish to obtain a generating function for u_m, the differential equation obtained from the above equation in differences does not lead to a simple explicit integral, but if we make $u_{2m} = (1\,.\,2\,.\,3 \ldots 2m)\,v_m$, as in the preceding cases, or, which is the same thing, $\omega_m = 2^m\,(1\,.\,2 \ldots m)\,v_m$, we get
$$4mv_m - 4\,(m-1)\,v_{m-1} - 2v_{m-1} + v_{m-2} = 0,$$
and, writing as before $\qquad y = v_0 + v_1 t + v_2 t^2 + \ldots,$
$$4\frac{dy}{dt} - 4t\frac{dy}{dt} - 2y + ty$$
will be found to be equal to zero. [This vanishing of the 3rd term in the differential equation being a feature common to all the cases we have considered, and due to the initial values of the v series in each case.] We have thus
$$\frac{4y'}{y} = \frac{1}{1-t} + 1, \quad y = \frac{e^{\frac{t}{4}}}{(1-t)^{\frac{1}{4}}}.$$
By way of verification, we may observe that
$$v_0 = 1, \quad v_1 = \frac{1}{2}, \quad v_2 = \frac{1}{4}, \quad v_3 = \frac{1}{6}, \ldots,$$
$$y = \left(1 + \frac{t}{4} + \frac{t^2}{32} + \frac{t^3}{384} + \ldots\right)\left(1 + \frac{t}{4} + \frac{5t^2}{32} + \frac{45t^3}{384} + \ldots\right),$$
and $\qquad \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad \frac{1}{32} + \frac{1}{16} + \frac{5}{32} = \frac{1}{4}, \quad \frac{45}{384} + \frac{5}{128} + \frac{1}{128} + \frac{1}{384} = \frac{1}{6}$*.

We may now proceed to calculate the number of distinct terms in an improper or vertebrated skew-determinant, which is interesting on account of its connection with the theory of orthogonal transformations. Using v_{2m}, instead of v_m, the generating function for the case last considered becomes $\dfrac{e^{\frac{t^2}{4}}}{\sqrt[4]{(1-t^2)}}$. Let $(1\,.\,2\,.\,3 \ldots m)\,V_m = U_m$ in general be used to denote the number of distinct terms in a vertebrate skew-determinant of the mth order. Then obviously
$$U_{2m} = u_{2m} + m\,.\,\frac{m-1}{2}\,u_{2m-2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{m-3}{4}\,u_{2m-4} + \ldots,$$

* The values of $v_1, v_2, v_3 \ldots$ are $\frac{1}{2}, \frac{2}{2\,.\,4}, \frac{8}{2\,.\,4\,.\,6}, \frac{50}{2\,.\,4\,.\,6\,.\,8}, \ldots$; that is, $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{25}{192}, \ldots$.

or
$$V_{2m} = v_{2m} + \frac{v_{2m-2}}{1\,.\,2} + \frac{v_{2m-4}}{1\,.\,2\,.\,3\,.\,4} + \dots.$$

Hence the generating function for V_{2m}

$$= \frac{e^{\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\left\{1 + \frac{t^2}{1\,.\,2} + \frac{t^4}{1\,.\,2\,.\,3\,.\,4} + \dots\right\} = \frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} + e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\},$$

and in like manner, since

$$U_{2m-1} = mu_{2m-2} + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}\,u_{2m-4} + \dots,$$

the generating function for V_{2m-1} will be

$$\frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} - e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\}.$$

Hence the number of distinct cross-products in the development of an orthogonal transformation-matrix of the mth order is

$$(1\,.\,2\,.\,3\,\dots\,m) \times \text{coefficient of } t^m \text{ in } \frac{e^{t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}.$$

POSTSCRIPT.—Let us consider the case of $2m$ elements; call the number of ways in which any disyntheme composed with them may be resolved into a pair of single synthemes one in each hand* its weight; furthermore, call the aggregate of those which appertain to an odd number of cycles the first class, and the other the second class. The entire sum of the weights we know is $1^2\,.\,2^2\,.\,3^2\,\dots\,(2m-1)^2$, but, furthermore, I find that the excess of the total weight of the first class over that of the second is

$$1^2\,.\,2^2\,.\,3^2\,\dots\,(2m-3)^2\,(2m-1);$$

or, in other words, the weights of the two classes are in the ratio of m to $m-1$.

The expressions for the sum and for the difference may, of course, by the *prolegomena* be translated into two theorems on the partition of numbers, neither of which, as far as I can see, is obvious upon the face of it†.

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* The two hands are introduced in order to double, by the effect of permutation, what the weight otherwise would be, except when the two component synthemes are identical, in which case the weight remains unity.

† REMARK [by F. Franklin].—The equation in differences for the number of double duadic synthemes may be obtained without recourse to determinants, as follows: Single out any element, 1; it may be paired in each of the component synthemes with any one of the remaining elements 2, 3, 4, ..., and there are two cases to be distinguished, namely, 1 may be paired either with the same element (2) or with two different elements (2, 3), in the two synthemes. The former may be done in $(m-1)$ ways, and, after having made our choice, we have still the choice of all the double synthemes that can be formed from 3, 4, ... m; 3, 4, ... m. The choice of two *different* elements may be made in $\frac{(m-1)(m-2)}{2}$ ways, and having chosen, we have still the choice of all the double

The properties of the ω series 1, 1, 2, 8, 50, ... [see p. 269] present some features of interest. These are the numbers of distinct terms in pure skew determinants of the order $2n$ divided by the product of the odd integers inferior to $2n$. Such numbers themselves may be termed the denumerants, and the quotients, when they are so divided, the reduced denumerants of the corresponding determinants; or for greater brevity we may provisionally call these reduced denumerants *skew numbers*. We have found, in what precedes, that

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{(1-t)}} = \omega_0 + \omega_1 \frac{t}{2} + \omega_2 \frac{t^2}{2 \cdot 4} + \omega_3 \frac{t^3}{2 \cdot 4 \cdot 6} + \dots .$$

From this we may easily obtain

$$\omega_x = \frac{Fx}{2^x},$$

where

$$Fx = 1 + 1 \cdot x + 1 \cdot 5 \frac{x(x-1)}{1 \cdot 2} + 1 \cdot 5 \cdot 9 \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + \dots + \dots + 1 \cdot 5 \cdot 9 \dots (4x-3),$$

which shows that Fx, for all values of x, contains 2^x as a factor, and that if we take x greater than unity, 2^{x+1} will be a factor of Fx. In general, it follows from the fundamental equation $\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2}$ that if two consecutive skew numbers ω_c, ω_{c+1} have a common factor, all those of superior orders, and consequently $\frac{Fx}{2^x}$, for all values of x from c upwards, will contain such factor. It becomes then a matter of interest to assign, if possible, a general expression for the greatest common measure of ω_x, ω_{x+1}.

In the first place I say these can have no common odd factor other than unity.

Lemma. It is well known that, in the development of $(1+a)^x$, all the coefficients except the first and last will contain x when it is a prime number. More generally it may easily be shown (and the mode of proof* is too obvious

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synthemes that can be formed from 3, 4, ... m; 3, 4, ... m. Now it is plain that the number of these can be obtained from the number of double synthemes that can be formed from 3, 4, ... m; 3, 4, ... m, by counting twice all except those in which 3 is paired twice with the same element; and is equal, therefore, from what precedes, to

$$2u_{m-2} - (m-3)\,u_{m-4}.$$

We have, therefore,

$$u_m = (m-1)\,u_{m-2} + \frac{(m-1)(m-2)}{2}[2u_{m-2} - (m-3)\,u_{m-4}]$$

$$= (m-1)^2 u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2} u_{m-4}.$$

* Some of the prolixity of the more obvious mode of proof of this lemma may be avoided by the substitution of the following method :

Call $(1+t)^n = 1 + A_1 t + A_2 t^2 + A_3 t^3 + \dots$, so that

$$n(1+t)^{n-1} = A_1 + 2A_2 t + 3A_3 t^2 + \dots$$

$$= B_0 + B_1 t + B_2 t^2 + \dots = \phi t.$$

to need setting out) that whatever x may be, any prime number contained in it must either divide any number r, or else the coefficient of a^r in the binomial expression above referred to. Hence we may prove that ω_x and x cannot have a common odd factor other than unity. For if possible, let $x = qp$, where p is a prime number contained in ω_x. Let the qp terms in Fx subsequent to the first term be divided into q groups, each containing p terms. Each of the terms in any one group (except the last) contains a binomial coefficient, which, by virtue of the lemma, will contain p. Moreover, the last term in the kth group will contain the factor

$$1 . 5 . 9 \dots (4kp - 1).$$

If p is of the form $4n - 3$, the nth term of the series 1, 5, 9, ... will be p, and if it is of the form $4n - 1$, the $(3n)$th term will be $3p$; and as $\frac{p+3}{4}$ and $3\frac{p+1}{4}$ are each not greater than p (and *à fortiori* not greater than kp) when p is greater than 1, it follows that the last coefficient, as well as all the others in any group, contains p. Hence $Fx = pP + 1$, and therefore ω_x, that is, $\frac{Fx}{2^x}$, cannot contain p. Hence the greatest common measure of ω_x and ω_{x+1} is a power of 2.

It will presently be shown by induction (waiting a strict proof)* that $\frac{\omega_{4x-2}}{2^x}$, $\frac{\omega_{4x-1}}{2^x}$, $\frac{\omega_{4x}}{2^x}$, $\frac{\omega_{4x+1}}{2^x}$ are all of them integers, and the first, third and fourth, odd integers; from this it will easily be seen that the greatest common measure of ω_x, ω_{x+1} is $2^{\theta\left(\frac{2x+1}{8}\right)}$, where, in general, $\theta(\mu)$ means the integer nearest to μ. Let us call the above fractions q_{4x-2}, q_{4x-1}, q_{4x}, q_{4x+1}, to which we may give the name of simplified skew numbers. In the subjoined table I have calculated the values of the residues of these numbers by a regular algorithm in respect to *moduli* beginning with 2^{23} and regularly decreasing according to the descending powers of 2. R stands for the words *residue of.*

Suppose $n = qp$: then designating the qth roots of unity by $\rho_1, \rho_2 \dots \rho_q$, we have

$$\frac{1}{q}\Sigma\rho^{q-k}\phi(\rho t) = B_k t^k + B_{k+q}t^{k+q} + B_{k+2q}t^{k+2q} + \dots + B_{k+(p-1)q}t^{k+(p-1)q},$$

and the left hand side of the equation is obviously a multiple of p. Hence, putting t successively equal to 0, 1, 2, 3, ... $(p-1)$, we obtain, by a well-known theorem of determinants,

$$\Delta B_{k+\lambda q} \equiv 0 \text{ [mod } p],$$

where Δ, being the product of the differences of 0, 1, 2, ... $(p-1)$, cannot contain p. Hence $B_{k+\lambda q} \equiv 0$ [mod p], and consequently giving k all values from 0 to $(q-1)$, and λ all values from 0 to $(p-1)$, we see that all the B's, from B_0 to B_{pq-1}, must contain p as a factor as was to be proved.

* Since the above was set up in print, I have found an easy proof, for which see *Postscript* [p. 279 below].

Modulus	x	Rq_{4x-2}	Rq_{4x-1}	Rq_{4x}	Rq_{4x+1}
8,388,608	0			1	1
4,194,304	1	1	4	25	209
2,097,152	2	1,087	13,504	194,951	1,088,983
1,048,576	3	929,451	442,068	992,179	576,715
524,288	4	287,913	118,168	393,089	71,201
262,144	5	201,913	14,228	126,417	179,945
131,072	6	51,071	56,656	46,407	127,767
65,536	7	56,531	24,452	15,131	46,739
32,768	8	12,521	29,928	22,753	29,729
16,384	9	14,289	5,412	15,209	14,305
8,192	10	1,119	2,784	4,063	4,751
4,096	11	3,283	3,156	2,331	3,059
2,048	12	1,721	1,632	425	1,801
1,024	13	913	84	1,001	385
512	14	215	240	479	239
256	15	91	132	99	219
128	16	81	8	9	9
64	17	41	36	1	57
32	18	23	0	31	15
16	19	3	4	11	3
8	20	1	0	1	1
4	21	1	0	1	1
2	22	1	0	1	1

From this table it appears that q_{8i-5} is 4 times an odd number, and that q_{8i-1} is 8 times a number which may be odd or even; thus we know the exact number of times that 2 will divide out all the skew numbers other than those whose orders are of the form $8i-1$, and an inferior limit to that number for that case.

It will further be noticed that, when x is of the form $4i$, or $4i+1$, the simplified skew numbers q_{4x-2}, q_{4x}, q_{4x+1} are all of the form $8\lambda+1$, that when x is of the form $4i+2$ the above named simplified skew numbers are of the form $8\lambda+7$, and when x is of the form $4i+3$, they are of the form $8\lambda+3$.

Before quitting this subject, I think it desirable briefly to refer to other series of integers closely connected with those which I have called *skew numbers.* To this end we may write, in general,

$$e^{\frac{t}{4}}(1-t)^{\frac{4\mu-1}{4}} = 1 + \omega_{1,\mu}\frac{t}{2} + \omega_{2,\mu}\frac{t^2}{2.4} + \omega_{3,\mu}\frac{t^3}{2.4.6} + \dots,$$

μ being any positive or negative integer, so that $\omega_{x,0}$ is the same as I have called hitherto ω_x. It may then easily be shown that $\omega_{x,\mu+1} = \dfrac{2\omega_{x+1,\mu} - \omega_{x,\mu}}{4\mu+1}$, that $\omega_{x,\mu-1} = \omega_{x,\mu} - 2x\omega_{x-1,\mu}$, and that the equation in differences for $\omega_{x,\mu}$, for μ constant, becomes

$$\omega_{x,\mu} = (2x + 2\mu - 1)\,\omega_{x-1,\mu} - (x-1)\,\omega_{x-2,\mu},$$

with the initial conditions $\omega_{0,\mu} = 1$, $\omega_{1,\mu} = 2\mu + 1$. Also, it is clear from the definition, that the explicit value of $\omega_{x,\mu}$ in a series becomes

$$\frac{1}{2^x}\left\{1 + (4\mu+1)\,x + (4\mu+1)(4\mu+5)\,x\frac{x-1}{2} \right.$$
$$\left. + (4\mu+1)(4\mu+5)(4\mu+9)\,x\frac{x-1}{2}\cdot\frac{x-2}{3} + \ldots\right\},$$

which is easily seen to verify the equation

$$2\omega_{x,\mu} - \omega_{x-1,\mu} = (4\mu+1)\,\omega_{x-1,\mu+1}.\text{*}$$

We might call the $\omega_{x,\mu}$ series skew numbers of the μth degree, and, as for the case of $\mu = 0$, so it may be shown in general that two consecutive skew numbers of the same degree can have no common odd factor. Also, it remains true that the greatest common factor of any two consecutive skew numbers of the same degree and the orders x, $x+1$, is $2^{\theta\left(\frac{2x+1}{8}\right)}$; $\omega_{4x-2,\mu}$, $\omega_{4x-1,\mu}$, $\omega_{4x,\mu}$, $\omega_{4x+1,\mu}$ being all divisible by 2^x, and the resulting quotients being, the first, third and fourth of them, always odd integers, and the second divisible by 4 or some higher power of 2 when μ is even, but only by the first power of 2 when μ is odd. But it would carry me too far away from the original object of this note, and from other investigations of more pressing moment to myself, to pursue further the theory of general skew numbers, which, however, seems to me to be well worthy of the study of arithmeticians.

I will only stop to point out that the rule for the greatest common measure of ω_x and ω_{x+1}, serves to prove the rule for the general case of $\omega_{x,\mu}$ and $\omega_{x+1,\mu}$. Thus suppose μ to be positive. Then since $\omega_{k,1} = 2\omega_{k+1} - \omega_k$, and $\omega_{4k-2} = 2^k(2\lambda+1)$, $\omega_{4k-1} = 2^{k+1}\tau$, $\omega_{4k} = 2^k(2\nu+1)$, $\omega_{4k+1} = 2^k(2\pi+1)$, $\omega_{4k+2} = 2^{k+1}(2\rho+1)$; it follows that

$\omega_{4k-2,1} = 2^k(2\lambda'+1)$, $\omega_{4k-1,1} = 2^{k+1}\tau'$, $\omega_{4k,1} = 2^k(2\nu'+1)$, and $\omega_{4k+1,1} = 2^k(2\pi'+1)$.

It is obvious further that, τ being even, τ' is odd. So again from these results we may, in like manner, deduce

$\omega_{4k-2,2} = 2^k(2\lambda''+1)$, $\omega_{4k-1,2} = 2^{k+1}\tau''$, $\omega_{4k,2} = 2^k(2\nu''+1)$, $\omega_{4k+1,2} = 2^k(2\pi''+1)$,

* And of course, in general, the equation

$$\lambda u_{x,y} - u_{x-1,y} + \phi y u_{x-1,y+\delta} = 0,$$

with the condition that $u_{0,y}$ is constant, has for its integral

$$u_{x,y} = \frac{c}{\lambda^x}\left\{1 - \phi y x + \phi y\phi(y+\delta)\,x\frac{x-1}{2} - \phi y\phi(y+\delta)\,\phi(y+2\delta)\,x\frac{x-1}{2}\,\frac{x-2}{3} + \ldots\right\}.$$

subject also to the remark that, τ' being odd, τ'' is even, and so on continually, τ being alternately even and odd. Again if μ is negative, we may, in like manner, by means of the formula $\omega_{k,\mu-1} = \omega_{k,\mu} - 2k\omega_{k-1,\mu}$, pass successively from the case of ω_k to that of $\omega_{k,-1} : \omega_{k,-2} : \ldots \omega_{k,-\mu}$, and establish precisely the same conclusion in regard to powers of 2 as for the case of μ positive, and it will be remembered that I have already shown how to establish that $\omega_{k,\mu}$ and $\omega_{k+1,\mu}$ have no common *odd* factor.

In the first note on this subject (Vol. II, No. 1, of the *Journal**) I showed how a general determinant could be completely represented by means of systems of cycles and that accordingly the terms in the total development would split up into families, as many in number as there are indefinite partitions of the index of the order of the determinant—the particular mode of aggregation depending upon the term chosen to represent the product of the elements in the principal diagonal, so that for the order n there would be $1 . 2 . 3 \ldots n$ distinct modes of distribution into families. This gives rise to a theory of transformation of cycles, corresponding to a transposition of the rows or columns of the matrix. Thus, for example, suppose the *umbrae* to be $1, 2, 3, \ldots n : r, s$ signifying the element in the rth row and sth column. Then if we interchange the mth and nth columns, this will have the effect of changing pm into pn and pn into pm.

Suppose now that a term of the developed determinant is expressed by a system of cycles such that m and n lie in two distinct cycles, say Xm and nY, where X, Y are each of them single elements, or aggregates of single elements; then the effect of the interchange will be to bring these cycles into the single cycle $XnYm$. If Xm, nY were both odd ordered or both even ordered cycles, their sum will be even ordered, and the number of *even* cycles will be increased or diminished by unity; so if one was of odd and the other of even order, their sum will be of odd order, and the number of even cycles will be diminished by unity. In either case, therefore, the sign, which depends on the *parity* of the number of even cycles, is reversed.

Again, suppose m and n to lie in the same cycle $mXnY$. Then the effect of the interchange will be to break this up into two cycles mX, nY, and for the same reason as above the sign will be reversed. Thus the sign of every term in the development will, we see, be reversed, as we know *à priori* ought to be the case.

[* p. 264 above.]

I shall conclude with applying the formula $\omega_x = \frac{Fx}{2^x}$ to determining the *asymptotic* mean value of the coefficients in a skew determinant of the order $2x$, that is, the function of x to which the mean value of the coefficients converges when x is taken indefinitely great. We know that all the coefficients, both in this case and in that of a symmetrical determinant, are different powers of 2; to find the mean of the indices of these powers would be seemingly an investigation of considerable difficulty, but there will be little or none in finding the ultimate expression for the mean of the coefficients themselves, or, which is the same thing, the first term in the function which expresses this mean in terms of descending powers of x. We shall find that, for symmetrical determinants, this is a certain multiple of the square root and, for skew determinants, of the fourth root of x, as I proceed to show.

From the equation

$$2^x \omega_x = 1 + x + 5x \frac{x-1}{2} + \dots + \{1 . 5 \dots (4x-3)\},$$

we have, when $x = \infty$,

$$2^x \omega_x = 1 . 5 . 9 \dots (4x-3) \left\{1 + \frac{x}{4x-3} + \frac{1}{2} \frac{x(x-1)}{(4x-3)(4x-7)} + \dots\right\}$$

$$= e^{\frac{1}{4}} . 1 . 5 . 9 \dots (4x-3).$$

The number of terms in the Pfaffian (the square root of the determinant taken with suitable algebraical sign) being $1 . 3 . 5 \dots (2x-1)$ and—as follows from what was shown in the first note—cancelling being out of the question, the sum of the coefficients all taken positively in the determinant itself will be $\{1 . 3 . 5 \dots (2x-1)\}^2$. Hence the mean value required is $\{1 . 3 . 5 \dots (2x-1)\}^2$ divided by $1 . 3 . 5 \dots (2x-1)\, \omega_x$, to express which quotient in exact terms we may make use of the formula

$$\frac{a(a+\delta)(a+2\delta)\dots(a+x\delta)}{b(b+\delta)(b+2\delta)\dots(b+x\delta)} = \frac{\Gamma\frac{b}{\delta}}{\Gamma\frac{a}{\delta}} x^{\frac{a-b}{\delta}}.$$

For the mean value is

$$\frac{1}{e^{\frac{1}{4}}} . \frac{1.3.5\dots(2x-1)}{2.4.6\dots(2x)} . \frac{4.8.12\dots(4x)}{1.5.9\dots(4x-3)} = \frac{1}{e^{\frac{1}{4}}} . \frac{1}{\Gamma\frac{1}{2}} x^{-\frac{1}{2}} . \Gamma\tfrac{1}{4}\, x^{\frac{3}{4}} = \frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{\pi}} x^{\frac{1}{4}}.$$

If we write this under the form $Qx^{\frac{1}{4}}$, we have

$$Q = \frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\Gamma\frac{1}{2}},$$

$$\log Q = \log\Gamma\tfrac{5}{4} + \log 2 - \log\Gamma\tfrac{3}{2} - \tfrac{1}{4}\log e$$
$$= 9{\cdot}9573211 + {\cdot}3010300 - 9{\cdot}9475449 - {\cdot}1085736$$
$$= {\cdot}2022326,$$

or $$Q = 1{\cdot}59306.$$

This result as may easily be seen remains unaffected when, instead of a pure skew determinant, one is taken in which the diagonal terms retain general values. The effect of this change will be to increase the numerator and denominator of the fraction which expresses the mean value, in the proportion of $\dfrac{e^2+1}{2e}$ to 1.

Finally, as regards the ultimate mean value of the coefficients of symmetrical determinants. This, for one of the order x, by virtue of Professor Cayley's formula previously given, will be the reciprocal of the coefficient of t^x in $\dfrac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$. It may readily be shown in general that, ϕt being any series of integer powers of t, the coefficient of t^x (when x becomes infinite) in $\dfrac{e^{\phi t}}{\sqrt{(1-t)}}$ is in a ratio of equality to the coefficient of t^x in $\dfrac{e^{(\phi 1)\,t}}{\sqrt{(1-t)}}$, so that in the present case this coefficient is the same as the coefficient of t^x in $\dfrac{e^{\frac{3}{4}t}}{\sqrt{(1-t)}}$, that is, in

$$\left\{1+\frac{1}{2}t+\frac{1\,.\,3}{2\,.\,4}t^2+\ldots+\frac{1\,.\,3\,.\,5\ldots(2x-1)}{2\,.\,4\,.\,6\ldots 2x}t^x+\ldots\right\}$$
$$\times\left\{1+\frac{3}{4}t+\left(\frac{3}{4}\right)^2\frac{t^2}{2}+\ldots+\left(\frac{3}{4}\right)^x\frac{t^x}{1\,.\,2\ldots x}+\ldots\right\},$$

which is obviously, when x is infinite, equal to $\dfrac{1\,.\,3\,.\,5\ldots(2x-1)}{2\,.\,4\,.\,6\ldots 2x}e^{\frac{3}{4}}$. Hence the ultimate mean value of the coefficients is $\dfrac{1}{e^{\frac{3}{4}}}\dfrac{2\,.\,4\,.\,6\ldots 2x}{1\,.\,3\,.\,5\ldots(2x-1)}$, or $\dfrac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$.

For a symmetrical determinant in which all the diagonal terms are wanting, the numerator of the fraction giving the mean value becomes $e^{-1}(1\,.\,2\,.\,3\ldots x)$ and the denominator is $(1\,.\,2\,.\,3\ldots x)$ into the coefficient of t^x in $\dfrac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$, which is the same as in $\dfrac{e^{-\frac{1}{4}t}}{\sqrt{(1-t)}}$. The result then is $\dfrac{\pi^{\frac{1}{2}}e^{\frac{1}{4}}}{e}\sqrt{x}$, or $\dfrac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$ as before. It may perhaps be just worth while to notice that the *skew numbers* (the ω's of the text) may be put under the form of a determinant, the nature of which is sufficiently indicated by the annexed diagram.

1	1	0	0	0	0	0
1	3	2	0	0	0	0
0	1	5	3	0	0	0
0	0	1	7	4	0	0
0	0	0	1	9	5	0
0	0	0	0	1	11	6
0	0	0	0	0	1	13

The successive principal minors in this matrix represent the successive skew numbers of all orders from 1 to 6 inclusive.

Postscript. [See p. 273 above, footnote.]

Since $\omega_{x+1} = (2x+1)\,\omega_x - x\omega_{x-1}$, we have

$\omega_{x+2} = (4x^2 + 7x + 2)\,\omega_x - (2x^2 + 3x)\,\omega_{x-1}$,

$\omega_{x+3} = (8x^3 + 32x^2 + 34x + 8)\,\omega_x - (4x^3 + 15x^2 + 13x)\,\omega_{x-1}$,

$\omega_{x+4} = (16x^4 + 116x^3 + 273x^2 + 231x + 50)\,\omega_x - (8x^4 + 56x^3 + 122x^2 + 82x)\,\omega_{x-1}$.

Suppose now that, for a given value of i, $q_{4i-2} = \dfrac{\omega_{4i-2}}{2^i} = 2\lambda + 1$, $q_{4i-1} = \dfrac{\omega_{4i-1}}{2^i} = 4\mu$, $q_{4i} = \dfrac{\omega_{4i}}{2^i} = 2\nu + 1$ and $q_{4i+1} = \dfrac{\omega_{4i+1}}{2^i} = 2\rho + 1$. Call $\omega_{x+4} = E_x\omega_x - F_x\omega_{x-1}$. Then when $x \equiv \pm 2$, $F_x \equiv 4$ [mod 8], and therefore, assuming that $q_{4i-3} = \dfrac{\omega_{4i-3}}{2^{i-1}}$ is odd, $\dfrac{F_{4i-2}\,\omega_{4i-3}}{2^{i+1}}$ is odd. Also, $E_{4i-2} \equiv 462 + 50 \equiv 0$ [mod 4], and consequently $\dfrac{E_{4i-2}\,\omega_{4i-2}}{2^{i+1}}$ is even; hence $q_{4i+2} = \dfrac{\omega_{4i+2}}{2^{i+1}}$ is integer and odd. Again when $x = 4i - 1$, $E_x \equiv 1 - 3 + 50 \equiv 0$ [mod 4], and $F_x \equiv 122 - 82 \equiv 0$ [mod 8]; hence $q_{4i+3} = \dfrac{\omega_{4i+3}}{2^{i+1}}$ is an integer divisible by 4. Again, when $x = 4i$, $E_{4i} \equiv 2$ and $F_{4i} \equiv 0$ [mod 4]; hence $q_{4i+4} = \dfrac{\omega_{4i+4}}{2^{i+1}}$ is integer and odd; and when $x = 4i + 1$, $E_{4i+1} \equiv 2$ and $F_{4i+1} \equiv 0$ [mod 4]; hence $q_{4i+5} = \dfrac{\omega_{4i+5}}{2^{i+1}}$ is integer and odd.

Thus it has been shown that if it be true up to $\lambda = i$ that $\frac{\omega_{4\lambda-2}}{2^\lambda}$, $\frac{\omega_{4\lambda-1}}{2^{\lambda+2}}$, $\frac{\omega_{4\lambda}}{2^\lambda}$, $\frac{\omega_{4\lambda+1}}{2^\lambda}$ are all integer, and the first, third and fourth odd integers, the same proposition can be affirmed for all superior values of i, and being true for ω_0, ω_1, ω_2, ω_3, the quotients corresponding to which are 1, 1, 1, 1, the theorem is true universally. It is inconceivable that it could have occurred to any human being to lay down so singular a train of induction as the one above employed, unless previously prompted to do so by an *à priori* perception of the law to be established, acquired through a preliminary study and direct inspection of the earlier terms in the series of numbers to which it applies. Here then we have a salient example (if any were needed) of the importance of the part played by the *faculty of observation* in the discovery and establishment of pure mathematical laws.

37.

ON THE COMPLETE SYSTEM OF THE "GRUNDFORMEN" OF THE BINARY QUANTIC OF THE NINTH ORDER.

[*American Journal of Mathematics*, II. (1879), pp. 98, 99.]

Enumeration of the Irreducible Invariants and Covariants of the Binary Quantic of the Ninth Order.

Degree in the Coefficients.	Order in the Variables.																				
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		3		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6			4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

The foregoing table has been calculated, out of the funds voted by the British Association, under my superintendence, by Mr Franklin, Fellow of Johns Hopkins University. A statement of the method employed will be given in a future number of the *Journal*.

The total number of irreducible forms will be seen from the table to be 415. The highest degree in the coefficients is 18, and the highest order in the variables 22. The *representative* generating function in this case (as in all others which have been hitherto treated, with the sole exception of the seventhic) has a *finite* numerator.

The total number of groundforms for the orders 0, 2, 4, 6 respectively (counting, as one ought to do, the absolute constant as one of them) is 1, 3, 6, 27, which becomes a regular series on increasing 6, which corresponds to a square index 4, in the proportion of 2 : 3. In like manner, for the orders 1, 3, 5, 7, 9, the series is 2, 5, 24, 125, 416, which, on increasing the last term corresponding to the square index 9 in the ratio 2 : 3, forms an almost regular progression 2, 5, 24, 125, 624, highly suggestive of the geometrical series 1, 5, 25, 125, 625. It seems then to be a not altogether improbable conjecture, that the number of groundforms for 10, which I hope very soon to get completely worked out, will be in the neighbourhood of a ratio of equality to 243*, and for 11, which there is not much prospect of calculating for some time to come, a number not very far out from a ratio of equality to 3125. In the next, or next but one, number of the *Journal* I hope to set out a synoptical table of the groundforms for all orders up to 10 inclusive, with their reduced and representative generating functions, as also for combinations of the orders: 2, 3; 2, 4; 3, 3; 3, 4; 4, 4; all the materials for which, with the exception of what pertains to the covariants *proper* of the tenthic, are already in existence.

* The number of groundforms for the Octavic (I quote from memory) is 70, not more inferior to 81 than might have been anticipated, when the composite form of the number 8 is taken into account. It seems likely that for 10, 243 is at all events a superior limit. [See below, p. 307.]

38.

TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS FOR THE BINARY QUANTICS OF THE FIRST TEN ORDERS.

[*American Journal of Mathematics*, II. (1879), pp. 223—251.]

IN what follows, "G. F." stands for the words *Generating Function.* In the Generating Functions, the exponents of the letter a refer to degree in the coefficients, and the exponents of the letter x to order in the variables. The Generating Functions for differentiants take account only of degree in the coefficients, without regard to the order in the variables of the covariant of which the differentiant is the "*source.*" In the *tabulated* numerators of the Generating Functions, the *minus* sign is placed *over* instead of *to the left of* the number which it affects.

QUADRIC.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-a^2)}$.

G. F. for covariants, $\dfrac{1}{(1-a^2)(1-ax^2)}$.

Groundforms: 1 of deg. 1, ord. 2; 1 of deg. 2, ord. 0.

CUBIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)(1-a^4)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax+a^2x^2}{(1-a^4)(1-ax)(1-ax^3)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^3}{(1-a^4)(1-a^2x^2)(1-ax^3)}$.

Groundforms: 1 of deg. 1, ord. 3; 1 of deg. 2, ord. 2; 1 of deg. 3, ord. 3; 1 of deg. 4, ord. 0.

QUARTIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)^2(1-a^3)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax^2+a^2x^4}{(1-a^2)(1-a^3)(1-ax^2)(1-ax^4)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^6}{(1-a^2)(1-a^3)(1-a^2x^4)(1-ax^4)}$.

Groundforms: 1 of deg. 1, ord. 4; 1 of deg. 2, ord. 0; 1 of deg. 2, ord. 4; 1 of deg. 3, ord. 0; 1 of deg. 3, ord. 6.

Quintic.

G. F. for differentiants,

$$\frac{1+a^2+3a^3+3a^4+5a^5+4a^6+6a^7+6a^8+4a^9+5a^{10}+3a^{11}+3a^{12}+a^{13}+a^{15}}{(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-ax)(1-ax^3)(1-ax^5)$.

Numerator: $1+a(-x-x^3)+a^2(x^2+x^4+x^6)-a^3x^7+a^4x^4+a^5(x+x^3-x^5)$
$+a^6(-1-x^4)+a^7(2x+x^3+x^5)+a^8(-x^2-x^4-2x^6)$
$+a^9(x^3+x^7)+a^{10}(x^2-x^4-x^6)-a^{11}x^3+a^{12}+a^{13}(-x-x^3-x^5)$
$+a^{14}(x^4+x^6)-a^{15}x^7$.

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{12})(1-a^2x^2)(1-a^2x^6)(1-ax^5)$.

Numerator: $1+a^3(x^3+x^5+x^9)+a^4(x^4+x^6)+a^5(x+x^3+x^7-x^{11})$
$+a^6(x^2+x^4)+a^7(x+x^5-x^9)+a^8(x^2+x^4)+a^9(x^3+x^5-x^7)$
$+a^{10}(x^2+x^4-x^{10})+a^{11}(x+x^3-x^9)+a^{12}(x^2-x^8-x^{10})$
$+a^{13}(x-x^7-x^9)+a^{14}(x^4-x^6-x^8)+a^{15}(-x^7-x^9)$
$+a^{16}(x^2-x^6-x^{10})+a^{17}(-x^7-x^9)+a^{18}(1-x^4-x^8-x^{10})$
$+a^{19}(-x^5-x^7)+a^{20}(-x^2-x^6-x^8)-a^{23}x^{11}$.

Table of Groundforms.

Degree in the Coefficients	Order in the Variables. 0	1	2	3	4	5	6	7	9
1						1			
2			1				1		
3				1		1			1
4	1				1		1		
5		1		1				1	
6			1		1				
7		1				1			
8	1		1						
9				1					
11		1							
12	1								
13		1							
18	1								

SEXTIC.

G. F. for differentiants, $\dfrac{1 + a^2 + 3a^3 + 4a^4 + 4a^5 + 4a^6 + 3a^7 + a^8 + a^{10}}{(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-ax^2)(1-ax^4)(1-ax^6)$.

Numerator: $1 + a(-x^2 - x^4) + a^2(-1 + x^4 + x^6 + x^8) + a^3(-1 + 2x^2 + x^4 - x^{10})$
$+ a^4(x^2 - x^6 - x^8) + a^5(-x^6 - x^8 + x^{10}) + a^6(1 - x^2 - x^8 + x^{10})$
$+ a^7(1 - x^2 - x^4) + a^8(-x^2 - x^4 + x^8) + a^9(-1 + x^6 + 2x^8 - x^{10})$
$+ a^{10}(x^2 + x^4 + x^6 - x^{10}) + a^{11}(-x^6 - x^8) + a^{12}x^{10}$.

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)(1-a^{10})(1-a^2x^4)(1-a^2x^8)(1-ax^6)$.

Numerator: $1 + a^3(x^2 + x^6 + x^8 + x^{12}) + a^4(x^4 + x^6 + x^{10}) + a^5(x^2 + x^4 + x^8 - x^{16})$
$+ a^6(x^4 + 2x^6) + a^7(x^2 + x^4 + x^8 - x^{12}) + a^8(x^2 + x^4 + x^6 - x^{14})$
$+ a^9(x^4 + x^6 - x^{10} - x^{12}) + a^{10}(x^2 + x^4 - x^{12} - x^{14}) + a^{11}(x^4$
$+ x^6 - x^{10} - x^{12}) + a^{12}(x^2 - x^{10} - x^{12} - x^{14}) + a^{13}(x^4 - x^8 - x^{12} - x^{14})$
$+ a^{14}(-2x^{10} - x^{12}) + a^{15}(1 - x^8 - x^{12} - x^{14}) + a^{16}(-x^6 - x^{10} - x^{12})$
$+ a^{17}(-x^4 - x^8 - x^{10} - x^{14}) - a^{20}x^{16}$.

Table of Groundforms.

Degree in the Coefficients. \ Order in the Variables.	0	2	4	6	8	10	12
1				1			
2	1		1		1		
3		1		1	1		1
4	1		1	1		1	
5		1	1		1		
6	1			2			
7		1	1				
8		1					
9			1				
10	1	1					
12		1					
15	1						

* This is not strictly the minimum form, its numerator and denominator being divisible by $1-a$; it is, however, the lowest form to which the fraction can be reduced when the factors of the denominator are all of the forms $1-a^r$, $1-a^r x^s$. The same remark applies to the "reduced form" in the case of the decimic.

SEPTIMIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$.

Numerator: $1+2a^2+6a^3+10a^4+19a^5+28a^6+44a^7+61a^8+79a^9$
$+102a^{10}+129a^{11}+156a^{12}+173a^{13}+196a^{14}+215a^{15}$
$+230a^{16}+231a^{17}+231a^{18}+230a^{19}+215a^{20}+196a^{21}$
$+173a^{22}+156a^{23}+129a^{24}+102a^{25}+79a^{26}+61a^{27}+44a^{28}$
$+28a^{29}+19a^{30}+10a^{31}+6a^{32}+2a^{33}+a^{35}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-ax)(1-ax^3)$
$(1-ax^5)(1-ax^7)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^0	1														
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$									
a^2			1		1		2		1		1				
a^3								$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^4					2				1						1
a^5		1		2						$\bar{1}$		$\bar{1}$			
a^6	$\bar{1}$		2		$\bar{1}$				$\bar{1}$		$\bar{1}$		1		
a^7		4		1		3				$\bar{1}$		1			
a^8	2		$\bar{1}$				$\bar{3}$		$\bar{3}$		$\bar{1}$		$\bar{1}$		
a^9		1		3		1		$\bar{1}$		2				2	
a^{10}	$\bar{1}$		4				$\bar{1}$		$\bar{2}$		$\bar{2}$				$\bar{1}$
a^{11}		5		3		2		$\bar{1}$		$\bar{2}$		$\bar{1}$		1	
a^{12}	5		1				$\bar{4}$		$\bar{6}$		$\bar{4}$		$\bar{1}$		2
a^{13}		1				$\bar{4}$		$\bar{4}$		$\bar{1}$		1		4	
a^{14}	2		5		1		1		$\bar{2}$				3		$\bar{1}$
a^{15}		8		$\bar{1}$		$\bar{1}$		$\bar{7}$		$\bar{5}$		$\bar{1}$		$\bar{1}$	
a^{16}	6		3		3		$\bar{4}$		$\bar{8}$				$\bar{1}$		5
a^{17}		$\bar{1}$		$\bar{2}$		$\bar{9}$		$\bar{8}$		$\bar{4}$		$\bar{3}$		4	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^{18}	2		6		1		2		2		1		6		2
a^{19}		4		$\bar{3}$		$\bar{4}$		$\bar{8}$		$\bar{9}$		$\bar{2}$		$\bar{1}$	
a^{20}	5		$\bar{1}$				$\bar{3}$		$\bar{4}$		3		3		6
a^{21}		$\bar{1}$		$\bar{1}$		$\bar{5}$		$\bar{7}$		$\bar{1}$		$\bar{1}$		3	
a^{22}	$\bar{1}$		3				$\bar{2}$		1		1		5		2
a^{23}		4		1		$\bar{1}$		$\bar{4}$		$\bar{4}$				1	
a^{24}	2		$\bar{1}$		$\bar{4}$		$\bar{6}$		$\bar{4}$				1		5
a^{25}		1		$\bar{1}$		$\bar{2}$		$\bar{1}$		2		3		5	
a^{26}	$\bar{1}$				$\bar{2}$		$\bar{2}$		$\bar{1}$				4		$\bar{1}$
a^{27}		2				2		$\bar{1}$		1		3		1	
a^{28}			$\bar{1}$		$\bar{1}$		$\bar{3}$		$\bar{3}$				$\bar{1}$		2
a^{29}				1		$\bar{1}$				3		1		4	
a^{30}			1		$\bar{1}$		$\bar{1}$				$\bar{1}$		2		$\bar{1}$
a^{31}				$\bar{1}$		$\bar{1}$						2		1	
a^{32}	1						1				2				
a^{33}		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$							
a^{34}					1		1		2		1		1		
a^{35}										$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^{36}															1

Owing to the non-existence of an irreducible invariant whose degree is 10, or any multiple of 10, no representative generating function with a *finite* numerator can be obtained for the septimic; the factor $1-a^{10}$ in the denominator has to be got rid of by dividing numerator and denominator by it, or, in other words, by striking it out of the denominator and multiplying the numerator by the infinite series $1+a^{10}+a^{20}+\ldots$. We thus obtain:

G. F. for covariants, representative form, (with infinite numerator),

Denominator: $(1-a^4)(1-a^8)(1-a^{12})^2(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})(1-ax^7)$.

Numerator: (Given to the terms containing the 45th power of a, inclusive; after which, each column can be continued by repeating *the last five coefficients* occurring in it, *ad inf.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^3				1		1		1		1		1				1								
a^4					2		1		2		1				1									
a^5		1		2		2		2		2								$\overline{1}$				$\overline{1}$		
a^6			3		2		3		3				2		$\overline{1}$		$\overline{1}$							
a^7		3		2		4		4				1				$\overline{2}$				1				1
a^8	2		3		4		6		$\overline{1}$		3		$\overline{1}$		$\overline{2}$				$\overline{1}$					
a^9		3		5		7		1		4				$\overline{2}$		$\overline{1}$		$\overline{2}$				1		
a^{10}			5		8		6		4		1		$\overline{4}$				$\overline{3}$		$\overline{1}$					
a^{11}		5		8		8		8		4		$\overline{4}$		$\overline{1}$		$\overline{5}$		$\overline{1}$						
a^{12}	4		9		9		12		4		$\overline{1}$		$\overline{3}$		$\overline{5}$		$\overline{6}$				$\overline{1}$		1	
a^{13}		9		9		12		6		$\overline{1}$		$\overline{3}$		$\overline{8}$		$\overline{9}$		$\overline{3}$		$\overline{1}$		1		
a^{14}	4		9		13		11		$\overline{1}$		$\overline{3}$		$\overline{9}$		$\overline{10}$		$\overline{7}$		$\overline{2}$				3	
a^{15}		9		12		16		3		2		$\overline{10}$		$\overline{11}$		$\overline{8}$		$\overline{3}$				3		2
a^{16}	5		14		15		12		1		$\overline{5}$		$\overline{16}$		$\overline{9}$		$\overline{9}$		$\overline{1}$		3		3	
a^{17}		12		15		16		6		$\overline{3}$		$\overline{17}$		$\overline{13}$		$\overline{15}$		$\overline{5}$		2		3		
a^{18}	9		14		15		14		$\overline{3}$		$\overline{13}$		$\overline{20}$		$\overline{15}$		$\overline{15}$		2		2		5	
a^{19}		15		16		18				$\overline{8}$		$\overline{18}$		$\overline{20}$		$\overline{19}$		$\overline{3}$		3		5		4
a^{20}	7		14		18		12		$\overline{10}$		$\overline{16}$		$\overline{25}$		$\overline{19}$		$\overline{12}$		2		5		9	
a^{21}		14		17		19		$\overline{1}$		$\overline{8}$		$\overline{27}$		$\overline{25}$		$\overline{16}$		$\overline{2}$		4		8		4
a^{22}	9		17		19		11		$\overline{8}$		$\overline{18}$		$\overline{31}$		$\overline{17}$		$\overline{15}$		-6		9		9	
a^{23}		17		19		18		$\overline{3}$		$\overline{13}$		$\overline{31}$		$\overline{25}$		$\overline{21}$		$\overline{4}$		9		9		5
a^{24}	8		17		17		10		$\overline{12}$		$\overline{27}$		$\overline{32}$		$\overline{22}$		$\overline{16}$		9		9		12	
a^{25}		18		17		19		$\overline{6}$		$\overline{17}$		$\overline{31}$		$\overline{28}$		$\overline{22}$		3		10		12		9
a^{26}	9		18		18		11		$\overline{17}$		$\overline{23}$		$\overline{34}$		$\overline{21}$		$\overline{10}$		10		14		15	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{27}		17		17		19		$\overline{9}$		$\overline{16}$		$\overline{36}$		$\overline{29}$		$\overline{19}$		8		13		14		7
a^{28}	8		17		18		9		$\overline{16}$		$\overline{26}$		$\overline{38}$		$\overline{18}$		$\overline{13}$		14		15		14	
a^{29}		18		19		17		$\overline{8}$		$\overline{16}$		$\overline{36}$		$\overline{25}$		$\overline{21}$		6		16		16		9
a^{30}	9		18		18		10		$\overline{18}$		$\overline{27}$		$\overline{35}$		$\overline{19}$		$\overline{11}$		16		15		17	
a^{31}		17		17		17		$\overline{8}$		$\overline{19}$		$\overline{36}$		$\overline{29}$		$\overline{19}$		8		15		17		8
a^{32}	9		18		18		8		$\overline{18}$		$\overline{26}$		$\overline{35}$		$\overline{19}$		$\overline{10}$		17		17		18	
a^{33}		18		18		18		$\overline{9}$		$\overline{18}$		$\overline{34}$		$\overline{26}$		$\overline{17}$		8		18		18		9
a^{34}	8		17		17		9		$\overline{17}$		$\overline{28}$		$\overline{36}$		$\overline{18}$		$\overline{8}$		17		17		17	
a^{35}		18		17		18		$\overline{9}$		$\overline{17}$		$\overline{35}$		$\overline{27}$		$\overline{18}$		9		18		17		8
a^{36}	9		19		18		9		$\overline{18}$		$\overline{25}$		$\overline{31}$		$\overline{17}$		$\overline{9}$		17		19		18	
a^{37}		17		17		18		$\overline{9}$		$\overline{18}$		$\overline{37}$		$\overline{26}$		$\overline{18}$		9		17		17		9
a^{38}	9		17		17		9		$\overline{18}$		$\overline{26}$		$\overline{37}$		$\overline{18}$		$\overline{9}$		18		17		17	
a^{39}		18		19		17		$\overline{9}$		$\overline{17}$		$\overline{34}$		$\overline{25}$		$\overline{18}$		9		18		19		9
a^{40}	9		17		18		9		$\overline{18}$		$\overline{27}$		$\overline{35}$		$\overline{17}$		$\overline{9}$		18		17		18	
a^{41}		17		17		17		$\overline{8}$		$\overline{18}$		$\overline{36}$		$\overline{28}$		$\overline{17}$		9		17		17		8
a^{42}	9		18		18		8		$\overline{17}$		$\overline{26}$		$\overline{34}$		$\overline{18}$		$\overline{9}$		18		18		18	
a^{43}		18		18		18		$\overline{9}$		$\overline{18}$		$\overline{34}$		$\overline{26}$		$\overline{17}$		8		18		16		9
a^{44}	8		17		17		9		$\overline{17}$		$\overline{28}$		$\overline{36}$		$\overline{18}$		$\overline{8}$		17		17		17	
a^{45}		18		17		18		$\overline{9}$		$\overline{17}$		$\overline{35}$		$\overline{27}$		$\overline{18}$		9		18		17		9

etc. etc. etc.

Table of Groundforms.

Degree in the Coefficients. \ Order in the Variables.	0	1	2	3	4	5	6	7	8	9	10	11	14	15
1								1						
2			1				1				1			
3				1		1		1		1		1		1
4	1				2		1		2		1		1	
5		1		2		2		2		2				
6			3		2		2		2					
7		3		2		4		2						
8	3		3		3		3							
9		3		5		2								
10			4		3									
11		5		3										
12	6		6											
13		7												
14	4													
15		3												
16	2													
17		2												
18	9													
22	1													

Octavic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)^2(1-a^4)(1-a^5)(1-a^7)$.

Numerator: $1+2a^2+6a^3+12a^4+19a^5+25a^6+31a^7+36a^8+38a^9+36a^{10}$
$+31a^{11}+25a^{12}+19a^{13}+12a^{14}+6a^{15}+2a^{16}+a^{18}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}
a^0	1									
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$						
a^2			1	1	2	1	1			
a^3			1			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^4			2							1
a^5		1	2		$\bar{1}$		$\bar{1}$			
a^6		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^7		2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^8	1	2			$\bar{2}$	$\bar{2}$	$\bar{2}$	1		
a^9	1	2	$\bar{2}$		$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	
a^{10}	1	1	$\bar{2}$		$\bar{2}$	$\bar{1}$			1	
a^{11}		1	$\bar{1}$		$\bar{1}$	$\bar{1}$		$\bar{1}$	1	
a^{12}		1			$\bar{1}$	$\bar{2}$		$\bar{2}$	1	1
a^{13}		1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{2}$	2	1
a^{14}			1	$\bar{2}$	$\bar{2}$	$\bar{2}$			2	1
a^{15}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1	2	
a^{16}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1	
a^{17}				$\bar{1}$		$\bar{1}$		2	1	
a^{18}	1							2		
a^{19}		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$			1		
a^{20}				1	1	2	1	1		
a^{21}							$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{22}										1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^2x^4)$
$(1-a^2x^8)(1-a^2x^{12})(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}
a^0	1															
a^3			1	1	1	1	1	1		1						
a^4			2	1	1	2	1	1		1						
a^5		1	2	2	1	3		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		
a^6		1	2	3	2	2		1		$\bar{1}$		$\bar{1}$				
a^7		2	2	3	2	2		$\bar{1}$		$\bar{2}$		$\bar{1}$				1
a^8	1	2	2	3	3	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{1}$					
a^9	1	3	1	3	2	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{1}$		1		
a^{10}	1	2	1	2	1	$\bar{2}$	$\bar{2}$	$\bar{5}$	$\bar{4}$	$\bar{4}$	$\bar{2}$	$\bar{1}$		1		
a^{11}		2	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{4}$	$\bar{4}$	$\bar{3}$			2		
a^{12}		1	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{3}$			2	1	
a^{13}		1	2			$\bar{3}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	1	
a^{14}			2			$\bar{3}$	$\bar{4}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	2	
a^{15}			1		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{2}$	$\bar{2}$	1	2	1	2	1
a^{16}			1		$\bar{1}$	$\bar{1}$	$\bar{4}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	2	3	1	3	1
a^{17}						$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	3	3	2	2	1
a^{18}	$\bar{1}$				$\bar{1}$		$\bar{2}$		$\bar{1}$		2	2	3	2	2	
a^{19}					$\bar{1}$		$\bar{1}$		1		2	2	3	2	1	
a^{20}			$\bar{1}$		$\bar{1}$		$\bar{1}$		1		3	1	2	2	1	
a^{21}							1		1	1	2	1	1	2		
a^{22}							1		1	1	1	1	1	1		
a^{25}																1

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	18
1					1				
2	1		1		1		1		
3	1		1	1	1	1	1	1	1
4	1		2	1	1	2	1	1	1
5	1	1	2	2	1	3		1	
6	1	1	2	3	1	1			
7	1	2	2	3					
8	1	2	2	2					
9	1	3	1						
10	1	2							
11		2							
12		1							

NONIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$
$(1-a^{14})(1-a^{16})$.

Numerator: $1+3a^2+10a^3+23a^4+49a^5+93a^6+172a^7+289a^8+457a^9$
$+701a^{10}+1036a^{11}+1477a^{12}+2023a^{13}+2720a^{14}+3568a^{15}$
$+4573a^{16}+5702a^{17}+7013a^{18}+8466a^{19}+10043a^{20}+11672a^{21}$
$+13400a^{22}+15155a^{23}+16880a^{24}+18487a^{25}+20013a^{26}$
$+21392a^{27}+22539a^{28}+23398a^{29}+24013a^{30}+24355a^{31}$
$+24355a^{32}+24013a^{33}+23398a^{34}+22539a^{35}+21392a^{36}$
$+20013a^{37}+18487a^{38}+16880a^{39}+15155a^{40}+13400a^{41}$
$+11672a^{42}+10043a^{43}+8466a^{44}+7013a^{45}+5702a^{46}+4573a^{47}$
$+3568a^{48}+2720a^{49}+2023a^{50}+1477a^{51}+1036a^{52}+701a^{53}$
$+457a^{54}+289a^{55}+172a^{56}+93a^{57}+49a^{58}+23a^{59}+10a^{60}$
$+3a^{61}+a^{63}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-a^{14})(1-a^{16})$
$(1-ax)(1-ax^3)(1-ax^5)(1-ax^7)(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$																
a^2			1		1		2		2		2		1		1									
a^3								$\bar{1}$		$\bar{1}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$		$\bar{1}$				
a^4	1				2		1		2				1		1		1		1		1		1	
a^5				2		1				$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$		$\bar{1}$						$\bar{1}$
a^6	$\bar{1}$		4		1		8										1		1		1			
a^7		5		5		5		1		$\bar{1}$		$\bar{8}$		$\bar{2}$		$\bar{2}$						$\bar{1}$		
a^8	5		8		4		$\bar{2}$		$\bar{8}$		$\bar{7}$		$\bar{5}$		$\bar{8}$		1		2				1	
a^9		5		8		2		1		$\bar{4}$		$\bar{2}$		$\bar{1}$		8		8		2		1		$\bar{1}$
a^{10}	8		15		6		5		$\bar{8}$		$\bar{7}$		$\bar{7}$		$\bar{2}$		1		$\bar{1}$		1		$\bar{2}$	
a^{11}		17		11		9		$\bar{2}$		$\bar{10}$		$\bar{16}$		$\bar{6}$		$\bar{8}$		2		4		$\bar{1}$		1
a^{12}	18		14		15		$\bar{2}$		$\bar{11}$		$\bar{24}$		$\bar{14}$		$\bar{8}$		8		8		8		1	
a^{13}		17		17		2		$\bar{12}$		$\bar{27}$		$\bar{21}$		$\bar{6}$		8		11		9		8		$\bar{6}$
a^{14}	15		89		21		6		$\bar{18}$		$\bar{26}$		$\bar{18}$		2		18		10		8		$\bar{7}$	
a^{15}		42		24		10		$\bar{28}$		$\bar{45}$		$\bar{62}$		$\bar{17}$		$\bar{5}$		18		11		$\bar{5}$		$\bar{8}$
a^{16}	44		41		81		$\bar{15}$		$\bar{88}$		$\bar{69}$		$\bar{26}$		8		28		81		18		$\bar{2}$	
a^{17}		44		28		$\bar{14}$		$\bar{52}$		$\bar{78}$		$\bar{63}$		$\bar{9}$		15		84		18		1		$\bar{18}$
a^{18}	43		77		83		$\bar{5}$		$\bar{85}$		$\bar{68}$		$\bar{11}$		28		51		84		20		$\bar{20}$	
a^{19}		79		82		$\bar{6}$		$\bar{82}$		$\bar{118}$		$\bar{108}$		$\bar{20}$		8		86		19		$\bar{17}$		$\bar{15}$
a^{20}	82		76		48		$\bar{89}$		$\bar{70}$		$\bar{109}$		$\bar{22}$		48		80		69		29		$\bar{13}$	
a^{21}		76		87		$\bar{48}$		$\bar{121}$		$\bar{159}$		$\bar{117}$				36		70		29		$\bar{10}$		$\bar{44}$
a^{22}	76		122		41		$\bar{85}$		$\bar{75}$		$\bar{112}$		6		88		118		76		88		$\bar{45}$	
a^{23}		120		87		$\bar{41}$		$\bar{163}$		$\bar{201}$		$\bar{165}$		$\bar{5}$		81		75		88		$\bar{43}$		$\bar{43}$
a^{24}	122		112		87		$\bar{86}$		$\bar{121}$		$\bar{161}$		2		120		160		123		40		$\bar{40}$	
a^{25}		109		81		$\bar{92}$		$\bar{205}$		$\bar{242}$		$\bar{164}$		89		88		120		87		$\bar{40}$		$\bar{82}$
a^{26}	107		151		25		$\bar{82}$		$\bar{116}$		$\bar{147}$		52		166		208		117		39		$\bar{82}$	
a^{27}		148		25		$\bar{85}$		$\bar{239}$		$\bar{267}$		$\bar{190}$		44		79		118		83		$\bar{84}$		$\bar{76}$
a^{28}	147		125		18		$\bar{186}$		$\bar{161}$		$\bar{188}$		50		206		237		158		87		$\bar{74}$	
a^{29}		121		14		$\bar{137}$		$\bar{265}$		$\bar{286}$		$\bar{152}$		107		185		157		85		$\bar{77}$		$\bar{122}$
a^{30}	119		153		$\bar{1}$		$\bar{128}$		$\bar{141}$		$\bar{151}$		111		243		268		187		28		$\bar{124}$	
a^{31}		149		1		$\bar{123}$		$\bar{281}$		$\bar{286}$		$\bar{165}$		108		123		188		27		$\bar{127}$		$\bar{107}$
a^{32}	147		112		$\bar{15}$		$\bar{167}$		$\bar{169}$		$\bar{164}$		109		270		280		166		18		$\bar{108}$	

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{33}		108		$\overline{13}$		$\overline{166}$		$\overline{280}$		$\overline{270}$		$\overline{109}$		164		169		167		15		$\overline{112}$		$\overline{147}$
a^{34}	107		127		$\overline{27}$		$\overline{188}$		$\overline{128}$		$\overline{108}$		165		286		281		123		$\overline{1}$		$\overline{149}$	
a^{35}		124		$\overline{28}$		$\overline{137}$		$\overline{263}$		$\overline{248}$		$\overline{111}$		161		141		123		1		$\overline{163}$		$\overline{119}$
a^{36}	122		77		$\overline{85}$		$\overline{157}$		$\overline{135}$		$\overline{107}$		152		286		265		137		$\overline{14}$		$\overline{121}$	
a^{37}		74		$\overline{87}$		$\overline{158}$		$\overline{237}$		$\overline{206}$		$\overline{50}$		188		161		186		$\overline{13}$		$\overline{125}$		$\overline{147}$
a^{38}	76		84		$\overline{83}$		$\overline{113}$		$\overline{79}$		$\overline{44}$		190		267		239		85		$\overline{25}$		$\overline{148}$	
a^{39}		82		$\overline{89}$		$\overline{117}$		$\overline{203}$		$\overline{166}$		$\overline{52}$		147		116		82		$\overline{25}$		$\overline{151}$		$\overline{107}$
a^{40}	82		40		$\overline{87}$		$\overline{120}$		$\overline{83}$		$\overline{89}$		154		242		205		92		$\overline{81}$		$\overline{109}$	
a^{41}		40		$\overline{40}$		$\overline{123}$		$\overline{160}$		$\overline{120}$		$\overline{2}$		161		121		86		$\overline{87}$		$\overline{112}$		$\overline{122}$
a^{42}	43		43		$\overline{33}$		$\overline{75}$		$\overline{81}$		5		165		201		163		41		$\overline{37}$		$\overline{120}$	
a^{43}		45		$\overline{88}$		$\overline{76}$		$\overline{118}$		$\overline{83}$		$\overline{6}$		112		75		85		$\overline{41}$		$\overline{122}$		$\overline{76}$
a^{44}	44		10		$\overline{29}$		$\overline{70}$		$\overline{36}$				117		169		121		43		$\overline{87}$		$\overline{76}$	
a^{45}		13		$\overline{29}$		$\overline{69}$		$\overline{80}$		$\overline{48}$		22		109		70		89		$\overline{48}$		$\overline{76}$		$\overline{82}$
a^{46}	15		17		$\overline{19}$		$\overline{86}$		$\overline{8}$		20		108		113		82		6		$\overline{32}$		$\overline{79}$	
a^{47}		20		$\overline{20}$		$\overline{84}$		$\overline{61}$		$\overline{28}$		11		68		85		5		$\overline{88}$		$\overline{77}$		$\overline{48}$
a^{48}	18		$\overline{1}$		$\overline{18}$		$\overline{84}$		$\overline{15}$		9		68		78		52		14		$\overline{28}$		$\overline{44}$	
a^{49}		2		$\overline{18}$		$\overline{81}$		$\overline{28}$		$\overline{8}$		26		59		83		15		$\overline{81}$		$\overline{41}$		$\overline{44}$
a^{50}	8		5		$\overline{11}$		$\overline{13}$		5		17		52		45		28		$\overline{10}$		$\overline{24}$		$\overline{42}$	
a^{51}		7		$\overline{8}$		$\overline{10}$		13		$\overline{2}$		18		26		18		$\overline{6}$		21		$\overline{39}$		$\overline{15}$
a^{52}	5		$\overline{8}$		$\overline{9}$		$\overline{11}$		$\overline{8}$		6		21		27		12		$\overline{2}$		$\overline{17}$		$\overline{17}$	
a^{53}		$\overline{1}$		$\overline{8}$		$\overline{8}$		$\overline{3}$		8		14		24		11		2		$\overline{15}$		$\overline{14}$		$\overline{18}$
a^{54}	$\overline{1}$		1		$\overline{4}$		$\overline{2}$		8		5		16		10		2		$\overline{9}$		$\overline{11}$		$\overline{17}$	
a^{55}		2		$\overline{1}$		1		$\overline{1}$		2		7		7		8		$\overline{5}$		$\overline{5}$		$\overline{15}$		8
a^{56}	1		$\overline{1}$		$\overline{2}$		$\overline{8}$		$\overline{8}$		1		2		4		$\overline{1}$		$\overline{2}$		$\overline{8}$		$\overline{5}$	
a^{57}		$\overline{1}$				$\overline{2}$		$\overline{1}$		8		5		7		8		2		$\overline{4}$		$\overline{8}$		$\overline{5}$
a^{58}			1						2		2		8		1		$\overline{1}$		$\overline{5}$		$\overline{5}$		$\overline{5}$	
a^{59}				$\overline{1}$		$\overline{1}$		$\overline{1}$										$\overline{8}$		$\overline{1}$		$\overline{4}$		1
a^{60}	1						1		1		2		1		2				$\overline{1}$		$\overline{2}$			
a^{61}		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				$\overline{2}$		$\overline{1}$		$\overline{2}$				$\overline{1}$
a^{62}					1		1		2		2		2		1		1							
a^{63}										$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		
a^{64}																	1		1		1		1	
a^{65}																								$\overline{1}$

G. F. for covariants, representative form,

Denominator : $(1-a^4)(1-a^8)(1-a^{10})(1-a^{12})^2(1-a^{14})(1-a^{16})(1-a^2x^6)$
$(1-a^2x^{10})(1-a^2x^{14})(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^0	1																			
a^3				1		1		1		2		1		1		1		1		
a^4	1				2		2		3		2		2		2		1		1	
a^5		1		3		4		4		3		4		2		2				
a^6			4		4		7		7		5		6		1		2			
a^7		4		8		9		10		11		7		6		2				
a^8	5		8		13		16		16		14		7		6		1		$\overline{1}$	
a^9		10		17		20		22		19		15		7		1		$\overline{3}$		$\overline{7}$
a^{10}	4		20		25		30		33		20		13		2		$\overline{3}$		$\overline{10}$	
a^{11}		21		32		41		43		40		20		11		$\overline{4}$		$\overline{14}$		$\overline{13}$
a^{12}	17		35		50		60		57		37		16				$\overline{18}$		$\overline{25}$	
a^{13}		39		57		75		71		57		28		6		$\overline{29}$		$\overline{34}$		$\overline{41}$
a^{14}	20		64		86		90		92		44		13		$\overline{31}$		$\overline{46}$		$\overline{59}$	
a^{15}		67		94		121		108		96		23		$\overline{11}$		$\overline{63}$		$\overline{73}$		$\overline{79}$
a^{16}	47		103		135		143		135		57		$\overline{7}$		$\overline{65}$		$\overline{91}$		$\overline{117}$	
a^{17}		108		142		181		154		116		3		$\overline{45}$		$\overline{139}$		$\overline{186}$		$\overline{148}$
a^{18}	61		152		195		191		181		37		$\overline{43}$		$\overline{149}$		$\overline{176}$		$\overline{198}$	
a^{19}		157		201		257		199		149		$\overline{38}$		$\overline{104}$		$\overline{239}$		$\overline{221}$		$\overline{222}$
a^{20}	97		211		270		260		225		21		$\overline{107}$		$\overline{252}$		$\overline{271}$		$\overline{302}$	
a^{21}		215		273		339		239		157		$\overline{108}$		$\overline{200}$		$\overline{391}$		$\overline{330}$		$\overline{338}$
a^{22}	120		281		348		308		262		$\overline{42}$		$\overline{206}$		$\overline{412}$		$\overline{410}$		$\overline{434}$	
a^{23}		284		348		418		269		159		$\overline{215}$		$\overline{327}$		$\overline{562}$		$\overline{462}$		$\overline{440}$

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{33}	x^{39}	
																				a^0
	1																			a^3
		1																		a^4
			$\overline{2}$				$\overline{1}$				$\overline{1}$									a^5
		$\overline{1}$		$\overline{1}$				$\overline{1}$												a^6
	$\overline{3}$				$\overline{1}$								1				1			a^7
$\overline{3}$		$\overline{4}$		$\overline{1}$		$\overline{2}$				1										a^8
	$\overline{4}$		$\overline{6}$		$\overline{3}$		$\overline{1}$				1				1				$\overline{1}$	a^9
$\overline{11}$		$\overline{9}$		$\overline{7}$		$\overline{2}$						1		1						a^{10}
	$\overline{16}$		$\overline{11}$		$\overline{6}$				2		2		3							a^{11}
$\overline{23}$		$\overline{24}$		$\overline{9}$		$\overline{4}$		1		3		5		2						a^{12}
	$\overline{36}$		$\overline{29}$		$\overline{9}$				4		7		2		2				$\overline{1}$	a^{13}
$\overline{55}$		$\overline{46}$		$\overline{20}$		$\overline{4}$		7		9		11		4		1		$\overline{1}$		a^{14}
	$\overline{65}$		$\overline{40}$		$\overline{9}$		8		20		15		12		4					a^{15}
$\overline{89}$		$\overline{78}$		$\overline{20}$				27		24		23		9		1		$\overline{4}$		a^{16}
	$\overline{102}$		$\overline{74}$		$\overline{5}$		25		38		30		17		7		$\overline{4}$		$\overline{5}$	a^{17}
$\overline{147}$		$\overline{121}$		$\overline{23}$		19		57		41		45		13				$\overline{10}$		a^{18}
	$\overline{150}$		$\overline{87}$		25		57		83		55		39		6		$\overline{8}$		$\overline{4}$	a^{19}
$\overline{202}$		$\overline{164}$		9		50		112		83		74		16		$\overline{3}$		$\overline{21}$		a^{20}
	$\overline{194}$		$\overline{113}$		63		109		137		86		48		6		$\overline{19}$		$\overline{17}$	a^{21}
$\overline{276}$		$\overline{202}$		43		107		194		121		112		16		$\overline{11}$		$\overline{39}$		a^{22}
	$\overline{230}$		$\overline{102}$		149		194		232		126		81		$\overline{2}$		$\overline{34}$		$\overline{20}$	a^{23}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{24}	165		353		419		366		278		$\overline{122}$		$\overline{338}$		$\overline{586}$		$\overline{555}$		$\overline{569}$	
a^{25}		353		417		490		275		115		$\overline{356}$		$\overline{481}$		$\overline{777}$		$\overline{593}$		$\overline{551}$
a^{26}	189		415		484		386		269		$\overline{247}$		$\overline{496}$		$\overline{800}$		$\overline{716}$		$\overline{692}$	
a^{27}		413		478		544		254		68		$\overline{519}$		$\overline{652}$		$\overline{976}$		$\overline{708}$		$\overline{622}$
a^{28}	223		471		529		403		235		$\overline{374}$		$\overline{669}$		$\overline{996}$		$\overline{839}$		$\overline{794}$	
a^{29}		464		521		570		211		$\overline{22}$		$\overline{694}$		$\overline{821}$		$\overline{1181}$		$\overline{795}$		$\overline{671}$
a^{30}	241		506		551		375		171		$\overline{530}$		$\overline{840}$		$\overline{1186}$		$\overline{959}$		$\overline{844}$	
a^{31}		499		538		568		139		$\overline{120}$		$\overline{859}$		$\overline{978}$		$\overline{1326}$		$\overline{832}$		$\overline{649}$
a^{32}	254		521		541		332		87		$\overline{669}$		$\overline{988}$		$\overline{1327}$		$\overline{998}$		$\overline{839}$	
a^{33}		510		529		534		49		$\overline{224}$		$\overline{1007}$		$\overline{1088}$		$\overline{1420}$		$\overline{809}$		$\overline{584}$
a^{34}	254		508		508		260		$\overline{5}$		$\overline{792}$		$\overline{1098}$		$\overline{1401}$		$\overline{991}$		$\overline{773}$	
a^{35}		499		492		474		$\overline{42}$		$\overline{322}$		$\overline{1104}$		$\overline{1144}$		$\overline{1432}$		$\overline{729}$		$\overline{459}$
a^{36}	241		475		449		183		$\overline{101}$		$\overline{877}$		$\overline{1143}$		$\overline{1406}$		$\overline{915}$		$\overline{650}$	
a^{37}		464		435		399		$\overline{132}$		$\overline{398}$		$\overline{1144}$		$\overline{1137}$		$\overline{1376}$		$\overline{593}$		$\overline{297}$
a^{38}	223		419		380		97		$\overline{184}$		$\overline{905}$		$\overline{1133}$		$\overline{1335}$		$\overline{788}$		$\overline{483}$	
a^{39}		413		367		311		$\overline{205}$		$\overline{446}$		$\overline{1122}$		$\overline{1076}$		$\overline{1240}$		$\overline{423}$		$\overline{128}$
a^{40}	189		357		297		16		$\overline{240}$		$\overline{891}$		$\overline{1062}$		$\overline{1203}$		$\overline{619}$		$\overline{306}$	
a^{41}		353		288		222		$\overline{251}$		$\overline{456}$		$\overline{1049}$		$\overline{956}$		$\overline{1051}$		$\overline{250}$		47
a^{42}	165		284		217		$\overline{40}$		$\overline{272}$		$\overline{825}$		$\overline{940}$		$\overline{1011}$		$\overline{441}$		$\overline{121}$	
a^{43}		284		210		147		$\overline{274}$		$\overline{446}$		$\overline{923}$		$\overline{801}$		$\overline{844}$		$\overline{80}$		191
a^{44}	120		213		147		$\overline{88}$		$\overline{278}$		$\overline{728}$		$\overline{780}$		$\overline{818}$		$\overline{264}$		34	
a^{45}		215		146		85		$\overline{270}$		$\overline{386}$		$\overline{769}$		$\overline{630}$		$\overline{619}$		65		297
a^{46}	97		152		91		$\overline{101}$		$\overline{256}$		$\overline{588}$		$\overline{615}$		$\overline{599}$		$\overline{107}$		145	
a^{47}		157		94		36		$\overline{242}$		$\overline{333}$		$\overline{604}$		$\overline{465}$		$\overline{427}$		158		338
a^{48}	61		102		46		$\overline{112}$		$\overline{219}$		$\overline{468}$		$\overline{452}$		$\overline{422}$		$\overline{7}$		215	
a^{49}		108		52		5		$\overline{203}$		$\overline{255}$		$\overline{446}$		$\overline{317}$		$\overline{253}$		209		359
a^{50}	47		62		17		$\overline{91}$		$\overline{175}$		$\overline{333}$		$\overline{309}$		$\overline{258}$		76		243	
a^{51}		67		25		$\overline{10}$		$\overline{158}$		$\overline{192}$		$\overline{307}$		$\overline{196}$		$\overline{136}$		224		321

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
$\overline{321}$		$\overline{224}$		136		196		307		192		158		10		$\overline{25}$		$\overline{67}$		a^{24}
	$\overline{243}$		$\overline{76}$		258		309		333		175		91		$\overline{17}$		$\overline{62}$		$\overline{47}$	a^{25}
$\overline{359}$		$\overline{209}$		253		317		446		255		203		$\overline{5}$		$\overline{52}$		$\overline{108}$		a^{26}
	$\overline{215}$		7		422		452		468		219		112		$\overline{46}$		$\overline{102}$		$\overline{61}$	a^{27}
$\overline{338}$		$\overline{158}$		427		465		604		333		242		$\overline{36}$		$\overline{94}$		$\overline{157}$		a^{28}
	$\overline{145}$		107		599		615		588		256		101		$\overline{91}$		$\overline{152}$		$\overline{97}$	a^{29}
$\overline{297}$		$\overline{65}$		619		630		769		386		270		$\overline{85}$		$\overline{146}$		$\overline{215}$		a^{30}
	$\overline{34}$		264		818		780		728		278		88		$\overline{147}$		$\overline{213}$		$\overline{120}$	a^{31}
$\overline{191}$		80		844		801		923		446		274		$\overline{147}$		$\overline{210}$		$\overline{284}$		a^{32}
	121		441		1011		940		825		272		40		$\overline{217}$		$\overline{284}$		$\overline{165}$	a^{33}
$\overline{47}$		250		1051		956		1049		456		251		$\overline{222}$		$\overline{288}$		$\overline{353}$		a^{34}
	306		619		1203		1062		891		240		$\overline{16}$		$\overline{297}$		$\overline{357}$		$\overline{189}$	a^{35}
128		423		1240		1076		1122		446		205		$\overline{311}$		$\overline{367}$		$\overline{413}$		a^{36}
	483		788		1335		1133		905		184		$\overline{97}$		$\overline{380}$		$\overline{419}$		$\overline{223}$	a^{37}
297		593		1376		1137		1144		398		132		$\overline{399}$		$\overline{435}$		$\overline{464}$		a^{38}
	650		915		1406		1143		877		101		$\overline{183}$		$\overline{449}$		$\overline{475}$		$\overline{241}$	a^{39}
459		729		1432		1144		1104		322		42		$\overline{474}$		$\overline{492}$		$\overline{499}$		a^{40}
	773		991		1401		1098		792		5		$\overline{260}$		$\overline{508}$		$\overline{508}$		$\overline{254}$	a^{41}
584		809		1420		1088		1007		224		$\overline{49}$		$\overline{534}$		$\overline{529}$		$\overline{510}$		a^{42}
	839		998		1327		988		669		$\overline{87}$		$\overline{332}$		$\overline{541}$		$\overline{521}$		$\overline{254}$	a^{43}
649		832		1326		978		859		120		$\overline{139}$		$\overline{568}$		$\overline{538}$		$\overline{499}$		a^{44}
	844		959		1186		840		530		$\overline{171}$		$\overline{375}$		$\overline{551}$		$\overline{506}$		$\overline{241}$	a^{45}
671		795		1181		821		694		22		$\overline{211}$		$\overline{570}$		$\overline{521}$		$\overline{464}$		a^{46}
	794		839		996		669		374		$\overline{235}$		$\overline{403}$		$\overline{529}$		$\overline{471}$		$\overline{223}$	a^{47}
622		708		976		652		519		$\overline{68}$		$\overline{254}$		$\overline{544}$		$\overline{478}$		$\overline{413}$		a^{48}
	692		716		800		496		247		$\overline{269}$		$\overline{386}$		$\overline{484}$		$\overline{415}$		$\overline{189}$	a^{49}
551		593		777		481		356		$\overline{115}$		$\overline{275}$		$\overline{490}$		$\overline{417}$		$\overline{353}$		a^{50}
	569		555		586		338		122		$\overline{278}$		$\overline{366}$		$\overline{419}$		$\overline{353}$		$\overline{165}$	a^{51}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{52}	20		34		2		$\overline{81}$		$\overline{126}$		$\overline{232}$		$\overline{194}$		$\overline{149}$		102		230	
a^{53}		39		11		$\overline{16}$		$\overline{112}$		$\overline{121}$		$\overline{194}$		$\overline{107}$		$\overline{43}$		202		276
a^{54}	17		19		$\overline{6}$		$\overline{48}$		$\overline{86}$		$\overline{137}$		$\overline{109}$		$\overline{63}$		113		194	
a^{55}		21		3		$\overline{16}$		$\overline{74}$		$\overline{83}$		$\overline{112}$		$\overline{50}$		$\overline{9}$		164		202
a^{56}	4		8		$\overline{6}$		$\overline{39}$		$\overline{55}$		$\overline{83}$		$\overline{57}$		$\overline{25}$		87		150	
a^{57}		10				$\overline{13}$		$\overline{45}$		$\overline{41}$		$\overline{57}$		$\overline{19}$		23		121		147
a^{58}	5		4		$\overline{7}$		$\overline{17}$		$\overline{30}$		$\overline{38}$		$\overline{25}$		5		74		102	
a^{59}		4		$\overline{1}$		$\overline{9}$		$\overline{23}$		$\overline{24}$		$\overline{27}$				20		78		89
a^{60}					$\overline{4}$		$\overline{12}$		$\overline{15}$		$\overline{20}$		$\overline{8}$		9		40		65	
a^{61}		1		$\overline{1}$		$\overline{4}$		$\overline{11}$		$\overline{9}$		$\overline{7}$		4		20		46		55
a^{62}	1				$\overline{2}$		$\overline{2}$		$\overline{7}$		$\overline{4}$				9		29		36	
a^{63}						$\overline{2}$		$\overline{5}$		$\overline{3}$		$\overline{1}$		4		9		24		23
a^{64}							$\overline{3}$		$\overline{2}$		$\overline{2}$				6		11		16	
a^{65}						$\overline{1}$		$\overline{1}$						2		7		9		11
a^{66}	1				$\overline{1}$				$\overline{1}$				1		3		6		4	
a^{67}										$\overline{1}$				2		1		4		3
a^{68}			$\overline{1}$				$\overline{1}$								1				3	
a^{69}												1				1		1		
a^{70}									1				1				2			
a^{71}																		$\overline{1}$		
a^{72}																			$\overline{1}$	
a^{75}																				

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
440		462		562		327		215		$\overline{159}$		$\overline{269}$		$\overline{418}$		$\overline{348}$		$\overline{284}$		a^{52}
	434		410		412		206		42		$\overline{262}$		$\overline{308}$		$\overline{348}$		$\overline{281}$		$\overline{120}$	a^{53}
338		330		391		200		108		$\overline{157}$		$\overline{239}$		$\overline{339}$		$\overline{273}$		$\overline{215}$		a^{54}
	302		271		252		107		$\overline{21}$		$\overline{225}$		$\overline{260}$		$\overline{270}$		$\overline{211}$		$\overline{97}$	a^{55}
222		221		239		104		38		$\overline{149}$		$\overline{199}$		$\overline{257}$		$\overline{201}$		$\overline{157}$		a^{56}
	198		176		149		43		$\overline{37}$		$\overline{181}$		$\overline{191}$		$\overline{195}$		$\overline{152}$		$\overline{61}$	a^{57}
148		136		139		45		$\overline{3}$		$\overline{116}$		$\overline{154}$		$\overline{181}$		$\overline{142}$		$\overline{108}$		a^{58}
	117		91		65		7		$\overline{57}$		$\overline{135}$		$\overline{143}$		$\overline{135}$		$\overline{103}$		$\overline{47}$	a^{59}
79		73		63		11		$\overline{23}$		$\overline{96}$		$\overline{108}$		$\overline{121}$		$\overline{94}$		$\overline{67}$		a^{60}
	59		46		31		$\overline{13}$		$\overline{44}$		$\overline{92}$		$\overline{90}$		$\overline{86}$		$\overline{64}$		$\overline{20}$	a^{61}
41		34		29		$\overline{6}$		$\overline{28}$		$\overline{57}$		$\overline{71}$		$\overline{75}$		$\overline{57}$		$\overline{39}$		a^{62}
	25		18				$\overline{16}$		$\overline{37}$		$\overline{57}$		$\overline{60}$		$\overline{50}$		$\overline{35}$		$\overline{17}$	a^{63}
13		14		4		$\overline{11}$		$\overline{20}$		$\overline{40}$		$\overline{43}$		$\overline{41}$		$\overline{32}$		$\overline{21}$		a^{64}
	10		3		$\overline{2}$		$\overline{13}$		$\overline{20}$		$\overline{33}$		$\overline{30}$		$\overline{25}$		$\overline{20}$		$\overline{4}$	a^{65}
7		3		$\overline{1}$		$\overline{7}$		$\overline{15}$		$\overline{19}$		$\overline{22}$		$\overline{20}$		$\overline{17}$		$\overline{10}$		a^{66}
	1		$\overline{1}$		$\overline{6}$		$\overline{7}$		$\overline{14}$		$\overline{16}$		$\overline{16}$		$\overline{13}$		$\overline{8}$		$\overline{5}$	a^{67}
				$\overline{2}$		$\overline{6}$		$\overline{7}$		$\overline{11}$		$\overline{10}$		$\overline{9}$		$\overline{8}$		$\overline{4}$		a^{68}
			$\overline{2}$		$\overline{1}$		$\overline{6}$		$\overline{5}$		$\overline{7}$		$\overline{7}$		$\overline{4}$		$\overline{4}$			a^{69}
				$\overline{2}$		$\overline{2}$		$\overline{4}$		$\overline{3}$		$\overline{4}$		$\overline{4}$		$\overline{3}$		$\overline{1}$		a^{70}
	$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{3}$		$\overline{2}$		$\overline{2}$				$\overline{1}$	a^{71}
		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				a^{72}
																			$\overline{1}$	a^{75}

Table of Groundforms.

Degree in the Coefficients	Order in the Variables: 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		3		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6			4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

Decimic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-a^8)(1-a^9)$.

Numerator: $1+3a^2+11a^3+27a^4+58a^5+112a^6+193a^7+318a^8+485a^9$
$+699a^{10}+951a^{11}+1245a^{12}+1541a^{13}+1842a^{14}+2108a^{15}$
$+2321a^{16}+2451a^{17}+2506a^{18}+2451a^{19}+2321a^{20}+2108a^{21}$
$+1842a^{22}+1541a^{23}+1245a^{24}+951a^{25}+699a^{26}+485a^{27}$
$+318a^{28}+193a^{29}+112a^{30}+58a^{31}+27a^{32}+11a^{33}+3a^{34}$
$+a^{36}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)$
$(1-a^9)(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^0	1														
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$										
a^2	$\bar{1}$		1	1	2	2	2	1	1						
a^3	$\bar{1}$	2	1	2	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$			
a^4		1	2			$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	1	1	1	1	
a^5		2	2			$\bar{1}$	$\bar{2}$		1	1	1	1			$\bar{1}$
a^6	3	1	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$			1		1	$\bar{1}$	$\bar{1}$	
a^7		1		1	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	1
a^8	2	3	4	2	1	2	$\bar{2}$	$\bar{1}$		2				$\bar{1}$	1
a^9	2	5		1	$\bar{2}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{3}$	$\bar{2}$					
a^{10}	4	3	3		$\bar{4}$	$\bar{6}$	$\bar{6}$	$\bar{3}$		4	5	4	2	1	
a^{11}		4	2	$\bar{3}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{4}$	2	2	5	1	1	$\bar{1}$	$\bar{3}$
a^{12}	6	5	4	1	$\bar{2}$	$\bar{5}$	$\bar{7}$	$\bar{2}$	2	5	4	3	1	$\bar{2}$	
a^{13}	$\bar{1}$	3	$\bar{1}$	$\bar{5}$	$\bar{11}$	$\bar{17}$	$\bar{12}$	$\bar{9}$		2	6	3	1	$\bar{1}$	$\bar{2}$
a^{14}	1	4	7	$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	5	10	14	11	7	4	$\bar{3}$	$\bar{2}$
a^{15}		5	1	$\bar{4}$	$\bar{9}$	$\bar{17}$	$\bar{12}$	$\bar{6}$	3	3	5	1	$\bar{4}$	$\bar{5}$	$\bar{4}$
a^{16}	3	1	2	$\bar{5}$	$\bar{11}$	$\bar{11}$	$\bar{6}$	3	10	17	13	8		$\bar{2}$	
a^{17}	$\bar{4}$	1	$\bar{3}$	$\bar{9}$	$\bar{10}$	$\bar{10}$	$\bar{2}$	4	15	13	8	1	$\bar{4}$	$\bar{6}$	$\bar{6}$
a^{18}	4		$\bar{1}$	$\bar{1}$	$\bar{1}$	2	3	13	13	14	4	1	$\bar{6}$	$\bar{8}$	1
a^{19}	$\bar{3}$	$\bar{5}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{7}$	1	2	4	4	1	$\bar{3}$	$\bar{9}$	$\bar{3}$	$\bar{1}$
a^{20}		$\bar{3}$	$\bar{1}$		4	14	13	16	13	14	4		$\bar{1}$	$\bar{3}$	

* Numerator and denominator divisible by $1-a$; see foot-note to reduced form for sextic.

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^{21}	$\overline{1}$	$\overline{3}$	$\overline{9}$	$\overline{3}$	1	4	4	2	1	$\overline{7}$	$\overline{8}$	$\overline{8}$	$\overline{8}$	$\overline{5}$	$\overline{3}$
a^{22}	1	$\overline{8}$	$\overline{6}$	1	4	14	13	13	3	2	$\overline{1}$	$\overline{1}$	$\overline{1}$		4
a^{23}	$\overline{6}$	$\overline{6}$	$\overline{4}$	1	8	13	15	4	$\overline{2}$	$\overline{10}$	$\overline{10}$	$\overline{9}$	$\overline{3}$	1	$\overline{4}$
a^{24}		$\overline{2}$		8	13	17	10	3	$\overline{6}$	$\overline{11}$	$\overline{11}$	$\overline{5}$	2	1	3
a^{25}	$\overline{4}$	$\overline{5}$	$\overline{4}$	1	5	3	3	$\overline{6}$	$\overline{12}$	$\overline{17}$	$\overline{9}$	$\overline{4}$	1	5	
a^{26}	$\overline{2}$	$\overline{3}$	4	7	11	14	10	5	$\overline{5}$	$\overline{6}$	$\overline{3}$	$\overline{1}$	7	4	1
a^{27}	$\overline{2}$	$\overline{1}$	1	3	6	2		$\overline{9}$	$\overline{12}$	$\overline{17}$	$\overline{11}$	$\overline{5}$	$\overline{1}$	3	$\overline{1}$
a^{28}		$\overline{2}$	1	3	4	5	2	$\overline{2}$	$\overline{7}$	$\overline{5}$	$\overline{2}$	1	4	5	6
a^{29}	$\overline{3}$	$\overline{1}$	1	1	5	2	2	$\overline{4}$	$\overline{7}$	$\overline{7}$	$\overline{6}$	$\overline{3}$	2	4	
a^{30}		1	2	4	5	4		$\overline{3}$	$\overline{6}$	$\overline{6}$	$\overline{4}$		3	3	4
a^{31}						$\overline{2}$	$\overline{3}$	$\overline{7}$	$\overline{7}$	$\overline{6}$	$\overline{2}$	1		5	2
a^{32}	1	$\overline{1}$				2		$\overline{1}$	$\overline{2}$	2	1	2	4	3	2
a^{33}	1	$\overline{1}$	$\overline{1}$		1	1		$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	1		1	
a^{34}		$\overline{1}$	$\overline{1}$	1		1			$\overline{2}$	$\overline{2}$	$\overline{1}$	1	1	1	3
a^{35}	$\overline{1}$			1	1	1	1		$\overline{2}$	$\overline{1}$			2	2	
a^{36}		1	1	1	1	1	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{2}$			2	1	
a^{37}				$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	1	2	1	2	$\overline{1}$
a^{38}							1	1	2	2	2	1	1		$\overline{1}$
a^{39}											$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	
a^{40}															1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)^2(1-a^8)(1-a^9)(1-a^{10})(1-a^{14})$
$(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-ax^{10})$.

Numerator:

x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}
1																								
	1		2	1	1	2	1	1	1	1		1												
		3	1	3	3	2	3	1	2	1	1		1											
	3	3	4	5	4	5	2	4		1		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$						
2	2	6	8	8	9	6	7	2	4				$\overline{2}$		$\overline{1}$		$\overline{1}$							
	7	10	11	13	11	11	7	6	1			$\overline{4}$		$\overline{2}$				1		1		1		
4	8	14	18	20	22	12	11	4	2	$\overline{2}$	$\overline{4}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{3}$				1					
4	15	21	27	30	24	23	12	7	$\overline{1}$	$\overline{8}$	$\overline{6}$	$\overline{9}$	$\overline{5}$	$\overline{5}$	$\overline{1}$	$\overline{1}$		2		1				$\overline{1}$
7	20	31	37	39	39	22	15	2	$\overline{8}$	$\overline{11}$	$\overline{18}$	$\overline{14}$	$\overline{15}$	$\overline{5}$	$\overline{4}$		2	1	2		1			
8	28	41	50	56	46	31	12	$\overline{2}$	$\overline{17}$	$\overline{28}$	$\overline{25}$	$\overline{26}$	$\overline{18}$	$\overline{13}$	$\overline{5}$	1	2	5	1	2		$\overline{1}$		
15	38	54	67	69	60	33	11	$\overline{12}$	$\overline{33}$	$\overline{41}$	$\overline{45}$	$\overline{36}$	$\overline{31}$	$\overline{12}$	$\overline{3}$	2	7	6	7	2	1		$\overline{1}$	
15	49	72	84	90	70	37	3	$\overline{26}$	$\overline{54}$	$\overline{66}$	$\overline{62}$	$\overline{56}$	$\overline{39}$	$\overline{21}$	$\overline{2}$	8	12	14	7	4	1	$\overline{1}$		
20	61	87	104	106	82	32	$\overline{9}$	$\overline{48}$	$\overline{86}$	$\overline{95}$	$\overline{93}$	$\overline{73}$	$\overline{55}$	$\overline{20}$	2	14	20	18	16	5	2	$\overline{3}$	$\overline{3}$	
27	75	108	127	128	92	32	$\overline{26}$	$\overline{76}$	$\overline{120}$	$\overline{134}$	$\overline{119}$	$\overline{100}$	$\overline{66}$	$\overline{25}$	11	27	32	32	18	9		$\overline{2}$	$\overline{2}$	$\overline{2}$
29	90	129	147	146	100	22	$\overline{49}$	$\overline{110}$	$\overline{165}$	$\overline{172}$	$\overline{157}$	$\overline{120}$	$\overline{77}$	$\overline{18}$	26	41	52	44	32	9	1	$\overline{6}$	$\overline{7}$	
35	105	148	168	164	103	5	$\overline{81}$	$\overline{153}$	$\overline{218}$	$\overline{227}$	$\overline{195}$	$\overline{150}$	$\overline{88}$	$\overline{15}$	44	67	70	63	37	14		$\overline{9}$	$\overline{8}$	$\overline{4}$
40	119	168	191	179	105	$\overline{11}$	$\overline{115}$	$\overline{201}$	$\overline{272}$	$\overline{274}$	$\overline{232}$	$\overline{169}$	$\overline{94}$	1	71	93	101	82	51	13	$\overline{6}$	$\overline{13}$	$\overline{15}$	$\overline{4}$
44	132	189	204	192	101	$\overline{36}$	$\overline{154}$	$\overline{254}$	$\overline{330}$	$\overline{330}$	$\overline{267}$	$\overline{190}$	$\overline{88}$	24	108	132	133	112	62	17	$\overline{7}$	$\overline{20}$	$\overline{20}$	$\overline{7}$
47	147	202	221	200	94	$\overline{64}$	$\overline{202}$	$\overline{305}$	$\overline{395}$	$\overline{379}$	$\overline{303}$	$\overline{203}$	$\overline{85}$	48	150	172	171	133	74	14	$\overline{16}$	$\overline{30}$	$\overline{28}$	$\overline{8}$
55	154	216	232	203	83	$\overline{98}$	$\overline{241}$	$\overline{365}$	$\overline{447}$	$\overline{431}$	$\overline{327}$	$\overline{208}$	$\overline{70}$	92	196	222	208	166	85	11	$\overline{26}$	$\overline{41}$	$\overline{38}$	$\overline{15}$
52	164	226	236	202	63	$\overline{127}$	$\overline{292}$	$\overline{413}$	$\overline{506}$	$\overline{470}$	$\overline{346}$	$\overline{210}$	$\overline{42}$	130	257	272	255	194	93	7	$\overline{37}$	$\overline{53}$	$\overline{49}$	$\overline{15}$
57	166	229	237	194	50	$\overline{168}$	$\overline{333}$	$\overline{465}$	$\overline{550}$	$\overline{502}$	$\overline{359}$	$\overline{193}$	$\overline{17}$	186	310	327	296	220	103	$\overline{4}$	$\overline{52}$	$\overline{73}$	$\overline{61}$	$\overline{20}$
56	172	228	236	187	22	$\overline{191}$	$\overline{372}$	$\overline{499}$	$\overline{585}$	$\overline{527}$	$\overline{353}$	$\overline{176}$	28	238	375	380	336	247	104	$\overline{13}$	$\overline{71}$	$\overline{88}$	$\overline{75}$	$\overline{27}$
57	166	227	225	168	7	$\overline{229}$	$\overline{401}$	$\overline{536}$	$\overline{610}$	$\overline{529}$	$\overline{347}$	$\overline{143}$	66	298	433	430	376	266	105	$\overline{31}$	$\overline{89}$	$\overline{109}$	$\overline{90}$	$\overline{29}$
52	164	217	211	155	$\overline{24}$	$\overline{249}$	$\overline{431}$	$\overline{551}$	$\overline{624}$	$\overline{536}$	$\overline{323}$	$\overline{114}$	119	346	487	474	403	281	98	$\overline{46}$	$\overline{111}$	$\overline{131}$	$\overline{105}$	$\overline{35}$
55	154	203	198	130	$\overline{38}$	$\overline{273}$	$\overline{442}$	$\overline{562}$	$\overline{620}$	$\overline{512}$	$\overline{296}$	$\overline{65}$	160	407	537	512	430	286	93	$\overline{68}$	$\overline{134}$	$\overline{150}$	$\overline{119}$	$\overline{40}$
47	147	190	176	112	$\overline{64}$	$\overline{281}$	$\overline{448}$	$\overline{556}$	$\overline{603}$	$\overline{490}$	$\overline{252}$	$\overline{26}$	216	448	578	541	443	296	76	$\overline{87}$	$\overline{155}$	$\overline{169}$	$\overline{132}$	$\overline{44}$

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x
a^{29}	44	132	169	155	87	$\overline{76}$	$\overline{296}$	$\overline{443}$	541	$\overline{578}$	$\overline{448}$	$\overline{216}$	26	252	490	603	556	448	281	64	$\overline{112}$	$\overline{176}$	$\overline{190}$	$\overline{147}$	
a^{30}	40	119	150	134	68	$\overline{93}$	$\overline{286}$	$\overline{430}$	512	$\overline{537}$	$\overline{407}$	$\overline{160}$	65	296	512	620	562	442	273	38	$\overline{130}$	$\overline{198}$	$\overline{203}$	$\overline{154}$	
a^{31}	35	105	131	111	46	$\overline{98}$	$\overline{281}$	$\overline{403}$	$\overline{474}$	$\overline{487}$	$\overline{346}$	$\overline{119}$	114	323	536	624	551	431	249	24	$\overline{155}$	$\overline{211}$	$\overline{217}$	$\overline{164}$	
a^{32}	29	90	109	89	31	$\overline{105}$	$\overline{266}$	$\overline{376}$	$\overline{430}$	$\overline{433}$	$\overline{298}$	$\overline{66}$	143	347	529	610	536	401	229	$\overline{7}$	$\overline{168}$	$\overline{225}$	$\overline{227}$	$\overline{166}$	
a^{33}	27	75	88	71	13	$\overline{104}$	$\overline{247}$	$\overline{336}$	$\overline{380}$	$\overline{375}$	$\overline{238}$	$\overline{28}$	176	353	527	585	499	372	191	$\overline{22}$	$\overline{187}$	$\overline{236}$	$\overline{228}$	$\overline{172}$	
a^{34}	20	61	73	52	4	$\overline{103}$	$\overline{220}$	$\overline{296}$	$\overline{327}$	$\overline{310}$	$\overline{186}$	17	193	359	502	550	465	333	168	$\overline{50}$	$\overline{194}$	$\overline{237}$	$\overline{229}$	$\overline{166}$	
a^{35}	15	49	53	37	$\overline{7}$	$\overline{93}$	$\overline{194}$	$\overline{255}$	$\overline{272}$	$\overline{257}$	$\overline{130}$	42	210	346	470	506	413	292	127	$\overline{63}$	$\overline{202}$	$\overline{236}$	$\overline{226}$	$\overline{164}$	
a^{36}	15	38	41	26	$\overline{11}$	$\overline{85}$	$\overline{166}$	$\overline{208}$	$\overline{222}$	$\overline{196}$	$\overline{92}$	70	208	327	431	447	365	241	98	$\overline{83}$	$\overline{203}$	$\overline{232}$	$\overline{216}$	$\overline{154}$	
a^{37}	8	28	30	16	$\overline{14}$	$\overline{74}$	$\overline{133}$	$\overline{171}$	$\overline{172}$	$\overline{150}$	$\overline{48}$	85	203	303	379	395	305	202	64	$\overline{94}$	$\overline{200}$	$\overline{221}$	$\overline{202}$	$\overline{147}$	
a^{38}	7	20	20	7	$\overline{17}$	$\overline{62}$	$\overline{112}$	$\overline{133}$	$\overline{132}$	$\overline{108}$	$\overline{24}$	88	190	267	330	330	254	154	36	$\overline{101}$	$\overline{192}$	$\overline{204}$	$\overline{189}$	$\overline{132}$	
a^{39}	4	15	13	6	$\overline{13}$	$\overline{51}$	$\overline{82}$	101	$\overline{93}$	$\overline{71}$	$\overline{1}$	94	169	232	274	272	201	115	11	$\overline{105}$	$\overline{179}$	$\overline{191}$	$\overline{168}$	$\overline{119}$	
a^{40}	4	8	9		$\overline{14}$	$\overline{37}$	$\overline{63}$	$\overline{70}$	$\overline{67}$	$\overline{44}$	15	88	150	195	227	218	153	81	$\overline{5}$	$\overline{103}$	$\overline{164}$	$\overline{168}$	$\overline{148}$	$\overline{105}$	
a^{41}		7	6	$\overline{1}$	$\overline{9}$	$\overline{32}$	$\overline{44}$	$\overline{52}$	$\overline{41}$	$\overline{26}$	18	77	120	157	172	165	110	49	$\overline{22}$	$\overline{100}$	$\overline{146}$	$\overline{147}$	$\overline{129}$	$\overline{90}$	
a^{42}	2	2	2		$\overline{9}$	$\overline{18}$	$\overline{32}$	$\overline{32}$	$\overline{27}$	$\overline{11}$	25	66	100	119	134	120	76	26	$\overline{32}$	$\overline{92}$	$\overline{128}$	$\overline{127}$	$\overline{108}$	$\overline{75}$	
a^{43}		3	3	$\overline{2}$	$\overline{5}$	$\overline{16}$	$\overline{18}$	$\overline{20}$	$\overline{14}$	$\overline{2}$	20	55	73	93	95	86	48	9	$\overline{32}$	$\overline{82}$	$\overline{106}$	$\overline{104}$	$\overline{87}$	$\overline{61}$	
a^{44}			1	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{14}$	$\overline{12}$	$\overline{8}$	2	21	39	56	62	66	54	26	$\overline{3}$	$\overline{37}$	$\overline{70}$	$\overline{90}$	$\overline{84}$	$\overline{72}$	$\overline{49}$	
a^{45}		1		$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	3	12	31	36	45	41	33	12	$\overline{11}$	$\overline{33}$	$\overline{60}$	$\overline{69}$	$\overline{67}$	$\overline{54}$	$\overline{38}$	
a^{46}			1		$\overline{2}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{1}$	5	13	18	26	25	28	17	2	$\overline{12}$	$\overline{31}$	$\overline{46}$	$\overline{56}$	$\overline{50}$	$\overline{41}$	$\overline{28}$	
a^{47}				$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{2}$		4	5	15	14	18	11	8	$\overline{2}$	$\overline{15}$	$\overline{22}$	$\overline{39}$	$\overline{39}$	$\overline{37}$	$\overline{31}$	$\overline{20}$	
a^{48}	1				$\overline{1}$		$\overline{2}$		1	1	5	5	9	6	8	1	$\overline{7}$	$\overline{12}$	$\overline{23}$	$\overline{24}$	$\overline{30}$	$\overline{27}$	$\overline{21}$	$\overline{15}$	
a^{49}						$\overline{1}$				3	1	6	3	4	2	$\overline{2}$	$\overline{4}$	$\overline{11}$	$\overline{12}$	$\overline{22}$	$\overline{20}$	$\overline{18}$	$\overline{14}$	$\overline{8}$	
a^{50}			$\overline{1}$		$\overline{1}$		$\overline{1}$				2		4			$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{11}$	$\overline{11}$	$\overline{13}$	$\overline{11}$	$\overline{10}$	$\overline{7}$	
a^{51}								1		1		2				$\overline{4}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{9}$	$\overline{8}$	$\overline{8}$	$\overline{6}$	$\overline{2}$	
a^{52}							1		1		2		1		$\overline{1}$		$\overline{4}$	$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{3}$	
a^{53}												$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{3}$		
a^{54}													$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\overline{2}$		$\overline{1}$	
a^{57}																									

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	16	18	20	22	24	26
1						1								
2	1		1		1		1		1					
3		1		2	1	1	2	1	1	1	1		1	
4	1		3	1	3	3	2	3	1	2	1	1		1
5		3	3	4	5	4	5	2	4		1			
6	4	2	5	8	6	8	2	3						
7		7	10	8	12	2	3							
8	5	8	11	15	4	5								
9	5	13	19	8	4									
10	8	20	12	10										
11	8	18	21											
12	12	30												
13	15	16												
14	13	17												
15	19													
16	5													
17	3													

The total number of irreducible invariants and covariants for the first 10 orders (counting in the absolute constant and the quantic itself), it appears from what precedes, is as follows:

Order of Quantic:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Number of Groundforms:	1,	2,	3,	5,	6,	24,	27,	125,	70,	416,	476.

For the benefit of those new to the subject, it may be well to recall the immediate algebraical meaning of either form of the generating function to a binary quantic $(x, y)^n$.

Suppose n an odd number, say 5, then if

$$\frac{1 - x^{-2}}{(1 - ax^{-5})(1 - ax^{-3})(1 - ax^{-1})(1 - ax)(1 - ax^3)(1 - ax^5)}$$

is expanded in a *bivergent* series, (that is, one going, as regards the powers of x, in two directions towards infinity), either generating function of the tables for the quintic is the sum of the terms which contain no negative powers of x. So if n be an even number, say 6,

$$\frac{1-x^{-2}}{(1-ax^{-6})(1-ax^{-4})(1-ax^{-2})(1-a)(1-ax^{2})(1-ax^{4})(1-ax^{6})}$$

being similarly expanded, either generating function of the tables for the sextic is, as before, the sum of the terms which contain only positive or zero powers of x. And so in general, for $(x, y)^n$, the numerator of the so-called *crude* generating function, being always $1-x^{-2}$ and its denominator a product of factors of the form $1-ax^{n-2i}$ (where i takes all values from nought up to n inclusive). Either generating function of the tables for the n^ic is the algebraic equivalent of the *positive* branch of the corresponding bivergent series, (that in which only positive powers of x appear), *plus* the *neutral* branch or term, namely, that which contains neither positive nor negative powers of x, or, which is the same thing, is a function only of a.

I subjoin a few reflexions which appear to me to be desirable on the foregoing tables.

It is scarcely necessary to state, that, in the development of the generating function, whether reduced or representative, the coefficient of $a^m x^\mu$ is the total number of linearly independent covariants of the degree m in the coefficients and the order μ in the variables.

Mr Franklin will probably, in a future number of the *Journal*, draw up a statement of the mode in which the tables have been calculated and the precautions taken to insure accuracy *; as regards the reduced form, three methods have been employed in calculating it, namely, Mr Sylvester's first method, Professor Cayley's method, fully explained in a preceding number of the *Journal* by its eminent author, and Mr Sylvester's second method, much briefer than his other, but, in general, not so brief as Professor Cayley's, which last, however, involves a delicate point in the expansion of series, the assumed principle of which, although its validity on moral grounds of evidence is unquestionable, cannot be regarded as *a priori* self-evident †.

The theory of the generating function, alike for single and simultaneous forms, depends on the law for determining the number of linearly indepen-

* In especial I wish to single out an ingenious device of Mr Franklin to check the operation of tamisage by introducing a common superfluous factor into the numerator and denominator of the representative generating function so selected as that the augmented denominator shall not cease to be representative; the effect of this will be to cause the groundforms obtained by tamisage of the augmented numerator to be the same as before, except that the groundform represented by the additional factor will not be found among them.

† In Prof. Cayley's method the crude generating function is regarded as a function of a; in my two methods as a function of x.

dent in- and co-variants of given order and degree or degrees belonging to a given quantic or system of quantics, a proof of which will be found at the end of a memoir by Mr Sylvester in *Borchardt's Journal* *, and also in the *London and Edinburgh Philosophical Magazine* †, that leaves nothing to be desired as regards rigour of demonstration. The law itself for the case of a single quantic was first stated by Professor Cayley whilst the theory was still in its infancy.

But besides this fundamental theorem, in order to deduce the tables of groundforms, a *fundamental postulate* still awaiting demonstration is necessary, which is, that no more linear relations between in- or co-variants are to be supposed to exist than are necessary in order to satisfy the *fundamental theorem*. The application of this principle in such a mode as to substitute a finite for an infinite process, leads to the use of representative generating functions and the simplified method of *tamisage*. The validity of the fundamental-postulate which is in accord with the law of parcimony is verified by its conducting to results which have been proved to be accurate for single binary quantics up to the sixth order inclusive, for pairs of binary quantics up to the fourth order inclusive, and also for systems of an indefinite number of linear and quadratic binary forms ‡.

The application of this principle discloses the remarkable singularity that for the quantic of the seventh order, there exists no finite representative generating function as shown in what follows.

The invariantive part of the numerator of the reduced form for the seventhic is

$$1 - a^6 + 2a^8 - a^{10} + 5a^{12} + 2a^{14} + 6a^{16} + 2a^{18} + 5a^{20} - a^{22} + 2a^{24} - a^{26} + a^{32},$$

and the invariantive part of the denominator is $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})$. Multiplying numerator and denominator by $(1+a^6)$, their invariantive portions§ become, respectively,

$$1 + 2a^8 - a^{10} + 4a^{12} + 4a^{14} + 5a^{16} + 7a^{18} + 7a^{20} + 5a^{22} + 4a^{24} + 4a^{26} - a^{28} + 2a^{30} + a^{38},$$

and $$(1-a^4)(1-a^8)(1-a^{10})(1-a^{12}).$$

[* p. 232 above.] [† p. 117 above.]

‡ If the *fundamental postulate* were called into question, this (it may be proved) would not affect the fact of the existence of the groundforms obtained by its aid, but only the possibility of the existence of other groundforms over and above those so obtained. Thus my tables of groundforms could only err (were that possible, which I do not believe it to be) in defect; and as those found by the German method can only err in excess, it follows that, whenever the tables coincide, both must be correct. The tables of groundforms here given, up to the sixth order, inclusive, and all those that follow, coincide exactly with those obtained by Clebsch, Gordan and Gundelfinger, when these latter are rectified by the omission of certain supposed groundforms which, in the *Comptes Rendus*, I have conclusively proved to be composite.

§ The factors in the denominator which involve x never offer any difficulty, as they represent the given quantic along with the complete system of covariants of the second degree, the several orders of which follow a well known rule.

The factors of the denominator are now, with the exception of $1-a^{10}$, representative factors; $1-a^{10}$ is not such, as a^{10} occurs in the numerator with the coefficient -1. If we multiply numerator and denominator by $1+a^{10}$, the factor $1-a^{20}$ will take the place of $1-a^{10}$ in the denominator, and the numerator will become

$$1+2a^8+4a^{12}+4a^{14}+5a^{16}+9a^{18}+6a^{20}+\ldots.$$

Here the coefficient of a^{20} is not negative, but it is less than the number (8) obtained by composition from the terms $2a^8$ and $4a^{12}$; hence, by the fundamental postulate there is no irreducible invariant of the degree 20. If, instead of multiplying numerator and denominator by $1+a^{10}$, we multiply them by the infinite series $1+a^{10}+a^{20}+\ldots$, the denominator becomes representative and the invariantive part of the numerator becomes the *recurrent* series given in the table (p. [288]), in which the coefficient of a^{30}, a^{40} and, in general, all powers of a whose exponents are multiples of and greater than 20, is 9; but 9 is less than the number obtained in the composition of a^{30}, a^{40} (and *a fortiori* of a^{50}, a^{60}, ...) out of the preceding terms; therefore, by the fundamental postulate, there is no irreducible invariant whose degree is any multiple of 10. It is a remarkable and significant fact that in this case the erroneous assumption of $1-a^{10}$ being a representative factor in the denominator of the complete generating function will be found to lead to no subsequent further error in the determination of the other groundforms of the seventhic.

A chorographical law obtains in the numerical tables of the numerators of the representative forms, which plays a considerable part in the complete theory of tamisage, and is too important to be passed over without notice, namely, it will be seen that all these tables consist of a small number of irregular but continuous bands or blocks of alternately positive and negative coefficients which can be drawn asunder without tearing or leaving any hole in the paper*. For the first four orders there is but one such block, for the

* In the operation of tamisage on the numerator of the representative groundforms the terms of the negative blocks are disregarded. In every case treated in these tables, and those to follow in the next number of the *Journal*, the only surviving terms will be found to be comprised in the first block. Had it turned out otherwise it would have been necessary to ascertain whether the surviving terms belonging to the other odd-numbered blocks would survive the operation of tamisage performed on the infinite aggregate of terms obtained by the development of the generating function; if not, they would have to be rejected. This is what I have found actually happens in a system of quadratic or linear forms when a sufficient number of such forms is employed. In that case, terms not confined to the first block emerge from the tamisage of the numerator of the representative groundforms, but disappear when the tamisage is performed on the infinite aggregate of terms of which the groundform is the sum. Such aggregate, it may be noticed, (I have proved elsewhere), consists exclusively of positive terms, the coefficients corresponding to non-existing types being always zero and never negative. It is very likely to be found true hereafter that in no case need any, except the first block of terms in the numerator of the representative groundforms, be submitted to tamisage in order to obtain the groundforms not represented in the denominator, and so in like manner that, in order to obtain the ground-syzygies of the first kind, that is, those that concern the groundforms, only the first

quintic and the sextic two, for the seventhic five, for the octavic three, and for the 9^{ic} and 10^{ic} four. A similar law obtains for systems of quantics, as for instance in the case of two simultaneous quantics, the corresponding tables consist of detachable solid blocks, alternately positive and negative, and small in number in comparison with the number of terms which they contain, as will be seen in the tables to appear in the next number of the *Journal* which will contain a complete set of them for all the systems that can be formed of two binary quantics of orders, m, n where neither m nor n exceeds 4.

It is my duty to state that the expense of calculating the tables for quantics of the 7th, 8th, 9th and 10th orders, has been defrayed out of a grant made by the British Association for the Advancement of Science, and I have pleasure in returning my thanks to that distinguished body for this act of aid in enabling me to bring to a successful issue an undertaking of such unusual magnitude and of such pith and moment to the progress of Algebraical Theory.

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positive and the first negative block need be considered, and so on for syzygies of the higher orders, each time a new block being taken into account until all are exhausted, it being quite conceivable that the number of blocks may designate the highest order of syzygy that occurs in any case, subject in the case of a linear or quadratic form (for which the block reduces to a single term, namely, unity) to the obvious exception that, for them, the syzygies become abortive.

To explain what is meant by syzygies of successive orders, suppose Z to be a rational and integral function of groundforms which, regarded as a function of the coefficients, is identically zero, then $Z=0$ is a syzygy and Z may be termed a syzygant of the first order and, if incapable of being resolved into a sum of products of syzygants multiplied respectively by rational algebraic functions of the groundforms, will be an irreducible or ground-syzygy of the first order. In like manner, if Z' is a function of ground-syzygants which, regarded as a function of the groundforms, vanishes identically $Z'=0$ is a syzygy and Z' is a counter-syzygant or a syzygant of the second order, and, if incapable of representation as a sum of products of other syzygants of the second order multiplied respectively by rational integral functions of syzygants of the first order, is a ground-syzygant of the second order; and so on indefinitely.

39.

ON CERTAIN TERNARY CUBIC-FORM EQUATIONS.

[*American Journal of Mathematics*, II. (1879), pp. 280—285, 357—393; III. (1880), pp. 58—88, 179—189.]

CHAPTER I. *On the Resolution of Numbers into the sums or differences of Two Cubes.*

SECTION 1.

M. LUCAS has written to inform me that in some one or more of a series of memoirs commencing with 1870, or elsewhere, the Reverend Father Pépin has made considerable additions to my published theorems * on the classes of numbers irresoluble into the *sum or difference* † of two rational cubes.

Using p, q to denote primes of the forms $18n+5$, $18n+11$, besides the 6 forms published by me, M. Pépin has found 10 other general classes of irresoluble numbers, the total number (as I understand from M. Lucas) known to the Reverend Father being as follows:

$$\begin{array}{cccccccc} p, & q^2, & p^2, & q, & 2p, & 2q^2, & 4p^2, & 4q, \\ 9p, & 9q^2, & 9p^2, & 9q, & 25p, & 25q^2, & 5p^2, & 5q, \end{array}$$

but the last four of these classes are special cases only, of three out of the four more general irresoluble classes pq, p^2q^2, $p_1p_2^2$, $q_1q_2^2$, where p_1, p_2 are any two numbers of the p class and q_1, q_2 any two of the q class. On making $p=5$ in the first two of these, and $p_1=5$, $p_2=p$, or $p_2=5$, $p_1=p$, in the third, Father Pépin's last four classes result. It is also true that the numbers in my four additional general classes respectively multiplied by 9 are still irresoluble. Hence the number of known classes of numbers (depending on p and q) irresoluble into the sum or difference of cubes may be arranged as follows:

$$\begin{array}{cccccccc} p, & q, & p^2, & q^2, & pq, & p^2q^2, & p_1p_2^2, & q_1q_2^2, \\ 9p, & 9q, & 9p^2, & 9q^2, & 9pq, & 9p^2q^2, & 9p_1p_2^2, & 9q_1q_2^2, \\ 2p, & 4q, & 4p^2, & 2q^2. & & & & \end{array}$$

[* See Vol. I. of this Reprint, pp. 107—118, and Vol. II. pp. 63, 107.]

† It is well to understand that a number resoluble into the sum is necessarily also resoluble into the difference of two positive cubes and *vice versâ*.

Moreover, I have ascertained the truth of the following two theorems of a somewhat different character:

1st. Let ρ, ψ, ϕ denote prime numbers respectively of the forms $18n+1$, $18n+7$, $18n+13$ and suppose ρ, ψ, ϕ to be *not* of the form f^2+27g^2 and consequently *not* to possess the cubic residue 2, then I say that all the numbers comprised in any one of the eight classes

$$2\rho,\quad 4\rho,\quad 2\rho^2,\quad 4\rho^2,\quad 2\psi,\quad 4\psi^2,\quad 4\phi,\quad 2\phi^2$$

are irresoluble into the sum of two cubes *.

2nd. Provided 3 is not a cubic residue to ν† [where ν, any $6n+1$ prime, is the same as ρ, ϕ, ψ taken collectively], 3ν and $3\nu^2$ are similarly irresoluble.

With the aid of these theorems and certain special cases of irresolubility noticed by Father Pépin, communicated to me by M. Lucas, supplemented by calculations of M. Lucas and my own as regards the non-excluded numbers, it follows (*mirabile dictu*) that of the first 100 of the natural order of numbers, there is only a single one, namely, 66, of which it cannot at present be affirmed with certitude either that it is or is not resoluble into the sum of two cubes, and of which, in the former case, the resolution cannot be exhibited.

The proof of these statements, and the resolutions into cubes in their lowest terms, when they exist, will be given in the next number of the *Journal*. For the present I limit myself to noticing (what I much regret not to have done before the paper was printed) a statement of M. Lucas which is capable of being misunderstood and might give rise to an erroneous conception.

It is where this distinguished contributor to our *Journal* speaks of deriving from one rational point on a cubic curve (defined by a cubic equation with integer coefficients) another by means of its intersections with a

* The exclusion of 2 as a cubic residue blocks out the possibility of the "distribution of the amplitude"; the form p^2+27q^2 blocks out the possibility of a solution in which x^2-xy+y^2 has a common factor with the amplitude, and thereby imposes upon the equation containing x, y, z (were it soluble in integers) the necessity of repeating itself perpetually with smaller numbers, which of course is impossible. But the two conditions thus separately stated are in fact mutually implicative, every number of the form f^2+27g^2 having 2 for a cubic residue and *vice versâ* every number of the form $6n+1$ to which 2 is a cubic residue being of the form f^2+27g^2. The sole condition, therefore, in order that a number coming under any of the eight categories in the text shall be known at sight to be irresoluble into the sum of two cubes, is that its variable part shall not be of the form p^2+27q^2, that is, shall not be 31, 43, 91, 109, 127, 157, 223, 229, 247, etc.

† If I am not mistaken this is tantamount to the proviso that ν shall not be of the form $f^2\pm 9fg+81g^2$. It is worth noticing that the above quantity multiplied by 3, say $3N$, is equal to $\frac{(9g\mp f)^3+(18g\pm f)^3}{27g}$, so that when g is a cube number N is immediately resoluble. The initial values of N will be found to be 61, 67, 73, 103, 151, 193, 271, 367, 547, etc., for each of which, up to 367 inclusive, $g=1$ or $g=-1$, so that their products by 3 are immediately resoluble.

conic drawn through five consecutive points situated at the given rational one; but, in fact, it follows from my theory of *residuation* that this point is collinear with the given point and its second tangential: just as a ninth point in which the cubic would be met by any other cubic passing through *eight* consecutive points situated at the given point would be the third tangential to the latter *.

Hence M. Lucas' third method amounts only to a combination of the other two; and in fact there is *but one single scale* of rational derivatives from any given point in a general cubic, the successive terms of which expressed in terms of the coordinates of the primitive are of the orders 1, 4, 16, 25, 49, ... the squares of the natural numbers with the multiples of 3 omitted †.

Scholium.

I term lmn the *amplitude* of the equation $lx^3 + my^3 + nz^3 = 0$, and if A cannot be broken up in any way into factors l, m, n, such that

$$lx^3 + my^3 + nz^3 = 0$$

shall be soluble in integers, I call the amplitude A of the equation

$$x^3 + y^3 + Az^3 = 0$$

undistributable.

When A is of the form $\dfrac{x^3 - 3x^2y + y^3}{3z^3}$, the equation $x^3 + y^3 + Az^3 = 0$ is always soluble, and when this equation is soluble, then, provided that its amplitude is undistributable and contains no prime factor of the form $6i + 1$, the equation $x^3 - 3x^2y + y^3 = 3Az^3$ must be soluble in integers, which cannot be the case when A contains any factor other than 3, or of the form $18i \pm 1$, inasmuch as *the cubic form* $x^3 - 3x \pm 1$ *contains no factors other than* 3 *or of the form* $18i \pm 1$.

* I make the important additional remark that at those special points of the cubic where this ninth point (sometimes elegantly called the subosculatrix) coincides with the point osculated, the scheme of rational derivatives returns upon itself, and instead of an infinite number there will be only two rational derivatives to such point. That is to say the infinite scheme becomes a system of 3 continually recurring points. The general theory of the special points which have only a finite number of rational derivatives will be given in the next number of the *Journal*.

† When the cubic is of the form $Ax^3 + Ay^3 + Cz^3 + Mxyz = 0$, where A, C, M are integers, then a rational point of inflection $x = 1$, $y = -1$, $z = 0$ is known and, in that case, from any other rational point *besides the ordinary ones* derivative rational points of the missing orders 9, 36, 81 can be found, but no others, and so universally if in the general cubic a rational point of inflection and a rational point (a, b, c) are given the scale of rational derivatives will be of the orders 1, 4, 9, 16, ... in a, b, c. This scale will of course be duplex, consisting of a series of points and a second series in which the radii drawn through the points of the first series and the point of inflection again meet the cubic.

This last theorem is a particular case of the following: If k be any integer and $F(x, y)$, the product of factors of the form $\left(x-2\cos\frac{2\lambda\pi}{k}\,y\right)$, where λ is every number prime to k up to $\frac{1}{2}(k-1)$, then Fx [$=F(x, 1)$] contains no prime factors excepting such as are contained in k or else are of the form $ki \pm 1$ *.

If it could be shown, in analogy with what holds for the quadratic forms Fx which result from making $k = 8$, 10, 12, that the cubic form $x^3 - 3xy^2 \pm y^3$ which results from making $k = 18$ may always be made to represent any prime number of the form $18n \pm 1$ itself, or else its treble (and for our purpose rational numbers would be as efficient as integers), we should then be able to affirm that any prime $18n \pm 1$ or else its nonuple could be resolved into the sum of two cubes. As a matter of fact I have ascertained that every prime number $18n \pm 1$ as far as 537 inclusive (and have no ground for supposing that the law fails at that point) can be represented by

$$x^3 - 3xy^2 \pm y^3$$

or else by its third part with *integer* values of x, y. Moreover, I find that the same thing is true of 17^2, 17.19, 19^2, 17.37, 19.37, 37^2, 17.53, 19.53, 37.53, that is, in fact for all the binary combinations of the natural progression of "r, ρ" numbers 17, 19, 37, 53, 71, 73, 89 (21 in all), as also 17^2, 19^2, 37^2 †. The number of *consecutive* r, ρ primes for which the law has been verified, that is, the number of those not exceeding 537 will be found to be 39, namely, 17, 19, 37, 53, 71, 73, 89, 107, 109, 127, 163, 179, 181, 197, 199, 233, 251, 269, 271, 307, 323, 341, 359, 361, 377, 379, 397, 413, 431, 433, 449, 451, 467, 469, 487, 503, 521, 523, 541, which according to the usual canons of induction would, I presume, be considered almost sufficient to establish the theorem for the case of $k = 9$.

* Thus, by making $k=8$ we learn that x^2-2 contains no factors except 2 and $8i \pm 1$ and by making $k=16$, that y^4-4y^2+2, none except 2 or $16i \pm 1$, by making $k=9$ that x^3-3x+1, by making $k=18$, that x^3-3x-1 contain no other factors but 3, or numbers of the form $18n \pm 1$. The theorem, I am aware, is well known for the case where k is a prime number and possibly is so for the general case. The proof of the irresolubility into two cubes of the 20 classes of numbers involving p's and q's, given at page [312], is an instantaneous consequence of the theorem for the case of $k=9$, for which case also there is no shadow of doubt of the theorem being true.

† 53^2 has not yet made its appearance. All the primes of that form themselves occurring in the first six hundred numbers have already occurred in my calculations except 557 and 593. I have worked with the formula $x^3-3xy^2 \pm y^3$ [x and y relative primes], giving to x and to y all the values possible from 1 to 36, and intend to extend the table to the limit of 50 or 100. The longer a moderate-sized number is in making its appearance, the longer it is likely to be before it appears, inasmuch as the large numbers of which it is the residuum or balance are becoming continually greater. It may very well then happen that the missing numbers alluded to may transcend all practicable limits of calculation to find them just as would be the case, for certain values of A, with finding values of x, y to satisfy the Pellian equation $x^2-Ay^2=1$, were there not a theoretical method of arriving at them.

The table of "*special* cases" of irresoluble numbers found by Father Pépin (according to the information most kindly communicated to me by M. Lucas) comprises the numbers

14, 21, 31, 38, 39, 52, 57, 60, 67, 76, 77, 93, 95 *,

all of which I have verified as irresoluble except the number 60, which I accept as such on the erudite and sagacious Father's authority.

Reverting to F, it is hardly necessary to recall that $F(x^2+y^2, xy)$ is the primitive factor of x^k-y^k, and that it is capable of very easy demonstration that this primitive factor contains no prime factors except such as are divisors of k or of the form $ki+1$, the linear divisor $ki-1$ being here excluded. It seems to be very probable that for $k=9$, $F(x, y)$ or else $3F(x, y)$ does represent any prime of the form $18n \pm 1$, and consequently that every such form of prime or else 9 times the same is the sum of two rational cubes †.

This last conjectural theorem, it will be noticed, is not in any real analogy to the theorem that every product of primes of the form $4n+1$, and also the double thereof, is the sum of two *integer* squares; the real analogy is between the fact, of which this theorem is a consequence, that $x^3-3xy^2 \pm y^3$ or its third part represents every number which is a product of primes of the form $18n \pm 1$, and each one of the facts that x^2-2y^2, x^2-5y^2 represent all numbers of the form $8i \pm 1$, $10i \pm 1$ respectively, and that x^2-3y^2 or its third part represents all numbers of the form $12i \pm 1$. On account of its importance to this theory it seems desirable to give a name to the law which governs the prime factors of $F(x, y)$, and I take advantage of the circumstance that $F(x^2+y^2, xy)$ contains prime factors of the form $ki+1$, but not of the form $ki-1$, whilst $F(x, y)$ contains prime factors of either of these forms indifferently, to characterize it as the Law of Twin Prime Factors. Let us suppose the circumference of a circle divided by points into k equal parts, and agree to designate the shorter arc between any two of the points a *primitive* division of the circle in respect to k, provided that no number less than k would be adequate to give rise to an equal length of arc, so that $\frac{2\lambda\pi}{k}$, when λ is prime to k and less than $\frac{k}{2}$, will serve to represent any such division. The assumed Law of Twin Factors (well known, I repeat, for the case of k a prime number and possibly in its extended form likewise) may then be enunciated as follows:

* Of these numbers all except 60, 31, 67, 77, 95 belong to some one or other of the general classes of irresoluble numbers given in the text.

† It may be and probably is true also that $x^3-3xy^2 \pm y^3$ will represent the product or else three times the product of any two primes each of which is of the form r or ρ, and possibly the square or else three times the square of any r or ρ; it cannot possibly represent three times *any cube*, for if it did we should be able to infer that a cube was resoluble into two cubes, which we know is not true.

That function of x whose first coefficient is unity and whose roots are the doubled cosines of all the primitive divisions of the circle in respect to k contains no prime factors except such as are divisors of, or else when increased or diminished by unity, are divisible by k. This may be called again the *Exclusional or Negative Theorem of Twin Factors;* and on the other hand the more extraordinary theorem which asserts (on evidence not yet conclusive) that the function of x above defined, when made homogeneous in x, y, will represent (at all events for the case of $k=9$) every prime number of the form $ki \pm 1$, or else certain specific multiples of any such number, may be called the *Inclusional or Representational Theorem of Twin Factors.*

EXCURSUS A. *On the Divisors of Cyclotomic Functions.*

Title 1. *Cyclotomic Functions of the* 1*st Species.* In the preceding section which should have been termed and will be hereafter referred to as the *Proem* of Chapter I., I stated that the proof of the first batch of theorems on the irresoluble cases of equations in numbers of the form $x^3+y^3+Az^3=0$, or, as we might say, of the forms of numbers A irresoluble into a pair of rational cubes, depends on the demonstration of the form of the numerical linear divisors of the function x^3-3x+1. At the time when this proem went to press I had reduced to a certainty the law of the divisors by numerical verifications without end, but had not obtained a rational demonstration of it, nor was I able to find such or even a statement of the law itself in any of the current text-books, such as Gauss, Legendre, Bachmann, Lejeune-Dirichlet or Serret. I was therefore compelled to seek out a demonstration for myself, and in so doing was unavoidably led to consider the general theory of the species of *cyclotomic* (*Kreistheilung*) functions of which the cubic function above written is an example of what may be called the second species and incidentally also the theory of the simpler or first species which, although discussed ever since the time of Euler, appears to me to remain still in a somewhat cloudy and incomplete condition. As this inquiry extends beyond the strict needs of the subject which called it forth, I entitle it an *excursus.* It will be necessary for me eventually to introduce another and still more important excursus or lateral digression on certain consequences of the Geometrical Theory of Residuation, which theory itself also took its rise in and is required for the purposes of the arithmetical theory which forms the subject of the entire memoir.

If fx is any rational integral function of the order ω in its variable, we know that in respect to a prime number p as modulus fx regarded as the subject of a congruence cannot have more than ω distinct real roots. If we take p^j as modulus, certain conditions increasing in number with the value of j, will have to be satisfied in order that fx may have a superfluity (that is, more than ω) of real roots.

One condition, the universal *sine quâ non,* will serve for the object I have in view, so that it will be sufficient to make $j = 2$. Obviously when this superfluity exists two of the roots must differ by a multiple of p since otherwise there would be a superfluity of roots *quâ* the first power of p as modulus. If then a and $a + \lambda p$ where $\lambda < p$ be two of the roots, we have $fa \equiv 0$ and $fa + \lambda f'a \,.\, p + Rp^2 \equiv 0$ mod. p^2. Hence $fa \equiv 0$ and $f'a \equiv 0$ mod. p, so that $fa + \lambda p = 0$ and $f'a + \mu p = 0$.

Applying the dialytic method to eliminate a it is obvious that the resultant of these two equations will differ only by a multiple of p from that of fa and $f'a$, that is, from the arithmetical discriminant of fa (I use the term arithmetical to distinguish it from the algebraical discriminant in obtaining which latter fx is supposed to be affected with binomial numerical coefficients ω, $\frac{1}{2}\omega(\omega - 1)$, ... and the factor ω to be struck out from each of the two equations $\dfrac{df(x, 1)}{dx} = 0$, $\dfrac{df(x, 1)}{d1} = 0$).

We see then that a rational integer function (the subject of a congruence) cannot have a superfluity of roots in respect to the power of a prime p^j as modulus, unless the strict (arithmetical) discriminant of the function contains p.

It is necessary for the purpose I have in view to express the strict relation between the arithmetical discriminant of a function, Δfx, and the product of the squares of the differences of its roots, $\zeta^2 fx$. I shall for greater simplicity suppose that the initial coefficient of fx is unity, as it is in the cases with which we shall have to deal.

We know that $\Delta f = \mu\zeta^2 f$ where μ is a function of n the order of f, so that to determine μ we may specialize f in any manner we please, provided the order is maintained. Let $fx = x^n - 1$. Then it is easily proved that, making $\rho = \cos\dfrac{2\pi}{n} + i \sin\dfrac{2\pi}{n}$,

$$(-)^{n\frac{n-1}{2}}\,\zeta^2 f = (\rho^{n-1})^{(n-1)\frac{n}{2}} \,.\, n^n,$$

so that
$$\zeta^2 f = (-)^{\frac{(n-1)(n-2)}{2}} \,.\, n^n,$$

and
$$\Delta f = (-)^{n-1} \,.\, n^{2n-2}.$$

Hence
$$\Delta f = (-)^{(n-1)\frac{n}{2}} \,.\, n^{n-2}\,\zeta^2 f\,*$$

expresses the universal relation between the arithmetical discriminant and the squared product of the root-difference of a function. If we had been

* As regards the application to be made of this result it was of course not necessary to determine the index of the power to which $(-)$ is raised, but it was hardly worth while to leave it undetermined.

dealing with the algebraical discriminant, it would have been necessary to replace n^{n-2} by n^{-n} in the above equation. It is furthermore to be observed that the discriminant is fixed in its sign by the condition that the term containing the highest power of the product of the expressed coefficients is to be taken positively.

So again it will be seen presently to be necessary to ascertain the strict relation between the resultant of two functions of degrees r, s and the product of the differences between the several roots ρ of the one and the several roots σ of the other of them, or, as we may say, between $R_{r,\,s}$ and $D_{\rho,\,\sigma}$, where if we choose to pay attention to algebraical signs that of $R_{r,\,s}$ may be understood to mean the resultant so taken that the term containing the highest power of the coefficient in the r-degreed function is positive and $D_{\rho,\,\sigma}$ to mean the product of the rs differences $(\rho-\sigma)$.

I shall again, for greater simplicity, suppose the initial coefficients of each of the two functions to be unity.

We know that $R_{r,\,s}=\mu D_{\rho,\,\sigma}$ where μ is a function of r and s exclusively. To determine it we may take x^r and x^s+1 as the two functions, it will be found without difficulty that

$$R_{r,\,s}=1^{*} \text{ and } D_{\rho,\,\sigma}=\{-(-1)^{\frac{1}{s}}\}^{rs}=(-)^{rs+r}.$$

Hence we have universally $R_{r,\,s}=(-)^{rs+r}D_{\rho,\,\sigma}$.

This seems to be the proper place to ascertain (what will be needed for future purposes) how far or under what qualifications the reciprocal connexion of the two facts: 1. Of two functions in x having a common root. 2. Of their resultant being zero, admits of being extended to roots of congruences in respect to a prime-number modulus.

Suppose fx, gx to be two in all respects (numerically† as well as algebraically) integer rational functions of the degrees i, j in x, then by eliminating dialytically $(i+j-1)$ powers of x between

$$fx,\ xfx,\ x^2fx\ \ldots\ x^{j-1}fx,\quad gx,\ xgx,\ \ldots\ x^{i-1}gx,$$

we may obtain the equation $\lambda xfx+\mu xgx=Rx^q$ (q having any integer value from 0 to $i+j-1$) where R is the resultant of f, g and λx, μx are in all respects integer functions of x of degrees $j-1$ and $i-1$ in x whose values

* Thus, for example, let $r=4$, $s=2$. Then $R_{r,s}$ is the dialytic resultant of

$$\begin{array}{cccccc}
x^5 & & & & & \\
 & x^4 & & & & \\
x^5 & & +x^3 & & & \\
 & x^4 & & +x^2 & & \\
 & & x^3 & & +x & \\
 & & & x^2 & & +1
\end{array}$$

which is obviously equal to unity.

† By which I mean that the coefficients are exclusively integer numbers.

depend on the value of q. If, then, fx and gx are simultaneously zero for some value of x, we must universally have $R = 0$ even if x should be zero, for thus we might make $q = 0$.

But this equation will not suffice to show that fx and gx will simultaneously vanish for some value of x, provided that $R = 0$; for every value of x which makes fx vanish, might, as far as this equation discloses (and for all values of g), have the effect of making μx vanish*. We may, however, prove the fact in question, on a certain hypothesis to be presently stated, by availing ourselves of the knowledge that R is, to a *numerical factor près*, the product of the differences between the roots of f and those of g.

The hypothesis I make is that $fx \equiv 0$ mod. p is a congruence *all whose roots are real;* in this case I shall show that if the resultant R of fx and gx satisfies the congruence $R \equiv 0$ mod. p (that is, if R contains p) then gx must have at least one real root in common with fx *quâ* modulus p.

From the congruence of $fx \equiv 0$ mod. p we may, by a well known principle, infer the existence of an equation $Fx = fx + p\phi x = 0$ whose roots are the same as those of the congruence above written, and the dialytic method of elimination renders it self-evident that the resultant of Fx and gx will differ only by a multiple of p from that of fx and gx, and will, therefore, be a multiple of p.

If, then, we call the roots of Fx (all real by hypothesis) $a_1, a_2, \dots a_i$, we shall have $ga_1 . ga_2 . ga_3 \dots ga_i \equiv 0$ mod. p, and, as all the factors on the left hand side of the equation are real, one of them must contain p. Hence, if $R\,(fx, gx) \equiv 0$ mod. p, and $fx \equiv 0$ mod. p *has all its roots real*, one of these roots must belong also to the congruence $gx \equiv 0$ mod. p.

Going back now to what precedes this investigation, let us determine strictly the relation between the arithmetical discriminants and resultant of two functions in x and the discriminant of their product.

Let ω, ω_1 be the degrees in x of two altogether integer functions fx, f_1x, and suppose $Fx = fx . f_1x$. Then obviously $\zeta^2 Fx = \zeta^2 fx . \zeta^2 f_1x . (D\,(fx, f_1x))^2$. Hence $\omega^{\omega-2} . \omega_1^{\omega_1-2} \Delta Fx = (\omega + \omega_1)^{\omega+\omega_1-2} \Delta fx . \Delta f_1x\,(R\,(fx, f_1x))^2$.

If, then, p any prime number is contained in Δfx, and ω, ω_1 are each less than p, p will necessarily be contained in ΔFx. And as a particular case of this theorem, if p were contained in the discriminant of any factor of $x^{p-1}-1$ it would be contained in the discriminant of $x^{p-1}-1$, that is, in a power of $(p-1)$, which is impossible. Hence, by a preceding theorem, no factor of $x^{p-1}-1$, regarded as the subject of a congruence, can contain *a superfluity* of real roots (that is, more real roots than there are units in its degree) in respect to the modulus p^j.

* I think it would not be incorrect to say that *in all cases* the fact of the resultant of two functions of x containing a prime number raises a strong presumption that the functions have a common congruence root in respect to that number.

It is easy to show, although I do not find it distinctly stated in any of the current text-books, that $x^{p-1} - 1 \equiv 0$ mod. p^j has $p - 1$ real roots.

For let $x = y^{p^{j-1}}$. Then the congruence becomes

$$y^{p^{j-1}\cdot(p-1)} - 1 \equiv 0 \text{ mod. } p^j,$$

where $p^{j-1}.(p-1)$ is what is commonly designated as the ϕ function of p^j, the number of numbers less than p^j and prime to it, (the so-called ϕ function of any number I shall here and hereafter designate as its τ function and call its Totient). This last congruence by Fermat's extended theorem has all its roots real. It is easy to see that they will consist of $(p-1)$ groups, each group containing p^{j-1} numbers for which the value of x *quâ* modulus p^j will be the same, but different for numbers belonging to two different groups. For let y_1 be any of the y roots, and $y_2^{p^{j-1}} - y_1^{p^{j-1}} \equiv 0$ mod. p^j. Then *quâ* mod. p, $y_2^{p^{j-1}} \equiv y_2$ and $y_1^{p^{j-1}} \equiv y_1$, because $p^{j-1} - 1$ contains $p - 1$.

All the values of y_2 will, therefore, be comprised in the series

$$y_1,\ y_1 + p,\ y_1 + 2p,\ \dots\ y_1 + (p^{j-1} - 1)p,$$

and
$$(y_1 + \lambda p)^{p^{j-1}} = y_1^{p^{j-1}} + p^{p^j}.Q.$$

Hence the p^j terms of the series (and no other values of z) all satisfy the congruence

$$z^{p^{j-1}} - y_1^{p^{j-1}} \equiv 0 \text{ mod. } p^j.$$

Hence $x = y^{p^{j-1}}$ has $(p-1)$ distinct real values *quâ* p^j or there are $(p-1)$ real roots to the congruence $x^{p-1} - 1 \equiv 0$ mod. p^j. Hence, if fx is any factor of $x^{p-1} - 1$, $fx \equiv 0$ [mod. p^j] will have all its roots real.

For let $fx.f_1x = x^{p-1} - 1$.

Then since $x^{p-1} - 1 \equiv 0$ mod. p^j has all its roots real, and fx and f_1x have no congruence root *quâ* mod. p in common*, if $fx \equiv 0$ to the modulus p^j has not *its full quota*, f_1x will have a *superfluity* of roots, but this has been shown to be impossible.

Now, let $p = mk + 1$. Then $x^k - 1$ is a factor of $x^{p-1} - 1$. Let $\chi_k x$ be the factor of $x^k - 1$, which contains all its primitive roots; this is what I term a *cyclotomic function of the first species* to the index k. $\chi_k x$ being a factor of $x^k - 1$ is a factor of $x^{p-1} - 1$, and will therefore, by what has just been shown, have all its roots real *quâ* the modulus p^j.

Hence a cyclotomic function of the 1st species to the index k contains, as a divisor, any power of any prime number of the form $mk + 1$, and, moreover, if ω is its degree, (where ω represents the *totient* of k), $(mk+1)^j$ will be an ω-fold divisor of the function, that is, will be a divisor thereof corresponding to ω distinct values of the variable of the function, that is, values incongruent with one another *quâ* the modulus p^j.

* For if this were the case two factors of $x^{p-1} - 1$ *quâ* mod. p having two roots in common $x^{p-1} - 1$ would not have its full quota of roots.

The divisors of the cyclotomic function to index h may be divided into two classes, namely, divisors which do not divide the index, which may be called superior or extrinsic divisors, and divisors which divide at the same time the function and its index which may be termed inferior or intrinsic divisors. I shall begin with showing that any prime number extrinsic divisor diminished by unity must contain the index, that is, that if p is an extrinsic divisor and k the index, we must have $p = mk + 1$ which is a reciprocal proposition to the one just established.

If possible let p, any prime such that $p - 1$ does not contain k nor k contain p, be a divisor of the cyclotomic function of the first species $\chi_k x$. And let δ be the greatest common divisor of $p - 1$ and k. Then we shall have $x^\delta - 1 \equiv 0$ mod. p. But we have also $\chi_k x \equiv 0$ mod. p. Hence the resultant of $x^\delta - 1$ and $\chi_k x$ must contain p, but $\dfrac{x^k - 1}{x^\delta - 1}$ contains $\chi_k x$; *à fortiori* therefore the resultant of this and $x^\delta - 1$ will contain p. But this resultant is evidently equal to the value of $\dfrac{x^k - 1}{x^\delta - 1}$ (where $x^\delta = 1$) raised to the power δ, that is, $= \left(\dfrac{k}{\delta}\right)^\delta$ and therefore, *ex hypothesi*, does not contain p.

It has thus been proved that every extrinsic divisor of $\chi_k x$ can only be of the form $mk + 1$.

Next let $k = k_1 p^j$ (k_1 being prime to p) and suppose p to be a divisor of $\chi_k x$.

Then p is a divisor of $(x^{p^j})^{k_1} - 1$ and, therefore, by what has been shown, must be of the form $mk_1 + 1$, unless $x^{p^j} - 1$ contained p in which case since $p^j - 1$ is divisible by $p - 1$, $x - 1$ must contain p and consequently p will be a divisor of $\chi_k 1$.

To find the value of $\chi_k 1$ we may proceed as follows:

Let $k = a^\alpha . b^\beta . c^\gamma . d^\delta . e^\epsilon$. Then the totient of k is

$$a^{\alpha-1} . b^{\beta-1} . c^{\gamma-1} . d^{\delta-1} . e^{\epsilon-1} \begin{Bmatrix} \alpha\beta\gamma\delta\epsilon + \Sigma\alpha\beta\gamma + \Sigma\alpha \\ - \Sigma\alpha\beta\gamma\delta - \Sigma\alpha\beta - 1 \end{Bmatrix},$$

and if we write this $L + M + N - P - Q - R$

$$\chi_k x = \frac{(x^L - 1)(x^M - 1)(x^N - 1)}{(x^P - 1)(x^Q - 1)(x^R - 1)},$$

and so in general the expression for $\chi_k x$, however many the distinct prime factors of k, imitates and follows *pari passu* the expression for the totient of k; and if $L, M, N, \ldots$ be the positive terms and $P, Q, R, \ldots$ be the negative ones in the algebraical representation of that totient, the common theory of vanishing fractions shows that $\chi_k 1 = \dfrac{L . M . N \ldots}{P . Q . R \ldots}$. There are two cases:

(1) When k contains i distinct prime factors, where $i > 1$. In that case supposing a to be one of them and α its index, the index of a in $L.M.N\ldots$ will be

$$\alpha\left\{1+\frac{(i-1)(i-2)}{1.2}+\frac{(i-1)(i-2)(i-3)(i-4)}{1.2.3.4}+\ldots\right\}$$

and in $P.Q.R\ldots$

$$\alpha\left\{(i-1)+\frac{(i-1)(i-2)(i-3)}{1.2.3}\ldots\right\},$$

so that the index in the quotient is $\alpha(1-1)^{i-1}$, that is, is zero. And so for $b, c, \ldots$. Hence $\chi_k 1 = 1$.

(2) When $i=1$ and $k=a^\alpha$, the value of $\chi_k x = \dfrac{x^{a^\alpha}-1}{x^{a^{\alpha-1}}-1}$, and consequently $\chi_k 1 = a$. Hence, when $k=k_1p^j$, and k_1 is not unity, p, if a divisor of $\chi_k x$, must be of the form mk_1+1. Moreover, the case of $k_1=1$ offers no exception to this conclusion, inasmuch as when $k_1=1, p$, (like every other number) comes under the form mk_1+1.

It now remains to show the converse that if $k=k_1p^j$ and $p=mk_1+1$, p will be a divisor of $\chi_k x$.

For the sake of greater simplicity, we may consider apart the case where $k=p^j$. Here $\chi_k x = \dfrac{x^{p^j}-1}{x^{p^{j-1}}-1} = 1 + x^{p^{j-1}} + x^{2p^{j-1}} + \ldots + x^{(p-1)p^{j-1}}$, which, (to modulus p) $\equiv 1 + x + x^2 + \ldots + x^{p-1} \equiv \dfrac{x^p-1}{x-1}$, and, therefore, can only contain p, if x^p-1, and, consequently, $x-1$ contains it. Hence, the only root of $\chi_k x \equiv 0$ [mod. p], for this case is $x=1$.

Moreover, only p itself, and no higher power of p, can be a divisor of the cyclotomic function in question, because

$$\frac{(1+\lambda p)^{p^j}-1}{(1+\lambda p)^{p^{j-1}}-1}=\frac{\lambda p^{j+1}+\ldots}{\lambda p^j+\ldots}=p+Bp^2+Cp^3+\ldots+Lp^{(p-1)p^{j-1}}$$

does not contain p^2*.

To save unnecessary fatigue of attention, about a small matter, to my readers and myself, I will take, as a representative of the general case, $k=k_1p$, $k_1=abc$, $p=mk_1+1$; it will easily be verified that the increase of the number of distinct prime factors a, b, c, or the affection of them or of p with indices, will in no manner affect the course of the demonstration or the validity of the conclusion.

* When $p=2$ and $j=1$ the third term will not be of a higher power in p than the second term in the development of the numerator, so that the conclusion ceases to hold; as ought to be the case for the cyclotomic of the 1st species to the index 2, namely, $x+1$ will obviously contain every power of 2 as a divisor.

In the above *special case*

$$\chi_k x = \frac{(x^{abcp}-1)(x^{ab}-1)(x^{ac}-1)(x^{bc}-1)(x^{ap}-1)(x^{bp}-1)(x^{cp}-1)(x-1)}{(x^{abc}-1)(x^{abp}-1)(x^{acp}-1)(x^{bcp}-1)(x^{a}-1)(x^{b}-1)(x^{c}-1)(x^{p}-1)}.$$

Let now $x^{k_1}-1=0$, so that $x^p = x$. Then obviously $\chi_k x = \dfrac{x^{abcp}-1}{x^{abc}-1} = p.$ Hence the resultant of $\chi_{k_1}x$ and $\chi_k x$ is $p^{\tau(k_1)}$ ($\tau(k_1)$ meaning the totient of k_1). Consequently since $\chi_{k_1}x \equiv 0$ [mod. p] has all its roots real, one root at least of $\chi_k x \equiv 0$ [mod. p] will be a root of the preceding congruence.

It will be noticed that if instead of $\chi_{k_1}x$ we took $\chi_{k'_1}x$ where k'_1 is a factor of k_1 it would not be true that the resultant of it and $\chi_k x$ would contain p.

For example, if $k'_1 = ab$ and $x^{k'_1}-1=0$ we should have

$$\chi_k x = \frac{x^{abcp+1}-1}{x^{abc}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1} = \frac{p}{p} = 1.$$

Or again, if $k'_1 = a$ and $x^{k'_1}-1=0$ we should have

$$\chi_k x = \frac{x^{abcp}-1}{x^{abc}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1}\cdot\frac{x^{ac}-1}{x^{acp}-1}\cdot\frac{x^{ap}-1}{x^{a}-1} = p\cdot\frac{1}{p}\cdot\frac{1}{p}\cdot p = 1$$

as before. So that the resultant instead of being p would, in each case, be 1, and consequently $x^k-1\equiv 0$ [mod. p] and $x^{k'_1}-1\equiv 0$ [mod. p] could not have a root in common. And so in general it may be shown that if $k = k_1 p^j$ and $k'_1 = \dfrac{k_1}{\delta}$ the resultant of $x^{k'_1}-1$ and $\chi_k x$ is 1, except when $\delta = 1$ in which case it is p.

Hence the roots of $\chi_k x \equiv 0$ [mod. p] are to be sought not among all the roots of $x^{k_1}-1\equiv 0$ [mod. p], but exclusively among only such of them as belong to the congruence $\chi_{k_1}x \equiv 0$ [mod. p].

We have seen that if p, a prime number, is an extrinsic divisor of a cyclotomic function to the index k, any power of p is also a divisor of the function. On the contrary, if p is an intrinsic divisor it will appear that p^2 cannot (and consequently no higher power of p than the 1st, can) be a divisor. For if x satisfies the congruence $\chi_{k_1}x\equiv 0$ [mod. p] we must have $x^{k_1} = 1+\lambda p$ and $x^p = x^{mk_1}.\,x = (1+mp)\,x$, where m represents a series of ascending powers of p. Hence

$$\chi_k x = \frac{x^{k_1 p}-1}{x^{k_1}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1}\cdot\frac{x^{ac}-1}{x^{acp}-1}\cdot\frac{x^{bc}-1}{x^{bcp}-1}\cdot\frac{x^{ap}-1}{x^{a}-1}\cdots,$$

where the first factor, being equal to $x^{k_1(p-1)} + x^{k_1(p-2)} + \ldots + 1$, will be of the form $p(1+Pp)$, P being a series containing only positive powers of p. Again,

$$\frac{x^{ab}-1}{(1+Qp)\,x^{ab}-1} = 1 + \frac{Qpx^{ab}}{1-x^{ab}} + \frac{Q^2p^2x^{2ab}}{(1+x^{ab})^2} + \ldots = 1 + Q_1 p$$

where Q_1 is an infinite series containing positive powers only of p and x.

In like manner $\dfrac{x^{ap}-1}{x^a-1}=\dfrac{(1+Rp)\,x^a-1}{x^a-1}=1+R_1p$ where R_1 (like R) is an infinite series of positive powers of p and x, and so for each separate factor.

On multiplying the product of these infinite series by $p(1+Pp)$, we shall necessarily obtain a finite series of the form $p(1+Gp)$. Consequently, the cyclotomic function will divide by p but not by p^2. And we might have used this method exclusively to have established the fact of the first power of p, under the conditions presupposed, being a divisor of the function. This method serves also to establish directly that *every* root of $\chi_{k_1}x\equiv 0$ is a root of the congruence $\chi_k x\equiv 0$ [mod. p]. And we thus see that the intrinsic divisor, when it exists, is a $\tau(k_1)$-fold divisor of the cyclotomic function.

When k is the index to a cyclotomic function, and $k=k_1p^j$, where p is a prime not contained in k, let us agree to call k_1 the sub-index to p. Then, from what precedes, we may draw the conclusion that a cyclotomic function of the first species has never more than one intrinsic divisor, which, if it exists, is the greatest prime number contained in the index, but is such only in the case when diminished by unity, it contains its own sub-index, (a conclusion necessarily satisfied when the index is a prime, for then its sub-index is unity), and, moreover, that the first power only of such intrinsic divisor, when it exists, is a divisor of the function.

It being true and capable of easy demonstration, that when a rational integer function contains, as a divisor, each of two numbers prime to one another, their product will also be a divisor of the function, it follows that any number, each of whose prime factors, diminished by unity, contains the index and also every such number multiplied by the highest prime number which is contained in the index (provided that when diminished by unity that prime contains its own sub-index) is a divisor of a cyclotomic function of the first species. This, as I have said, is only another name for that irreducible factor of a binomial x^k-1 whose degree in x is the *totient* of k.

When the cyclotomic function of any species is made homogeneous by the introduction of a second variable y, relatively prime to x, it becomes a form, (in the technical sense of the word), and may then very conveniently be designated a *cyclo-quantic*.

Title 2. *Cyclotomic Functions of the Second Species* (*Conjugate Class*)*. I pass on to the theory of the divisors of the function which has for roots the sum of the binomial (*zweigliedrig*) groups of the primitive roots of x^k-1,

* When, in the matter comprehended under this title, by inadvertence, cyclotomic functions of the second species are spoken of without a qualification annexed, it is to be understood, in all cases, that only those of the conjugate class or, in other words, those whose roots are all real, are intended. For brevity I shall usually call this class of functions cyclotomics of the second *sort*.

or, in other words, all the distinct values, $\frac{1}{2}\tau(k)$ in number, of $2\cos\frac{2\lambda\pi}{k}$ where λ is any number less than $\frac{1}{2}k$ and prime to k.

Such a function, in which the coefficient of the highest power of the variable is supposed to be unity, I call a cyclotomic function, or simply a cyclotomic, of the second species and conjugate class to the index k. It may be found most readily by dividing the corresponding one of the first species, whose variable say is x, by $x^{\frac{1}{2}\tau(k)}$, substituting u for $x+\frac{1}{x}$, and applying for successive values of m the trigonometrical formula for expressing $\cos m\theta$ in terms of powers of $\cos\theta$, except when the index is a prime number, in which case the function in u is given more expeditiously at once by the well-known formula

$$u^m + u^{m-1} - \frac{m-1}{1}u^{m-2} - \frac{m-2}{1}u^{m-3} + \frac{(m-2)(m-3)}{1\,.\,2}u^{m-4} + \frac{(m-3)(m-4)}{1\,.\,2}u^{m-5} - \ldots,$$

which last coefficient, in the French edition of the *Disq. Arith.*, 1807, it may be worth noting, is written erroneously $\frac{(m-1)(m-4)}{1\,.\,2}$.

I have thought it would be useful and convenient for many of my readers to be able to see before them the functions of the two sorts, and I accordingly annex a table of their values for all indices up to 36 inclusive.

To the index 1 or 2, the cyclotomic of the second species has no existence. Those of the first species to the index 1 or 2, and of the second to the index 3, 4 or 6 are linear, and of course as forms, have no arithmetical properties, but contain every number as a divisor, linear forms being, as it were, the protoplasm out of which the higher forms are organized.

Table of Cyclotomic Functions of the first species and the conjugate class of the second species for all values of the index from 1 *to* 36 *inclusive.*

Index	1st Species	2nd Species, Conjugate Class
1	$x-1$	
2	$x+1$	
3	x^2+x+1	$u+1$
4	x^2+1	u
5	$x^4+x^3+x^2+x+1$	u^2+u-1
6	x^2-x+1	$u-1$
7	$x^6+x^5+x^4+x^3+x^2+x+1$	u^3+u^2-2u-1
8	x^4+1	u^2-2
9	x^6+x^3+1	u^3-3u-1
10	$x^4-x^3+x^2-x+1$	u^2-u+1

Index	1st Species	2nd Species, Conjugate Class
11	$x^{10}+x^9+\ldots+x+1$	$u^5+u^4-4u^3-3u^2+3u+1$
12	x^4-x^2+1	u^2-3
13	$x^{12}+x^{11}+\ldots+x+1$	$u^6+u^5-5u^4-4u^3+6u^2+3u-1$
14	$x^6-x^5+x^4-x^3+x^2-x+1$	u^3-u^2+2u+1
15	$x^8-x^7+x^5-x^4+x^3-x+1$	$u^4-u^3-4u^2+4u+1$
16	x^8+1	u^4-4u^2+2
17	$x^{16}+x^{15}+\ldots+x+1$	$u^8+u^7-7u^6-6u^5+15u^4+10u^3-10u^2-4u+1$
18	x^6-x^3+1	u^3-3u+1
19	$x^{18}+x^{17}+\ldots+x+1$	$u^9+u^8-8u^7-7u^6+21u^5+15u^4$ $+10u^3-10u^2+5u+1$
20	$x^8-x^6+x^4-x^2+1$	u^4-5u^2+5
21	$x^{12}-x^{11}+x^9-x^8+x^6-x^4$ $+x^3-x+1$	$u^6-u^5-6u^4+6u^3+8u^2-8u+1$
22	$x^{10}-x^9+\ldots-x+1$	$u^5-u^4-4u^3+3u^2-3u+1$
23	$x^{22}+x^{21}+\ldots+x+1$	$u^{11}+u^{10}-10u^9-9u^8+36u^7+28u^6-56u^5$ $-35u^4+35u^3+15u^2-6u-1$
24	x^8-x^4+1	u^4-4u^2+1
25	$x^{20}+x^{15}+x^{10}+x^5+1$	$u^{10}-10u^8+35u^6+u^5-50u^4-5u^3+25u^2-5u-1$
26	$x^{12}-x^{11}+\ldots-x+1$	$u^6-u^5-5u^4+4u^3+6u^2-3u-1$
27	$x^{18}-x^9+1$	$u^9-9u^7+27u^5-30u^3+9u-1$
28	$x^{12}-x^{10}+x^8-x^6+x^4-x^2+1$	$u^6-7u^4+14u^2-7$
29	$x^{28}+x^{27}+\ldots+x+1$	$u^{14}+u^{13}-13u^{12}-12u^{11}+66u^{10}+55u^9-165u^8$ $-120u^7+210u^6+126u^5-126u^4-56u^3+28u^2$ $+7u-1$
30	$x^{16}-x^{14}+x^{10}-x^8+x^6-x^2+1$	$u^8-9u^6+26u^4-26u^2+1$
31	$x^{30}+x^{29}+\ldots+x+1$	$u^{15}+u^{14}-14u^{13}-13u^{12}+78u^{11}+66u^{10}-220u^9$ $-165u^8+330u^7+210u^6-252u^5-126u^4+84u^3$ $+28u^2-4u-1$
32	$x^{16}+1$	$u^8-8u^6+20u^4-16u^2+2$
33	$x^{20}-x^{19}+x^{17}-x^{16}+x^{14}-x^{13}$ $+x^{11}-x^{10}+x^9-x^7+x^6-x^4$ $+x^3-x+1$	$u^{10}-u^9-10u^8+10u^7+34u^6-34u^5$ $-43u^4+43u^3+12u^2-12u-1$
34	$x^{16}-x^{15}+x^{14}-\ldots+x^2-x+1$	$u^8-u^7-7u^6+6u^5+15u^4-10u^3-10u^2+4u+1$
35	$x^{24}-x^{23}+x^{19}-x^{18}+x^{17}-x^{16}$ $+x^{14}-x^{13}+x^{12}-x^{11}+x^{10}-x^8$ $+x^7-x^6+x^5-x+1$	$u^{12}-u^{11}-12u^{10}+11u^9+54u^8-43u^7-113u^6$ $+71u^5+110u^4-46u^3-40u^2+8u+1$
36	$x^{12}-x^6+1$	$u^6-6u^4+9u^3-3$

A very good test (or, in most cases, pair of tests) of the correctness of the figures is to write $u=\pm 2$* corresponding to $x=\pm 1$ and see if the values for the same index agree. Our interest will presently be concentrated on the single entry in the right hand column, that which expresses the conjugate class of the second species of cyclotomic to the index 9, but the function for the neighbouring case of the index 8 is worthy of arresting the reader's attention for a moment, inasmuch as the general theory of cyclotomic divisors applied to it will be seen to supply an instantaneous proof that all prime

* And a further double test is given by taking $u=0$, $x=i$, as we ought to find $\chi i=\pm i^{\frac{1}{2}\tau k}\psi 0$.

numbers of the form $8n \pm 1$, and no other prime numbers have 2 for a quadratic residue*.

It is hardly necessary to observe that, when the index is a prime number, it may be duplicated without affecting the character of either set of functions, the only effect produced thereby being the entirely unimportant one of a change in the sign of the variable.

The formula which I have employed for computing $\cos n\theta$ is that which, beginning with the *highest* power of $\cos\theta$, admits of a uniform scheme of setting down the work, which is not the case when the series is started from the other end. It, and the series used for $\dfrac{\sin\frac{p\theta}{2}}{\sin\frac{\theta}{2}}$, also required for my purposes, may be obtained by a much simpler method than any I have seen given in the text-books as follows.

In general, the denominator of $\frac{1}{a_1 -}\,\frac{1}{a_2 -}\,\ldots\,\frac{1}{a_n}$, say the procumulant $[a_1, a_2, \ldots a_n] = A_0 - A_1 + A_2$ etc., where A_0 is $a_1 . a_2 . \ldots . a_n$, A_1 is the sum of the quotients of A_0 by any pair of consecutive elements $a_i . a_{i+1}$, A_2 of the quotients of A_0 by the product of any two such pairs as $a_i . a_{i+1} . a_j . a_{j+1}$, and so on. If we call the *number* of such quotients in A_i, $D_i n$, it is obvious that

$$D_{i+1} n = \Sigma_{t=0}^{t=n-2} D_i t.$$

Hence $D_0 n = 1$, $D_1 n = n - 1$, $D_2 n = (n-2)\dfrac{n-3}{2}$, $D_3 n = \dfrac{(n-3)(n-4)(n-5)}{1.2.3}$, and so on.

On making $a_1 = a_2 = \ldots = a_n = 2\cos\theta$, it will immediately be seen that the procumulant $[2\cos\theta,\ 2\cos\theta \ldots$ to n terms$]$ expresses $\dfrac{\sin(n+1)\theta}{\sin\theta}$, because, calling this u_n, the equation in difference for finding it is

$$u_{n+1} = 2\cos u_n - u_{n-1} \text{ and } u_0 = 1.$$

Consequently

$$\frac{\sin(n+1)\theta}{\sin\theta} = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{(n-1)(n-2)}{2}(2\cos\theta)^{n-4}\ldots.$$

Hence $2\cos n\theta = 2\left(\dfrac{\sin(n+1)\theta}{\sin\theta} - \cos\theta\,\dfrac{\sin n\theta}{\sin\theta}\right) = (2\cos\theta)^n - n(2\cos\theta)^{n-2}$ $+\, n\,\dfrac{n-3}{2}(2\cos\theta)^{n-4}\ldots$. Also, $\dfrac{\sin\frac{2n+1}{2}\theta}{\sin\frac{\theta}{2}} = \dfrac{\sin(n+1)\theta}{\sin\theta} + \dfrac{\sin n\theta}{\sin\theta} = (2\cos\theta)^n$ $+ (2\cos\theta)^{n-1} - n(2\cos\theta)^{n-2} - (n-1)(2\cos\theta)^{n-4} + \ldots$†

* So, under the third Title, it will be found that $u^2 + 2$ is a *non-conjugate* cyclotomic of the second species to the index 8, of which, according to the general cyclotomic law, the odd prime divisors are of the form $8m+1$ or $8m+3$.

† This expansion Gauss (*Rech. Arith.*, Paris, 1757, p. 431) suggests deriving by means of the

Writing u in place of $2\cos\theta$ these are the two expansions which I have used to express $x^n + \frac{1}{x^n}$ and $\frac{x^{\frac{p-1}{2}} - x^{-\frac{p-1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}$ in terms of powers of $x + \frac{1}{x}$ in calculating the cyclotomics of the 2nd sort whose values are given in the preceding table.

Since $(x^{p-1} - 1)(x^{p+1} - 1) = x^{2p} - x^{p+1} - x^{p-1} + 1$, if, for convenience, we write $x + \frac{1}{x} = u = 2\cos\theta$, it is evident that $\cos p\theta - \cos\theta$, regarded as an algebraical function of $\cos\theta$, will contain all the cyclotomic functions of the second species (conjugate class) whose indices are divisors of $p-1$ or $p+1$ and in addition to these $\left(x - \frac{1}{x}\right)^2$ or $u^2 - 4$ derived from the factor $x^2 - 1$ which is common to $x^{p-1} - 1$ and $x^{p+1} - 1$, but does not give rise to a cyclotomic of this sort until it is squared; $\cos p\theta - \cos\theta$ is thus a product exclusively of cyclotomics of the second sort.

It is well known that $\cos p\theta - \cos\theta \equiv 0 \pmod{p}$ regarded as a congruence in $\cos\theta$ has the p roots $\cos\theta = 0, 1, 2, 3, \ldots (p-1)$, p being supposed to be a prime number.

But more generally the congruence $\cos p^j\theta - \cos p^{j-1}\theta \equiv 0$ mod. p^j has its full complement of p^j real roots—a theorem, this, which is the analogue of the theorem of Fermat extended to powers of prime numbers put under the form of affirming that $x^{p^j} - x^{p^{j-1}} \equiv 0$ mod. p^j has its full complement of real roots; but, as I do not recall seeing the *cosine* theorem for modulus p^j anywhere stated, and as it is wanted for the theory I am developing, and its truth is not obvious, I shall proceed to prove it. For greater simplicity of notation let us begin with the case where $j = 2$. We have then

$$\cos p^2\theta = (\cos\theta)^{p^2} - p^2\frac{p^2-1}{2}(\cos\theta)^{p^2-2}.(\sin\theta)^2 + \frac{p^2(p^2-1)(p^2-2)(p^2-3)}{1.2.3.4}(\cos\theta)^{p^2-4}.(\sin\theta)^4 \ldots$$

and $$\cos p\theta = (\cos\theta)^p - p\frac{p-1}{2}(\cos\theta)^{p-2}.(\sin\theta)^2 + \frac{p.(p-1)(p-2)(p-3)}{1.2.3.4}(\cos\theta)^{p-4}.(\sin\theta)^4 \ldots$$

where of course all the powers of $(\sin\theta)^2$ are regarded as functions of $\cos\theta$. It will easily be recognized that every coefficient in the first series will be

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exceedingly awkward and unmanageable process indicated by the formula $\frac{\sqrt{(1-\cos n\theta)}}{1-\cos\theta}$, $\cos n\theta$ being previously supposed to be expanded in terms of powers of $\cos\theta$. *Quandoque bonus dormitat Homerus.*

divisible by p^2 with the exception of those terms in which a new multiple of p first makes its appearance among the factors of the denominator, which will lose one power of p; the next coefficient to any such as last named taking up a new factor of p into the numerator, the fraction to which it belongs will recover the lost p and be again divisible by p^2.

The difference, therefore, between the two series *quâ mod.* p^2 will be

$$(\cos\theta)^{p^2} - (\cos\theta)^p$$

$$+\frac{p^2(p^2-1)\dots(p^2-2p+1)}{1.2\dots 2p}(\cos\theta)^{p^2-2p}.(\sin\theta)^{2p} - p\frac{p-1}{2}(\cos\theta)^{p-2}(\sin\theta)^2$$

$$+\frac{p^2(p^2-1)\dots(p^2-4p+1)}{1.2\dots 4p}(\cos\theta)^{p^2-4p}.(\sin\theta)^{4p}$$

$$-\frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\cos\theta)^{p-4}(\sin\theta)^4$$

....

It may be shown that every pair of terms in the above is divisible by p^2 for all real values of $\cos\theta$.

(1) $(\cos\theta)^{p^2} - (\cos\theta)^p$ contains p^2 by Fermat's extended theorem.

(2) *Quâ* p, $(\cos\theta)^{p^2-2p} \equiv (\cos\theta)^{p-2}$ and $(\sin\theta)^{2p} \equiv (\sin\theta)^2$.

Hence *quâ* p^2, the sum of the second pair of terms

$$\equiv p\frac{p-1}{2}\left\{\frac{(p+1)(p-2)(p-3)\dots(p^2-2p+1)}{2.3\dots(2p-1)} - 1\right\} \equiv 0$$

$$\equiv p\frac{p-1}{2}\left\{\frac{2.3\dots(2p-1)}{2.3\dots(2p-1)} - 1\right\} \equiv 0.$$

(3) *Quâ* p, inasmuch as

$p^2-5p+4=(p-1)(p-4)$, $(\cos\theta)^{p^2-4p} \equiv (\cos\theta)^{p-4}$ and $(\sin\theta)^{4p} \equiv (\sin\theta)^4$.

Also, $\quad p^n - 1 \equiv p-1,\ p^2-2 \equiv p-2$ and $p^2-3 \equiv p-3$.

Hence the sum of the 3rd pair of terms *quâ* p^2

$$\equiv \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\left\{\frac{(p^2-4)(p^2-5)\dots(p^2-4p+1)}{4.5\dots(4p-1)}\right\} \equiv 0.$$

And so each pair of terms may be proved to be congruous to zero *quâ* p^2.

The same form of demonstration may be shown to apply to the case of the modulus p^j*, and we may regard as proved the important theorem that $\cos p^j\theta - \cos p^{j-1}\theta \equiv 0$ [mod. p^j] contains the maximum number of roots p. It follows that $\cos p\theta - \cos\theta \equiv 0$ mod. p^j will contain p distinct roots. For, if we make $\theta = p^{j-1}\phi$, the congruence becomes $\cos p^j\phi - \cos p^{j-1}\phi \equiv 0$ mod. p^j,

* The reader will please bear in mind that in the expansion of $(a+b)^{p^j}$ the number of coefficients in which p enters to the power $j, j-1, \dots 2, 1, 0$ respectively is $p^j - p^{j-1}$, $p^{j-1}-p^{j-2}$, ... p^2-p, $p-1$, 2.

which has p^j roots. These roots will separate into p groups of p^{j-1} each, such $\cos(p^{j-1}\phi)$ will be the same for all the $(\cos\phi)$'s in the same group, but different (*quâ* mod. p^j) for any two belonging to distinct groups. For if $\cos\phi_1$ be one of the values regarded as given, and $\cos(p^{j-1}\phi_2) \equiv \cos(p^{j-1}\phi_1)$ mod. p^j,

and
$$\left.\begin{array}{l}\cos(p^{j-1}\phi_2) \equiv \cos\phi_2 \\ \cos(p^{j-1}\phi_1) \equiv \cos\phi_1\end{array}\right\} \text{mod. } p.$$

If, then, we form the series

$$\cos\phi_1,\ \cos\phi_1 + p,\ \cos\phi_1 + 2p,\ \ldots\ \cos\phi_1 + (p^{j-1}-1)\,p,$$

all the values of $\cos\phi_2$ must be included among the terms of this series. Conversely, if we make $\cos\phi_2 = \cos\phi_1 + \lambda p$, we shall have

$$\cos p^{j-1}\phi_2 - \cos p^{j-1}\phi_1 \equiv 0 \text{ mod. } p^j.$$

For, writing q for p^{j-1},

$$\cos q\phi_2 = (\cos\phi_2)^q - q\frac{q-1}{2}(\cos\phi_2)^{q-2}.(\sin\phi_2)^2 + \ldots.$$

If in this development we take the term containing $(\cos\phi_2)^{q-2t}.(\sin\phi_2)^{2t}$, its coefficient will contain q, except in the case where t contains p^i, in which case the coefficient will contain $\frac{q}{p^i}$ but not q, and the index of $(\cos\phi_2)$ and $(\sin\phi_2)^2$ will each contain p^i. Hence, since $\cos\phi_2 = \cos\phi_1 + \lambda p$, and consequently $(\sin\phi_2)^2$ is of the form $(\sin\phi_1)^2 + \Delta p$, it follows that the difference between this term and the corresponding one in the development of $\cos q\phi_1$ will in the one case contain qp and in the other $\frac{q}{p^i}p^{i+1}$, in either case therefore it contains $p.q$, that is, p^j, and consequently making $\cos\phi_2$ equal to any of the p^{j-1} terms of the series, we shall have $\cos(p^{j-1}\phi_2) \equiv \cos(p^{j-1}\phi_1)$ mod. p^j as was to be shown. Hence $\cos p\theta - \cos\theta \equiv 0$ mod. p^j will have p real roots.

Again no algebraical factor of $\cos p\theta - \cos\theta$ can have a *superfluity* of real roots *quâ* mod. p^j, for if it had then by the same reasoning as applied to the cyclotomics of the first species, it would be necessary for p to be contained in the discriminant of $\cos p\theta - \cos\theta$ regarded as a function of $\cos\theta$, but *quâ* mod. p, this is the same as the discriminant of $(\cos\theta)^p - \cos\theta$ in regard to $\cos\theta$ or of $x^p - x$ in regard to x which is the discriminant of $x^{p-1} - 1$ multiplied by the squared resultant of x and $x^{p-1} - 1$, and is therefore a power of $(p-1)$. Hence every algebraical factor of $\cos p\theta - \cos\theta$ *quâ mod.* p^j contains *its full quota* of real roots, that is, as many roots as there are units in its degree.

If then $p = mk + \epsilon$, where $\epsilon = \pm 1$, since $\cos p\theta - \cos\theta$ will contain the cyclotomic of the second sort to the index k, such cyclotomic equivalented to zero [mod. p^j] will have all its roots real, so that $(mk \pm 1)^j$ will be a $\frac{1}{2}\tau(k)$-fold divisor of such function.

As in the case of cyclotomics of the 1st species we may separate the divisors of those of the 2nd sort into intrinsic and extrinsic, according as they are or are not divisors of the index.

First, as regards the extrinsic divisors, we may prove that no other prime numbers except those of the form $k \pm 1$ can be divisors of the 2nd species of cyclotomics to the index k.

To show this I proceed as follows: $\psi_k u$ is contained algebraically in $\dfrac{\sin \frac{k}{2}\theta}{\sin \frac{\theta}{2}}$, and *à fortiori* in its square, that is, in $\dfrac{1-\cos k\theta}{1-\cos\theta}$, so that if $2\cos\theta$ is a value of u, which makes $\psi_k u$ contain p,

$$\cos k\theta \equiv 1 \text{ mod. } p,$$

but also $\cos p\theta \equiv \cos\theta$ mod. p, and if $\dfrac{\sin p\theta}{\sin\theta} \equiv a + bp$,

$$1 = (\cos\theta)^2 + a^2(1-\cos\theta)^2 + cp,$$

and $(1-a^2)(1-\cos\theta)^2 = cp$, and, therefore, $a = \pm 1$ mod. p, for $\dfrac{1-\cos k\theta}{1-\cos\theta}$ does not contain $(1-\cos\theta)$, and if $(1-\cos k\theta)$ contains $1-(\cos\theta)^2$, which is only the case when k is even, $\dfrac{1-\cos k\theta}{1-(\cos\theta)^2}$, does not contain either $1-\cos\theta$ or $1+\cos\theta$, and, therefore, $\psi_k u$, which, in that case, is contained in $\dfrac{1-\cos k\theta}{1-(\cos\theta)^2}$, will not contain either $1-\cos\theta$ or $1+\cos\theta$.

Hence $1-(\cos\theta)^2$ is not zero, and, consequently, $a \equiv \pm 1$, and, therefore, $\dfrac{\sin p\theta}{\sin\theta} \equiv \pm 1$ mod. p.

Hence, either

$$\left.\begin{array}{l} \cos(p-1)\theta = \cos p\theta \,.\, \cos\theta + \dfrac{\sin p\theta}{\sin\theta}(\sin\theta)^2 \equiv (\cos\theta)^2 + (\sin\theta)^2 \equiv 1 \\ \text{or} \\ \cos(p+1)\theta = \cos p\theta \,.\, \cos\theta - \dfrac{\sin p\theta}{\sin\theta}(\sin\theta)^2 \equiv (\cos\theta)^2 + (\sin\theta)^2 \equiv 1 \end{array}\right\} \text{mod. } p,$$

and writing $\epsilon = \pm 1$, we must have

$$\cos(p-\epsilon)\theta \equiv 1 \text{ mod. } p.$$

If possible, let $(p-\epsilon)$ not contain k, and δ (less than k) be the greatest common measure of k and $(p-\epsilon)$.

Let $\lambda(p-\epsilon) - \mu k = \delta$. Then

$$\left.\begin{array}{ll} \cos\lambda(p-\epsilon)\theta \equiv 1 & \dfrac{\sin\lambda(p-\epsilon)\theta}{\sin\theta} \equiv 0 \\[2ex] \cos\mu k\theta \equiv 1 & \dfrac{\sin\mu k\theta}{\sin\theta} \equiv 0 \end{array}\right\} \text{mod. } p.$$

Hence $\cos\delta\theta \equiv 1$ mod. p, and, consequently, the resultant of $\psi_k u$ and $\cos\delta\theta - 1$ in respect to $\cos\theta$ must contain p. But $\psi_k u$, when δ is any divisor of k other than k itself, is an algebraical factor of $\dfrac{\cos k\theta - 1}{\cos\delta\theta - 1}$ *à fortiori*, therefore, the resultant of this last named function of $\cos\theta$ and of $\cos\delta\theta - 1$ must contain p.

This resultant will be the product of the values of $\dfrac{\cos k\theta - 1}{\cos\delta\theta - 1}$ for every root of $\cos\delta\theta - 1$, it is therefore the δth power of the value of the vanishing fraction $\dfrac{\cos\mu\phi - 1}{\cos\phi - 1}$ $\left[\text{where } \mu = \dfrac{k}{\delta}\right]$ when $\cos\phi = 1$, that is, of $\left(\dfrac{\sin\frac{\mu}{2}\phi}{\sin\frac{\phi}{2}}\right)^2$ when $\phi = 0$. The resultant is, therefore, $\left(\dfrac{k}{\delta}\right)^{2\delta}$, which cannot contain p, since, by hypothesis, p is not contained in k. Hence $p - \epsilon = mk$, or $p = mk \pm 1$. So that, for the extrinsic divisors, the law, both as regards what numbers are and what are not such divisors, is precisely the same as for the cyclotomics of the first species, except that $mk \pm 1$ takes the place of $mk + 1$.

Next, for the intrinsic divisors. Suppose p to be any such, and that $k = k_1 p^j$, where k_1 is prime to p. Then p is a divisor of $\cos k_1 p^j\theta - 1$, and, therefore, by what has been shown, must be of the form $mk_1 \pm 1$, unless $(\cos p^j\theta - 1)$ contains p, in which case, since

$$\cos p^j\theta = (\cos p^j\theta - \cos p^{j-1}\theta) + (\cos p^{j-1}\theta - \cos p^{j-2}\theta) + \ldots + \cos\theta,$$

$\cos\theta - 1$ must contain p, and, consequently, p must be a divisor of $\psi_k 2$, that is, of $\chi_k 1$, which we have seen is equal to 1, except when $k_1 = 1$. Hence, p must be of the form $mk_1 \pm 1$. To show the converse, that when $k = k_1 p^j$ and $p = mk_1 \pm 1$, p will be a divisor of $\psi_k u$. Taking, first, the case of $k_1 = 1$ or $k = p^j$, $\psi_k u$, for $u = 2$ will be equal to $\chi_k 1$, which, as we have seen, will divide by p, and not by p^2.

To ascertain if there is any other value of u which will make the function divisible by p, I observe that, for this case, $(\psi_k u)^2 = \dfrac{\cos p^j\theta - 1}{\cos p^{j-1}\theta - 1}$, which is of the form $\dfrac{\cos\theta - 1 + Lp}{\cos\theta - 1 + lp}$, and if this function contains p, we must obviously have $\cos\theta \equiv 1$ mod. p.

Proceeding to the more general case where $k = k_1 p^j$ and k_1 is other than unity, taking as I did for the first species the specimen case $k = k_1 p$, $k_1 = abc$, $p = mk_1 \pm 1$, we shall have

$$(\psi_k u) = \frac{(\cos abcp\theta - 1)(\cos ab\theta - 1)(\cos ac\theta - 1)(\cos bc\theta - 1)(\cos ap\theta - 1)(\cos bp\theta - 1)(\cos cp\theta - 1)(\cos\theta - 1)}{(\cos abc\theta - 1)(\cos abp\theta - 1)(\cos acp\theta - 1)(\cos bcp\theta - 1)(\cos a\theta - 1)(\cos b\theta - 1)(\cos c\theta - 1)}$$

If, now, $\cos k_1\theta - 1 = 0$, and we suppose $\cos\theta$ to be a root of $\psi_k u = 0$, $\cos p\theta = \cos(\pm\theta) = \cos\theta$, $(\psi_k u)^2$ becomes equal to $\dfrac{\cos pk_1\theta - 1}{\cos k_1\theta - 1} = p$, and paying no attention to the algebraical sign which is immaterial to our object, we shall have $\psi_k u = p$, and the resultant of $\psi_{k_1} u$ and $\chi_k u$ will be $p^{\frac{1}{2}\tau k_1}$, and, consequently, since $\chi_{k_1} u \equiv 0$ mod. p has all its roots real, one of them, at all events, will belong to $\chi_k u \equiv 0$ mod. p, and precisely in like manner, as in the case for cyclotomics of the 1st species, it may be shown that this reasoning ceases to apply if $\cos\theta$, although satisfying $\cos k_1\theta - 1 = 0$, does not satisfy $\chi_{k_1} u = 0$, in which case the resultant, instead of being a power of p, would become unity, so that the value of $\cos\theta$, satisfying $\cos k_1\theta - 1 \equiv 0$ mod. p, could not be a congruence root of $\chi_k u \equiv 0$ mod. p. Finally, as for the case of the 1st species, it may be shown that *every* congruence root of $\chi_{k_1} u \equiv 0$ [when $k = k_1 p^j$ and $p = mk_1 \pm 1$] will satisfy the congruence $\chi_k u \equiv 0$ mod. p, and that only p, and not p^2, will be a divisor of $\chi_k u$, subject, however, to an exception for the case of $p = 2$, when $k = 2$ or $k = 4$, and also for the case of $p = 2$ and $p = 3$ when $k = 6$*. As regards these intrinsic divisors, it is clear that any root must be the highest prime factor of the index unless its sub-index is 3, in which case it may be 2. It is obvious, then, that except the index is 6 or 12, the second cyclotomic function can have only one intrinsic divisor. When the index is 6, the function is simply $u - 1$, and contains of course *every power* of 2 and 3, as well as every power of $6i \pm 1$ as a divisor.

Leaving out of consideration the three known cyclotomics, whose indices are 3, 4, 6, and the one just referred to, $u^2 - 3$, whose index is 12, we may combine the results obtained into the statement that any number, each of whose factors, diminished or increased by unity, contains the index, and any such number, multiplied by the highest prime number in the index, provided that that number, when increased or diminished by unity, contains its sub-index, and no other numbers but such as satisfy one or the other of these two descriptions, will be a divisor of a non-linear cyclotomic function of the conjugate class of the second species whose index is other than 12. As regards the index 12, any number, whose factors are all of the form $12m \pm 1$, as also the double, treble and sextuple of any such number, will be a divisor of the function.

By way of example let us consider the indices 15, 21, 35.

$\chi_{15}x$ will contain neither 3 nor 5, $\psi_{15}x$ will contain 5 but not 3.

$\chi_{21}x$ will contain 7 but not 3, $\psi_{21}x$ will contain 7 but not 3.

$\chi_{35}x$ will contain neither 5 nor 7, $\psi_{35}x$ will contain neither 5 nor 7.

* I may probably show this in full in a future note. But since the vast and dazzling theory for cyclotomics of all species, with an indefinite number of classes to each species, has loomed into view, I must confess to a certain feeling of impatience as regards working out these small details for a single class of a single species. The inordinately augmented amplitude of the subject calls for some broader method of treatment.

To find a value of x which makes $\psi_{15}x$ contain 5, write $\psi_3 u = u + 1 \equiv 0$ mod. 5, then $u \equiv -1$.

To find values of x which make $\psi_{21}x$ contain 7, write $u + 1 \equiv 0$ mod. 7, then $u \equiv 6$; and to find values of x which make $\chi_{21}x$ contain 7, write $x^2 + x + 1 \equiv 0$ mod. 7, then $x \equiv 2$ or $x \equiv 4$.

On turning to the table p. [327] it will be seen that

$$\psi_{15}(-1) = 1 + 1 - 4 - 4 + 1 = -5,$$

$$\psi_{21}(-1) = 1 + 1 - 6 - 6 + 8 + 8 + 1 = 7,$$

$$\psi_{21}2 = \left.\begin{array}{l} 4096 + 512 + 64 + 8 + 1 \\ \quad - 2048 - 256 - 16 - 2 \end{array}\right\} = 4681 - 2322 = 2359 = 7 \,.\, (16 \,.\, 21 + 1),$$

and of course since $\chi_{21}x^2$ contains $\chi_{21}x$ as an algebraical factor, $\chi_{21}4$ will also contain the intrinsic divisor 7 on the general principle that if λ be any number prime to k, $\chi_k x^\lambda$ must contain $\chi_k x$ as an algebraical factor, as admits of easy demonstration.

Also $\psi_{21}6 \equiv \psi_{21}\left(2 + \frac{1}{2}\right) \equiv \chi_{21}2$ [mod. 7] will also contain 7. Lastly, to mod. 5, for $x = 0, 1, 2, 3, 4$

$$\chi_{35}(x) \equiv 1, 1, 1, 1, 1; \quad \psi_{35}(x) \equiv 1, 1, 1, 1, 1;$$

and to mod. 7, for $x = 0, 1, 2, 3, 4, 5, 6$,

$$\chi_{35}(x) \equiv 1, 1, 1, 1, 1, 1, 1; \quad \psi_{35}(x) \equiv 1, 2, 1, 3, 3, 1, 2;$$

so that neither 5 nor 7 is a divisor of either function to index 35.

Title 3. *On Cyclotomic Functions of Any Species and Class.* The cyclotomic functions, called by me, of the second *sort* or conjugate class of the second species discussed under the preceding title, constitute the leading class of a much more general kind of binomial (*zweigliedrig*) cyclotomics, which it would mislead were I to suppress all allusion to.

Suppose k to contain θ distinct odd prime factors, then we know that the number of square roots of unity to the modulus k is 2^θ, except when k is divisible by 4, in which case it is $2^{\theta+1}$, or $2^{\theta+2}$, according as $\frac{k}{8}$ is fractional or integer, or, setting apart unity, the number remaining is $2^\theta - 1$, $2^{\theta+1} - 1$, $2^{\theta+2} - 1$ in the three cases respectively. Let $\sqrt{1}$ (one of the totitives to k) denote any *specific one* of these square roots. Then, if we call ρ any primary kth root of unity and make $x = \rho + \rho^{\sqrt{1}}$, we shall obtain a completely integer function of the degree $\frac{1}{2}\tau k$ in x, which may be called a binomial cyclotomic. When k is divisible by 4, one value of $\sqrt{1}$ will be $\frac{k}{2} + 1$, and the value of $\rho + \rho^{1+\frac{k}{2}}$ being zero, the cyclotomic function that ought to be, degenerates

into a power of x. Hence, when k is not divisible by 4, the number of binomial cyclotomics is $2^{\theta}-1$, when it is divisible by 4, $2^{\theta+1}-2$, or the double of the former value, and when by 8, $2^{\theta+2}-2$.

All these binomial cyclotomics will be found to possess similar properties to those which have been demonstrated under Title 2 concerning their leading class, as the annexed examples will serve to demonstrate, where the odd prime extrinsic factors it will be seen are of the form $mk+1$ or $mk+\sqrt{1}$; that is to say, in respect to the index, are congruous to one or the other of the *primordinal* totitives 1 and $\sqrt{1}$ where the latter quantity has a definite value for each of the cyclotomics in question.

Thus, suppose $k=15$, the square roots of unity (*quâ* 15) are ± 1, ± 4. Let $\sqrt{1}=4$, and make $x=\rho+\rho^4$, then it will be found that $x^4-x^3+2x^2+x+1$ will contain the four roots of x and all the odd prime divisors of this function are of the form $15m+1$, 4.

Or, again, let $\alpha=\rho+\rho^{11}$, then it will be found that x is a root of the function $x^4+x^3+x^2+x+1$, the prime factors of which, other than 5, are of the form $15m+1$, 11, which is, in effect, the same as the form $5m+1$.

Again, let $k=20$. The values of $\sqrt{1}$ [mod. 20] are ± 1, ± 9. If we were to put $x=\rho+\rho^{11}$, its value would be zero, but writing $x=\rho+\rho^9$, we shall find it will be the root of x^4+3x^2+1, all the prime factors of which, other than the intrinsic one 5, are of the form $20m+1$, 9*.

We may now proceed to generalize these results by considering cyclotomics of every possible numerosity of grouping for a given index, and of every possible order of conjunction for a given numerosity—in a word, we are brought face to face with the most general theory of ν-nomial cyclotomic functions†.

I have accordingly calculated cyclotomic functions for the cases following:

$k=15$	$\mu=2$	$\nu=4$
$k=21$	$\mu=4$	$\nu=3$
	$\mu=3$	$\nu=4$
	$\mu=2$	$\nu=6$
$k=26$	$\mu=4$	$\nu=3$
	$\mu=2$	$\nu=6$
$k=28$	$\mu=4$	$\nu=12$
	$\mu=2$	$\nu=6$
$k=25$	$\mu=5$	$\nu=4$
$k=33$	$\mu=5$	$\nu=4$
	$\mu=4$	$\nu=5$
	$\mu=2$	$\nu=10$

* If $k=8$ and we take $x=\rho+\rho^3$ it will be a root of x^2+2 of which the odd extrinsic factors will be of the form $8m+1$, 3.

† All the species with their several classes here referred to form but a single genus of cyclotomic functions. The second genus will arise from the subdivision of groups into smaller groups and so on continually.

Understanding by the "totitives" of k the numbers less than k and prime to it, these totitives may be arranged in (among others) the natural groups hereunder written.

Totitives to 15 for $\mu = 2$, $\nu = 4$

1	4	11	14
2	7	8	13

„ to 21 for $\mu = 4$, $\nu = 3$

1	4	16
2	8	11
5	17	20
10	13	19

„ „ for $\mu = 3$, $\nu = 4$

1	8	13	20
2	5	16	19
4	10	11	17

„ „ for $\mu = 2$, $\nu = 6$

1	4	5	16	17	20
2	8	10	11	13	19

„ to 26 for $\mu = 4$, $\nu = 3$

1	3	9
5	15	19
7	11	21
17	23	25

„ „ for $\mu = 3$, $\nu = 4$

1	5	21	25
3	11	15	23
7	9	17	19

„ to 28 for $\mu = 4$, $\nu = 3$

1	9	25
3	27	19
5	17	13
11	15	23

„ „ for $\mu = 2$, $\nu = 6$

1	3	9	19	25	27
8	10	11	17	18	23

„ to 25 for $\mu = 5$, $\nu = 4$

1	7	18	24
2	11	14	23
3	4	21	22
6	8	17	19
9	12	13	16

To save space, I omit the groupings to $k=33$.

If, in any of the above tables, we call the totitives of the several rows,

$$\begin{array}{l} \tau_{1,1},\ \tau_{1,2}\ \dots\ \tau_{1,\nu} \\ \tau_{2,1},\ \tau_{2,2}\ \dots\ \tau_{2,\nu} \\ \dots\dots\dots\dots\dots\dots \\ \tau_{\mu,1},\ \tau_{\mu,2}\ \dots\ \tau_{\mu,\nu} \end{array}$$

and if ρ be a primitive root of x^k-1, and we write $R_\theta=\rho^{\tau_{\theta,1}}+\rho^{\tau_{\theta,2}}+\dots\rho^{\tau_{\theta,\nu}}$, R_1, R_2, ... R_μ will be the roots of a cyclotomic of the νth species to the index k, or, as we may say, the index k and *nome* ν.

The values of the cyclotomic functions may be found most easily by calculating all the values of σ_i (the sum of the ith powers of its roots), from $i=1$ to $i=\mu$ where $\mu=\dfrac{\tau(k)}{\nu}$.

The value of $X_{k,\nu}$ will then be the sum of the terms not containing negative powers of x in the development of $x^\mu\left\{e^{-\frac{\sigma_1}{x}-\frac{\sigma_2}{2x^2}-\dots-\frac{\sigma_\mu}{\mu x^\mu}}\right\}$.

It will, of course, be recognized that the first row of numbers (the primordinal totitives, as we may term them) in any of the foregoing natural schemes of decomposition of the kth primitive roots of unity into groups are νth roots (not necessarily comprising any primitive root) of unity in respect to the index k as modulus.

The values of the cyclotomics are exhibited in the annexed table.

Index	Nome	Cyclotomic function	Primordinal Totitives
15	4	x^2-x-1	1, 4, 11, 14
21	3	$x^4-x^3-x^2-2x+4$	1, 4, 6
,,	4	x^3-x^2-2x+1	1, 8, 13, 20
,,	6	x^2-x-5	1, 4, 5, 16, 17, 20
26	3	$x^4-x^3+2x^2+4x+3$	1, 3, 9
,,	4	x^3-x^2-4x-1	1, 5, 21, 25
28	3	x^4-3x^2+4	1, 9, 25
,,	6	x^2-7	1, 3, 9, 19, 25, 27
25	4	$x^5-10x^3+5x^2+10x+1$	1, 7, 18, 24
33	4	$x^5-x^4-4x^3+3x^2+3x-1$*	± 1, ± 10
,,	5	$x^4-x^3-2x^2-3x+9$	1, -2, 4, -8, 16
,,	10	x^2-x-8	± 1, ± 2, ± 4, ± 8, ± 16

In each of the above cases calling the index k, its totient $\mu\nu$, the nome ν and the primordinal totitives θ_1, θ_2 ... θ_ν it will be found that all the *odd* extrinsic prime number divisors (that is, primes dividing the function but not its index) are of the form $mk+\theta_1$, θ_2, ... θ_ν.

* The values of σ_2, σ_3, σ_4, σ_5 in this case follow the noticeable progression 9, 4, 25, 16.

Here, for the present, I must be content to leave this great theory, or I should be in danger of never finding my way back from it to the original object of the memoir which, although its parent, it transcends in importance; for Bachmann's work, as it seems to me, gives proof, that Cyclotomy is to be regarded not as an incidental application, but as the natural and inherent centre and core of the arithmetic of the future.

Remark on the intrinsic divisors of cyclotomic functions of the 1*st species.*

It has been seen that if $k = \frac{p-1}{m} p^{j-1} = k_1 p^{j-1}$, $\chi_k x \equiv 0$ *mod.* p^j has all its roots the same as those of $\chi_{k_1} x \equiv 0$ mod. p and does not contain p^2. If, then, we make j successively 0, 1, 2 ... $j-1$ it will follow that

$$\chi_{k_1}, \chi_{k_1 p}, \chi_{k_1 p^2}, \dots \chi_{k_1 p^{j-1}}$$

will each contain p, but only in the first power for the same τk_1 values of x. Hence $x^{\frac{(p-1)p^{j-1}}{m}} - 1$, which contains all the above written cyclotomics, will contain p^j, so that $x^{\frac{\tau p^j}{m}} - 1 \equiv 0$ mod. p^j will have $\tau \left(\frac{p-1}{m}\right)$ primitive roots, and it is easy to see that $x^{\frac{k}{n}} - 1$ will not have any congruence root in common with $x^{k_1} - 1$ in respect to the modulus p^j.

The theory of intrinsic divisors, it will thus be seen, contains within itself the whole theory of primitive roots, which I notice because it induces me to withdraw the remark made in a previous footnote that the exact determination of the properties of the intrinsic cyclotomic divisors is a matter of comparatively small importance.

Notes to Proem.

1. *On the rational in- and- exscribed triangle to the cubic curve*
$$x^3 - 3xy^2 - y^3 + 3z^3 = 0.$$

In the proem it was, under another form of expression, intimated in advance of what will be shown in the second section of this chapter, that the curve $x^3 + y^3 + Az^3 = 0$ has a correspondence with the curve

$$x^3 - 3xy^2 - y^3 + 3Az^3 = 0,$$

of such a kind that whenever the second equation has a rational solution, the same must be true of the first, so that, for example, on making $A = 1$, the solubility of $x^3 - 3xy^2 - y^3 + 3z^3 = 0$ in integers implies the like of the equation $x^3 + y^3 + z^3 = 0$. Hence it might, at first sight, be rashly inferred (which is what happened to me when writing the 2nd footnote to page [316] from a sick bed) that since a cube number cannot be broken up into the sum of two others, the former of these last written equations is insoluble in

integers. But the fact stares one in the face that it has three solutions in integers, namely,

$$x:y:z::\;1:\;1:1$$
$$x:y:z::-2:\;1:1$$
$$x:y:z::\;1:-2:1.$$

In general, (except at points of inflexion or at points whose ith tangentials are points of inflexion*), one rational point in a cubic gives rise to an infinite series of rational derivatives, but in this case the three points $1:1:1$, $-2:1:1$, $1:-2:1$ are the angles of a triangle in- and- exscribed to the curve $x^3-3xy^2-y^3+3z^3$, and are the only rational points on the curve. Each of them is its own third tangential, so that, at any one of the three, an infinite number of cubic curves can be made to pass having plethoric, or, so to say, pluperfect contact with each other (9-point contact) and accordingly will not intersect each other in any other point.

To these three points will be found to correspond (as will presently be shown in § 2) points for which x or y is zero in the curve $x^3+y^3+z^3=0$. This perfectly explains the seeming paradox.

The sides of the rational in- and- exscribed triangle are easily seen to be $y-z=0$, $x+y+z=0$, $x-z=0$.

In general, if any cubic be thrown into the form $x^2y+y^2z+z^2x+\lambda xyz$, it will obviously be in- and- exscribed to the triangle x, y, z†. In the present instance, if we write $x-z=u$, $y-z=v$, $x+y+z=-w$, it will be found that the curve $x^3-3xy^2-y^3+3z^3$ becomes simply $uv^2+vw^2+wu^2$, of which the Hessian is the three straight lines $u^3+v^3+w^3-3uvw$. If we take the sides of an equilateral triangle whose area is $\frac{1}{2}\Delta$ for the axes of u, v, w, we shall have $u+v+w=\Delta$, and the three real points of inflexion being in the line $u+v+w$, will pass off to infinity, so that the curve will possess three infinite branches. Writing $\omega=\frac{2\pi}{9}$, each asymptote will cut the sides of the angles of reference in three pairs of segments abutting at the several angles, such that the ratio to each other of the segments in the several pairs, taken in regular order, will be (for the three asymptotes respectively),

$$\frac{\cos\omega}{\cos 2\omega},\quad \frac{\cos 2\omega}{\cos 4\omega},\quad \frac{\cos 4\omega}{\cos\omega},$$
$$\frac{\cos 2\omega}{\cos 4\omega},\quad \frac{\cos 4\omega}{\cos\omega},\quad \frac{\cos\omega}{\cos 2\omega},$$
$$\frac{\cos 4\omega}{\cos\omega},\quad \frac{\cos\omega}{\cos 2\omega},\quad \frac{\cos 2\omega}{\cos\omega}.$$

* Thus we have the following distinction of cases as regards the algebraically rational derivatives of any point on a cubic curve: (1) An infinite succession of links. (2) A finite open chain reducing in the case of inflexions to a single point. (3) A closed chain with a finite number of links.

† For x will touch the cubic at x, y; y at y, z; z at z, x.

These ratios, of course, remain the same, for the conjugate cubic $u^2v + v^2w + w^2u$, except that the order of the readings has to be reversed.

According to my departed friend, (of cherished memory), Otto Hesse's dictum, I suppose it may almost be taken for granted without proof, which would obviously be easy, that the two sets of real asymptotes for the conjugate cubics will envelop one and the same conic.

In a future excursus I propose to demonstrate the existence of an infinite number of polygons in- and- exscribable about any given cubic, and to determine the number of such polygons for any existent number of sides. Since $wv^2 + vu^2 + uw^2 = 0$ is equivalent to $(2uw + v^2)^2 + (4u^3v - v^4) = 0$, we are able to deduce, from the fact that one cube cannot be the sum of two others, the theorem that the equation $v^4 - 4u^3v = t^2$ has no solution in integers*, (zeros excluded) which seems to me (the way in which it is got, I mean, not the theorem itself) a very surprising inference.

SCHOLIUM. *On triangles and polygons in- and- exscribable to a general cubic.*

The apices of any such triangle must be points which are their own 3rd tangentials. Any such point, it may be shown, is completely defined by the condition that two right lines, drawn, the first through it and any one chosen at will, of the 9 points of inflexion, the second through its tangential and some other point of inflexion, shall meet the curve in the same point.

If, then, the cubic be written under its canonical form, and we select the point of inflexion (I), for which $x = 1$, $y = 1$, and through the point $P(x, y, z)$, which is to be its own 3rd tangential, and I draw a ray meeting the curve in P', and through P' and Q, the tangential to P, [that is, the point whose coordinates are $x(y^3 - z^3)$, $y(z^3 - x^3)$, $z(x^3 - y^3)$] draw a ray, the point (X, Y, Z), where that ray meets the curve, must be a point of inflexion, and, *vice versâ*, if the condition is fulfilled, P is its own 3rd tangential.

* Suppose the equation $u^2v + v^2w + w^2u = 0$ is resoluble in non-zero integers. We may regard u, v, w as having no common measure, as any such, if it existed, could be driven out of the equation by division. Suppose p to be any prime number entering exactly α times into u and β times into v; then writing $u = p^\alpha u_1$, $v = p^\beta v_1$, since w^2u contains p^α, and v^2w, $p^{2\beta}$, we must have $\alpha = 2\beta$ and $p^{3\beta}u_1^2v_1 + v_1^2w + w^2u_1 = 0$, and proceeding similarly with each prime common measure of u, v, of v, w and of w, u, it is obvious that, calling the greatest common measure of these three pairs δ, ϵ, θ, we must have $\delta^3u'^2v' + \epsilon^3v'^2w' + \theta^3w'^2u' = 0$, where u', v', w' have no two of them any common measure. Hence, apart from algebraical sign u', v', w' must be each of them unity, and the above equation may be written $\delta_1^3 + \epsilon_1^3 + \theta_1^3 = 0$, the same in form as that which gave birth to the equation $\xi^3 - 3\xi\eta^2 + \eta^3 = 0$, of which $u^2v + v^2w + w^2u = 0$ is a transformation. It is worthy also of remark that the two equations $u^2v + v^2w + w^2u = 0$ and $x^3 + y^3 + z^3 = 0$ pass into one another through the medium of the self-reciprocal substitution-matrix

$$\begin{matrix} 1 & 1 & 1 \\ \rho^{\frac{1}{3}} & \rho^{\frac{4}{3}} & \rho^{\frac{7}{3}} \\ \rho^{\frac{2}{3}} & \rho^{\frac{8}{3}} & \rho^{\frac{5}{3}} \end{matrix}$$

where ρ is a primitive cube root of unity.

It will be found that

$$\begin{aligned} &X: - x^6y^3 - y^6z^3 - z^6x^3 + 3x^3y^3z^3 \\ ::\ &Y: - x^3y^6 - y^3z^6 - z^3x^6 + 3x^3y^3z^3 \\ ::\ &Z: \quad xyz\,(x^6 + y^6 + z^6 - x^3y^3 - y^3z^3 - z^3x^3), \end{aligned}$$

and we must have $X = 0$ or $Y = 0$ or $\dfrac{Z}{xyz} = 0$, the factor which figures in Z being disregarded, because it would lead to the 9 points of inflexion, which may be thrown out of account, as for each of them the in- and- exscribed triangle reduces to a point.

Combining each of the above equations taken separately with the equation to the cubic, we see that there will be $3 \times (9 + 9 + 6)$, that is 72 points forming the apices of 24 in- and- exscribed triangles to the cubic. It may be shown further that these 24 triangles consist of 12 pairs of conjugate triangles, every pair being so situated that each is a threefold perspective representation of the other, the three perspective centres being some one of the 12 sets of 3 collinear points of inflexion*.

The 24 in- and- exscribed triangles may therefore be distributed into 4 groups, each containing 3 pairs of conjugate triangles. This theory and the general one of in- and- exscribed polygons with any number of sides to a cubic curve will be treated more fully in a future excursus. It may, however, be remarked here that the equation $\dfrac{Z}{xyz} = 0$ is equivalent to the two $x^3 + \rho y^3 + \rho^2 z^3 = 0$, and $x^3 + \rho^2 y^3 + \rho z^3 = 0$, so that 18 of the points xyz may be found by solving two cubic equations between x^3, y^3 or y^3, z^3 or z^3, x^3. The

* ABC, LMN are in threefold perspective when AL, BM, CN; AM, BN, CL; AN, BL, CM meet in three several points. If ABC be taken as the triangle of reference and the coordinates of L, M, N are a, b, c; a', b', c'; a'', b'', c'' respectively, the triple "perspectivische lage" requires only the satisfaction of two conditions, namely, $ab'c'' = bc'a'' = ca'b''$, so that there is nothing between single and triple perspective relation. This statement constitutes a porism. The double condition $ba'c'' = cb'a'' = ac'b''$ of course corresponds to the contrary relation of triple perspective where AM, BL, CN; AL, BN, CM; AN, BM, CL meet in three several points.

Let I, I', I'', J, J', J'', K, K', K'' denote three points of collinear inflexions and P, Q the 3rd point collinear with P and Q, any two points on the cubic. If Q is the tangential to P, one of the vertices in question, it may be proved that any inflexion I, being assumed, another J may be found such that $IP = JQ$. From this it follows that PQ will satisfy the 10 equations

$$\begin{array}{lll} & PP = Q & \\ IP = JQ & JP = KQ & KP = IQ \\ I'P = J'Q & J'P = K'Q & K'P = I'Q \\ I''P = J''Q & J''P = K''Q & K''P = I''Q. \end{array}$$

These will necessarily continue to be satisfied when I and J are interchanged, provided that $4P$, Q be written KP and KQ or $K'P$ and $K'Q$ or $K''P$ and $K''Q$, and, consequently, to P, Q, R one in- and- exscript, will correspond another denotable indifferently by KP, KQ, KR, $K'P$, $K'Q$, $K'R$, $K''P$, $K''Q$, $K''R$, which will obviously therefore be in triple *perspectivische lage* with the first named one.

remaining 54 may be found by substituting for x, y, z respectively (in the simple equations which express their ratios)

$$\begin{array}{llll} 1^\circ. & x+y+z & x+\rho y+\rho^2 z & x+\rho^2 y+\rho z \\ 2^\circ. & x+y+\rho z & x+\rho y+z & \rho x+y+z \\ 3^\circ. & x+y+\rho^2 z & x+\rho^2 y+z & \rho^2 x+y+z \end{array}$$

(these substituted values, together with the original values of x, y, z, representing the sides of the 4 triangles which contain 3 points of inflexion on each side)*.

We may thus neglect altogether the equations $X=0$, $Y=0$, the values of x, y, z, to which they would lead, being comprised among those resulting from the above method†.

In like manner, as we have found the number of in- and- exscribable triangles, it may be shown that the number of quadrilaterals in- and- exscribable to a cubic is 54, and of p-laterals, when p is a prime number, $8(2^{p-1}-1)(2^{p-2}+1)$. For a k-sided polygon, where k is any number whatever, the rule is as follows. Let

$$\phi x = 8\,(2^{x-1}-(\bar{1})^{x-1})\,(2^{x-2}-(\bar{1})^{x-2}),$$

and let the totient of k, (supposed to contain i distinct prime factors) be expressed in the usual manner as the sum of 2^{i-1} positive terms P and the like number 2^{i-1} negative terms Q.

Then it may be proved (for it requires proof) that $\Sigma\phi P - \Sigma\phi Q$ will contain k; the quotient will contain the number of k-sided polygons in- and- exscribable about a cubic.

This theorem does not accord with the formula given by Professor Cayley in the *Phil. Tr.* for 1871, as quoted in the *Math. Fortschr.*, Vol. III.

* When the cubic is $x^3+y^3+z^3$, X, Y, Z become $x^9+6x^6y^3+3x^3y^6-y^9$, ..., $xyz\,(x^6+x^3y^3+y^6)$ $X=0$ then gives $\dfrac{x^3}{y^3}=t-t^2$ if $t^3-3t+1=0$, that is, $t=2\cos\dfrac{2\pi}{9}$, $2\cos\dfrac{4\pi}{9}$, $2\cos\dfrac{8\pi}{9}$; calling the three values of $\dfrac{x^3}{y^3}$ thus obtained τ_1, τ_2, τ_4, one of the two real in- and- exscribed triangles will have at its vertices $\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x} = \tau_1^{\frac{1}{3}}, \tau_2^{\frac{1}{3}}, \tau_4^{\frac{1}{3}} = \tau_2^{\frac{1}{3}}, \tau_4^{\frac{1}{3}}, \tau_1^{\frac{1}{3}} = \tau_4^{\frac{1}{3}}, \tau_1^{\frac{1}{3}}, \tau_2^{\frac{1}{3}}$ respectively, and the triangle conjugate to it will have at its vertices $\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x}$ equal to the same three systems of ratios.

† If $x^3+y^3+z^3+3mxyz$ be the given cubic, one set of 9 points will be found from the equation

$$[(1-\rho)\,y^3+(1-\rho^2)\,z^3]^3+27m^3\,(\rho y^6z^3+\rho^2y^3z^6)=0,$$

or

$$y^9-3\,\{(1-\rho^2)\,m^3-\rho^2\}\,y^6z^3-\{(1-\rho)\,m^3-\rho\}\,y^3z^6+z^9=0,$$

and the fellow set by interchanging y and z. The disadvantage of this method consists in its leading to equations with imaginary coefficients for finding *inter alia* real roots which the equations $Y=0$ or $Z=0$, being of odd degrees, show must necessarily always exist.

The number of triangles in- and- exscribable to a curve whose order is x, whose class is X and whose number of cusps + three times its class is ξ, is there stated to be

$$X^4 + (2x^3 - 18x^2 + 52x - 46) X^3 + (\overline{18}x^3 + 162x^2 - 420x + 221) X^2$$
$$+ (52x^3 - 420x^2 + 704x + 172) X + (x^4 - 46x^3 + 221x^2 + 172x)$$
$$+ \xi \{\overline{9}X^2 + (\overline{12}x + 135) X + (\overline{9}x^2 + 135x - 600)\}.$$

On making $x = 3$, $X = 6$ and $\xi = 18$ we ought to have 24 the number of in- and- exscribable triangles to a general cubic, but on making these substitutions the result will be found to be zero. It is *quite certain*, therefore, that this formula requires some correction which has been overlooked by its illustrious author. For I have actually, in the text, given a cubic and a triangle in- and- exscribable to it, not to add that it is manifestly impossible for a general cubic to refuse to pass under the form $xy^2 + yz^2 + zx^2 + mxyz$.

Before quitting this subject I wish to call attention to the fact that the formula above given for composite numbers is a form deduced from the form ϕk precisely as in the excursus, the expression for $\log \chi_k x$ was deduced from $\log (x^k - 1)$*. It is clear from general logical considerations that this sort of deduction must be continually liable to occur and a name is imperatively called for to express it as much as one was formerly wanted to express the kind of deduction which leads from an algebraical form to its Hessian. Here the deduction depends on the arithmetical constitution of the subject of the form, and it is a great impediment to the free course of ratiocination not to be able to pass at once, in language and in thought, from the form to its deduct. I intend then in future to call such deduct the *functional totient* of the form, say ϕk, from which it is derived, and to denote it by $(\phi\tau)\,k$. This constitutes a very important gain to arithmetical nomenclature.

I would further call attention to the fact of an arithmetical theorem, of some considerable difficulty to demonstrate (by means of Fermat's extended theorem) in the general case, as any one, who goes through the process of the proof for the single case of $k =$ the product of two primes, will easily satisfy himself, (I mean the theorem that the *functional totient* of $8\,(2^{k-1} - (\overline{1})^{k-1})$ $(2^{k-2} - (\overline{1})^{k-2})$ is always divisible by k) should admit of an intuitional proof through the intervention of a pure property of cubic curves without any recourse to concepts drawn from reticulated arrangements, as in the applications of geometry to arithmetic made by Dirichlet and Eisenstein. This example of the possibility of such application (akin to that whereby the binomial theorem is made to prove that $\dfrac{\pi\,(m + m')}{\pi m\,.\,\pi m'}$ is an integer) is, as far as I can recall, without a precedent in mathematical history.

* The expression actually there given is for $\chi_k x$ and not its logarithm; using the notation explained above, and calling $\phi k = \log (x^k - 1)$ the cyclotomic of the 1st species to the index k, is $e^{(\phi\tau)k}$.

Postscriptum.

Mr Franklin obtains my result as follows: The condition that the $(i-1)$th tangential shall lie on the first polar is of the degree $2 \cdot 4^{i-1}+1$; the number of points on the cubic (exclusive of inflexions) satisfying this condition is $3(2 \cdot 4^{i-1}+1)-27=24(4^{i-2}-1)$. But the $(i-1)$th tangential will be on the first polar, not only when it is a true antitangential, but also when it is the original point itself or the consecutive point; so that we have to deduct from the above number twice the number of points (exclusive of inflexions) whose $(i-1)$th tangentials are the points themselves; that is, denoting by u_i the number of vertices of in- and- exscribed i-laterals, we have

$$\begin{aligned} a_i &= 24(4^{i-2}-1)-2u_{i-1} \\ &= 24\{2^{2i-4}-2^{2i-5}+\ldots+(-2)^{i-1}-(1-2+2^2-\ldots+(-2)^{i-3})\} \\ &= \ 8(2^{i-1}+(-1)^{i-2})(2^{i-2}-(-1)^{i-2}), \end{aligned}$$

which will be the number of the vertices, not only of true i-laterals, but also of all the $\frac{i}{\delta}$-laterals, (δ being any divisor of i except i itself) as well. Mr Franklin further suggests that the discrepancy between this result for $i=3$ and Prof. Cayley's formula may be due to the latter not taking account of the peculiar kind of in- and- exscription in which the curve is in- and- exscribed at the same points. Finally, let us call the *summant* of a number k of the form $a^\lambda \cdot b^\mu \cdot c^\nu$ (a, b, c being primes) the well-known quantity consisting of $(1+\lambda)(1+\mu)(1+\nu)\ldots$ terms which represents the sum of the divisors of k. We may speak of a *functional summant* to ϕk obtained by prefixing ϕ to each monomial term in the *development* of the summant and denote it by $(\phi\sigma)k$. The equation $(\phi\sigma)k=\omega(k)$ has for its solution $fk=(\omega\tau)k$. My method gives at once, for the *functional summant* of u^k (*without exclusion* of inflexions) $(2^k-\tau^k)^2$, and accordingly, the functional totient to this form divided by k is the simplest expression for the number of ex- and- inscribed k-laterals to the cubic. Thus, for $k=1, 2, 3, 4, 5, 6$, that number is 9, 0, 24, 54, 216, 648 respectively.

2. *On 2 and 3 as cubic residues.*

For the benefit of those among my readers in this country who may not have access to the later works on arithmetic, it may be as well to point out how with the aid of their Gauss or Legendre they may verify the conditions which, later on, I shall have need to employ of 2 or 3 being cubic residues to k, a prime of the form $6i+1$. The cyclotomic function of the third degree in the variable, to the index k, if we make $4k=m^2+27n^2$, is known to be $x^3+x^2-\frac{k-1}{2}x-\frac{3k-1+\epsilon mk}{27}$, where $\epsilon^2=\pm 1$ and $m-\epsilon$ contains 3. Connecting this with the same function formed in the manner in which the

cyclotomics in the Excursus under Title 3 have been calculated, calling U the number of solutions of the congruence $1+\beta+\gamma\equiv 0$ (mod. k), where β, γ are any two unequal cubic residues to k, and θ the number of solutions (1 or 0) of the congruence $1+2\beta\equiv 0$ (mod. k), it will easily be found, by comparing the constant terms in the two expressions, that

$$U+\frac{3\theta}{2}=\frac{k-8+\epsilon m}{18}.$$

Hence, when $\theta=1$, that is when 2 is a cubic residue, m (and therefore also n) must be even, and consequently when $\theta=0$, or 2 is not a cubic residue, m must be odd, and *vice versâ*.

Again, if we compare the values of the sum of the 4th powers of the roots of the cyclotomic as found by the general method with that deducible from the given function, we shall find

$$V+\frac{2}{3}\vartheta=\frac{k^2+3k-66-4m\epsilon k}{162},$$

where V is the number of solutions of the congruence $1+\beta+\gamma+\delta\equiv 0$, plus the number of solutions of the congruence $1+\beta+2\gamma\equiv 0$ (β, γ, δ being cubic residues to k) and ϑ the number of solutions of the congruence $1+3\beta\equiv 0$ (mod. k), that is 1 or 0, according as 3 is, or is not, a cubic residue to k.

The numerator is necessarily divisible by 54, but the criterion of ϑ being 0 or 1 depends on its being divisible or not by 81. On substituting for k its value in terms of m and n, it will be found that 16 times the numerator to modulus 81 is congruous with 54 times $(n^2-1)+\epsilon\left\{\left(\frac{m-\epsilon}{3}\right)^3-\frac{m-\epsilon}{3}\right\}$, and consequently is divisible or not by 81 according as n is not, or is, divisible by 3. Hence $\vartheta=1$ when n is divisible by 3 and otherwise is 0.

The joint effect of these two results may be translated into the following statement, which is better adapted than the more complete* form of enunciation would be to the purposes of this memoir.

If $k=f^2+3g^2$, *when* $(f\pm g)$ *contains* 9, 3 *is, and* 2 *is not, a cubic residue; when* g *contains* 3, *but not* 9, 2 *is, and* 3 *is not, a cubic residue; when* g *contains* 9, 2 *and* 3 *are each of them cubic residues, and in any other case neither* 2 *nor* 3 *is a cubic residue to* k†.

The equation $U+\frac{3\theta}{2}=\frac{3k-1+\epsilon mk}{18}$ contains a complete solution of the interesting question, "How many times, if the cubic residues to a given

* I mean more complete in the sense of fixing the cubic character in the case of 3 being a non-residue, which is unimportant to the matter in hand.

† In other words, if $4p=m^2+27n^2$ [an equation always possible when $p=6i+1$], n divisible by 2 is the necessary and sufficient condition of 2, and n divisible by 3 is the necessary and sufficient condition of 3, being a cubic residue to p.

modulus are set out in a regular ascending series, will consecutive terms differ from one another by a single unit?" When 2 is not a cubic residue, the answer is obviously $2U$, for $1+\alpha+\beta=n$ gives two sequences, α, $n-\beta$ and β, $n-\alpha$, differing by units. But when 2 is a cubic residue, there will be three extra sequences not contained among the $2U$ just spoken of, namely,

$$1,\ 2;\ \frac{k-1}{2},\ \frac{k+1}{2};\ k-2,\ k-1.$$

Hence, in each case, the number is $2U+3\theta$, that is $\dfrac{k-8+\epsilon m}{9}$, or, if we count in 0 as a residue, $\dfrac{k+\epsilon m+1}{9}$.

SECTION 2.

On certain numbers and classes of numbers that cannot be resolved into the sum or difference of two rational cubes.

Title 1. *Theorem on irresoluble numbers whose prime factors other than* 2 *or* 3 *are of the form* $18n+5$ *or* $18n+11$*. I propose to prove the following collective theorem. If A represents any one of the numbers 1, 2, 3, 4, 18, 36 or any number of the form

$$\begin{array}{cccc} p, & q, & p^2, & q^2, \\ 9p, & 9q, & 9p^2, & 9q^2, \\ 2p, & 4q, & 4p^2, & 2q^2, \\ pq, & p_1p_2^2, & q_1q_2^2, & p^2q^2, \end{array}$$

(where any p means a prime number of the form $18n+5$, and any q a prime of the form $18n+11$) A will be irresoluble into the sum of two unequal rational cubes.

Lemma. If we decompose A (when it is not a prime) into any factors f, g, h, prime to each other, other than 1, 1, A, the equation $fx^3+gy^3+hz^3=0$ will be irresoluble in integers.

I prove this by showing that the above equation converted into a congruence to *modulus* 9 is irresoluble in integers.

x^3, y^3, z^3, each of them to this modulus is equivalent to one or the other of the three numbers $\bar{1}$, 0, 1.

$$\begin{array}{llllr} p,\ p_1,\ p_2 & \text{to this modulus is} & \text{equivalent} & \text{to} & \bar{4} \\ q,\ q_1,\ q_2 & \text{,,} & \text{,,} & \text{,,} & 2 \\ p^2,\ p_1^2,\ p_2^2 & \text{,,} & \text{,,} & \text{,,} & \bar{2} \\ q^2,\ q_1^2,\ q_2^2 & \text{,,} & \text{,,} & \text{,,} & 4, \end{array}$$

* This theorem includes and transcends all the cases of irresolubility that had been discovered prior to the date of publication of the Proem in the last number of the *Journal*, with the exception of certain specific numbers whose irresolubility had been determined by the Abbé Pépin.

and on inspection, it will easily be verified that the limited linear congruence $f\lambda + g\mu + h\nu \equiv 0$ [mod. 9], where λ, μ, ν must each be picked out of the three numbers $\bar{1}$, 0, 1, has no solution.

Hence, if $fx^3 + gy^3 + hz^3 = 0$ and $f.g.h = A$, and x, y, z are supposed to be prime to each other, two of the quantities f, g, h will be unities and the third equal to A.

Let, now, $x^3 + y^3 + Az^3 = 0$ be supposed soluble in integers. Then, since A contains no $6n + 1$ prime, we must have

$$\left.\begin{array}{l} x + y = A\zeta^3 \\ x^2 - xy + y^2 = \omega^3 \\ z = -\zeta\omega \end{array}\right\} \text{when } x + y \text{ does not contain 3,}$$

and

$$\left.\begin{array}{l} x + y = 9A\zeta^3 \\ x^2 - xy + y^2 = 3\omega^3 \\ z = -3\zeta\omega \end{array}\right\} \text{when } x + y \text{ contains 3.}$$

If $x + y$ is even, since $x^2 - xy + y^2 = \left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2$, we must have $\frac{x+y}{2} + \sqrt{(-3)}\frac{x-y}{2} = \{\xi + \sqrt{(-3)}\eta\}^3$, when $x + y$ does not contain 3, and $\frac{x-y}{2} + \sqrt{(-3)}\frac{x+y}{6} = \{\xi + \sqrt{(-3)}\eta\}^3$, when $x + y$ contains 3. In the one case $\frac{x+y}{2} = \xi^3 - 9\eta^3$, $\frac{x-y}{2} = 3\xi^2\eta - 3\eta^3$, and in the other $\frac{x-y}{2} = \xi^3 - 9\eta^2\xi$, $\frac{x+y}{6} = 3\xi^2\eta - 3\eta^3$.

In the one case, then, $2\xi(\xi - 3\eta)(\xi + 3\eta) = A\zeta^3$, and in the other $2\eta(\xi - \eta)(\xi + \eta) = A\zeta^3$. In either case, therefore, there is an equation-system of the form $\rho\sigma\tau = -A\zeta^3$, $\rho + \sigma + \tau = 0$, to be satisfied; therefore, disregarding permutations of ρ, σ, τ, we must have

$$\rho = fx_1^3, \quad \sigma = gy_1^3, \quad \tau = hz_1^3$$
$$f.g.h = A, \quad x_1y_1z_1 = -\zeta$$
$$fx_1^3 + gy_1^3 + hz_1^3 = 0,$$

and consequently by the Lemma $x_1^3 + y_1^3 + Az_1^3 = 0$ (or the same equation with x_1, y_1, z_1 interchanged) where $x_1y_1z_1$ is a factor of z.

Continuing the same process perpetually, as long as the new x and y have the same parity, each new x, y, z being contained in the immediately preceding z, must perpetually decrease, and if the process could be indefinitely continued, x and y must each evidently become unity, since otherwise z could go on decreasing without limit. This could only happen when $A = 2$, and even then is excluded by the condition that the cubes are to be unequal

as well as rational*. Hence, if the proposed equation is soluble at all, it must contain solutions in which x and y are one even and the other odd.

On this hypothesis, let us consider separately case (1), where $x+y$ does not, and case (2) where $x+y$ does contain 3.

Case (1). Here $(x+y)^2+3(x-y)^2=4(L^2+3M^2)=4\omega^3$, and all the solutions of this equation are necessarily included in those of the system $L^2+3M^2=\omega^3$, $x+y=L+3M$, $x-y=L-M$.

Hence $x+y=\xi_1^3+9\xi_1^2\eta_1-9\eta_1^2\xi_1-9\eta_1^3=A\zeta^3$. On making $\xi_1=\xi-3\eta_1$, this becomes $\xi^3-36\xi\eta_1^2+72\eta_1^3=A\zeta^3$, or, making $\eta'=6\eta_1$, $3\xi^3-3\xi\eta'^2+\eta'^3=3A\zeta^3$, which, on writing $\eta'=\eta+\xi$, becomes $\eta^3-3\eta\xi^2+\xi^3=3A\zeta^3$, where A unless it is unity contains at least one factor that is not of the form $18n\pm1$, or else (in the case when $A=3$) the square of 3. Hence, by virtue of the cyclotomic law for index 9, species 2 (conjugate class) (see Table, p. [327]), the above equation is insoluble in integers†.

Case (2). Here, using L and M in the same sense as above, $\frac{x+y}{3}=L-M$ and $x-y=L+3M$ or $\xi_1^3-3\xi_1^2\eta_1-9\xi_1\eta_1^2+3\eta_1^3=3A\zeta^3$. Here writing $2\eta_1=-\xi$, $\xi_1=\eta+2\xi$, the equation becomes $\eta^3-3\eta\xi^2+\xi^3=3A\zeta^3$, and is insoluble in integers as before. Hence, since by hypothesis $x+y$ is not even, and it has been shown that it cannot be odd, *the number A when not unity is irresoluble into the sum or difference of two unequal rational cubes*‡.

When A is unity the equation above written becomes $\eta^3-3\eta\xi^2+\xi^3=3\zeta^3$, the necessity for discussing which may be avoided by choosing the x, y out of x, y, z (which in this case are indistinguishable) so as to make $x+y$ always

* To prove this, let ξ, η, ζ be the system of variables, for which $\xi=1$, $\eta=1$ and x, y, z the system immediately preceding it. Then we have $A=2$, $\xi=1$, $\eta=1$, $\zeta=-1$, and either $x-y=0$, or $x+y=0$. The latter of these equations would imply $z=0$ and the former $x:y:z::1:1:-1$, and so continually until we fall back on the original equation in x, y, z. Hence the only possible resolution of 2, if $x+y$ is even, is into two equal cubes.

† $3A$ not containing any cube, ξ and $3A$ must be prime to each other, since otherwise η, ξ, ζ would have a common measure. Hence we may make $\eta=\xi\mu-3A\lambda$, and, consequently, $(\mu^3-3\mu+1)\xi^3\equiv0$ mod. $3A$, and, therefore, $\mu^3-3\mu+1$ must contain $3A$.

This conclusion would not hold if $3A$ were of the form A_1B^3 where A_1 contained no cube. We could then only infer $\mu^3-3\mu+1\equiv0$ mod. A_1. Thus, in the case of $A=9$, $3A=B^3$, and our inference would become $\mu^3-3\mu+1\equiv0$ mod. 1, which, of course, is satisfied, and, accordingly, 9 ought to be resoluble into two cubes, as it obviously is, namely, into 1 and 8. Thus, the equation $x^3-3xy^2+y^3=3Az^3$, when $A=9$ has an infinite number of solutions, when $A=3$ has no solution, and when $A=1$ has just 3 solutions.

It may be worth noting that, in general, if $(x, y)^n=Az^n$, and $A=A_1B^n$, where A_1 contains no nth power of a number, $(x, 1)^n$ will contain A_1 as a divisor, provided that the coefficient of x^n in $(x, y)^n$ is a prime to A_1. Cases of this inference being drawn of course frequently occur, but the general principle, obvious as it is, I do not recollect to have seen formulated in the text books. It may be made more precise by the statement that any factor of A_1, prime to the coefficient of x^n, will be a divisor of $(x, 1)^n$.

‡ The equations of substitution are: for case 1, $\xi=\xi_1+3\eta_1$, $\eta=-\xi_1+3\eta_1$; and for case 2, $\xi=-2\eta_1$, $\eta=\xi_1-\eta_1$.

even, which is the ordinary and easier method; but it is not without interest to show how the desired conclusion may be arrived at by keeping $x+y$ always odd. This may be done as follows: The equation between ξ, η, ζ, on writing $\eta+\zeta=u$, $\zeta-\xi=v$, $-\eta+\xi+\zeta=w$ * becomes $uv^2+vw^2+wu^2=0$ which, as shown in footnote to p. [341], involves the relations $u=y'^2z'$, $v=z'^2x'$, $w=x'^2y'$ and consequently $x'^3+y'^3+z'^3=0$ where $x'y'z'=\sqrt[3]{(uvw)}$.

Let us use in general two or more separate letters enclosed within a parenthesis to denote the absolute value of the *greatest one of them* (their *dominant* as I am wont to call it).

When $x+y$ does not contain 3, $x+y=\zeta^3$, $x^2-xy+y^2=(\xi_1^2+3\eta_1^2)^3$. Hence $\zeta<2^{\frac{1}{3}}(x^{\frac{1}{3}}, y^{\frac{1}{3}})$ $(\xi_1, \eta_1)<3^{\frac{1}{3}}(x^{\frac{1}{3}}, y^{\frac{1}{3}})$. Therefore $(\xi_1, \eta_1, \zeta)<3^{\frac{1}{3}}(x, y, z)^{\frac{1}{3}}$, and consequently since $\xi=\xi_1+3\eta_1$ and $\eta=-\xi_1+3\eta_1$, $(\xi, \eta, \zeta)<4\,.\,3^{\frac{1}{3}}(x, y, z)^{\frac{1}{3}}$ and therefore $(u, v, w)<4\,.\,3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}}$. Hence $x'\,.\,y'\,.\,z'<(u, v, w)<4\,.\,3^{\frac{4}{3}}$ $(x, y, z)^{\frac{1}{3}}$.

In like manner when $x+y$ does contain 3, from the equations $\xi=-2\eta_1$, $\eta=\xi_1-\eta_1$, $x+y=9\zeta^3$, $x^2-xy+y^2=3(\xi_1^2+3\eta_1^2)^3$, follow $\zeta<\left(\frac{1}{3}\right)^{\frac{2}{3}}(x, y)^{\frac{1}{3}}$ $(\xi_1, \eta_1)<(x, y)^{\frac{1}{3}}$, $(\xi_1, \eta_1, \zeta)<(x, y, z)^{\frac{1}{3}}$, $(\xi, \eta, \zeta)<(x, y, z)^{\frac{1}{3}}$, $x'\,.\,y'\,.\,z'<(u, v, w)$ $<3(x, y, z)^{\frac{1}{3}}$.

In any case therefore $x'\,.\,y'\,.\,z'<4\,.\,3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}}<18(x, y, z)^{\frac{1}{3}}$. But the difference between any two cubes except 8 and 1 being greater than 8, the smallest of the numbers x', y', z' cannot be less than 3, and, since neither 3^3+4^3 nor 3^3+5^3 is a cube, it follows that $\dfrac{x'\,.\,y'\,.\,z'}{(x', y', z')}>18$, and therefore $(x', y', z')<(x, y, z)^{\frac{1}{3}}$, or the dominant of the quantities x, y, z which satisfy $x^3+y^3+z^3=0$ is continually replaced by another similar dominant less than the cube root of its predecessor, which is impossible.

Hence $x^3+y^3+z^3=0$ is insoluble. Let us see how this is reconcilable with the existence of the 3 rational solutions of $\eta^3-3\eta\xi^2+\xi^3+3\zeta^3=0$, namely, $\xi, \eta, \zeta=\bar{1}, 1, 1$ or $2, 1, 1$ or $1, 2, \bar{1}$ respectively.

In case (1) $\xi=\xi_1+3\eta_1$ $\eta=-\xi_1+3\eta_1$ $\xi, \eta=\bar{1}, 1$ gives $\eta_1=0$ $\xi, \eta=2, 1$ gives $\eta_1=-\xi_1$ $\xi, \eta=1, 2$ gives $\eta_1=\xi_1$. In each instance therefore $M=3\eta_1(\xi_1^2-\eta_1^2)=0$ and consequently $x+y=L=x-y$ and $y=0$.

In case (2) $\xi=-2\eta_1$ $\eta=\xi_1-\eta_1$ $\xi, \eta=\bar{1}, 1$ gives $\xi_1=3\eta_1$ $\xi, \eta=2, 1$ gives $\xi_1=-3\eta_1$ and $\xi, \eta=1, 2$ gives $\xi_1=0$.

In each instance therefore $L=\xi_1(\xi_1^2-9\eta_1^2)=0$ and therefore $x=0$. Thus the rational solutions of the equation in ξ, η, ζ in both cases correspond to rational but futile solutions of the equation in x, y, z.

* From these equations it is obvious that the dominant, that is, the arithmetically greatest of the quantities u, v, w, is less than 3 times the dominant of ξ, η, ζ.

CHAPTER I.

EXCURSUS B.—ON THE CHAIN RULE OF CUBIC RATIONAL DERIVATION.

I think it desirable, while the colours, so to say, are still wet on the palette, and my mind is still dwelling upon the subject which has been casually introduced in the note to the proem contained in the last number of the *Journal* (and there made use of to determine the number of in-and-exscribed k-laterals to a cubic), without waiting to put forth the titles which in natural order of sequence, perhaps, should immediately follow Title 1 of Section 2, to proceed at once to develop the theory of derivation which, irrespective of the casual use of it alluded to, will be found to be of essential importance when I reach that part of my proposed task which deals with soluble cubic-form equations, nor less so when, in Chapter II., I have to treat of insoluble cases of certain classes of cubic-form equations with four or more terms.

Title 1.—*On the Natural or Discontinuously Numbered Scale of Rational Derivatives to a Point on a Cubic Curve.*

Let us take any point on a cubic curve along with its successive tangentials *ad infinitum.* We may, by drawing straight lines through any two of these points, either contiguous or apart, to meet the curve, obtain an additional set of points, and thus form an enlarged system which may again be subjected to a like process of collineation or tangentialization, and such method of augmentation and amplification may be continued indefinitely. Every point thus obtained will obviously be a rational derivative of the original point (that is, its co-ordinates will be rational integral functions of those of that point), and, at first sight, it would seem as if we might in this way obtain a network, or spread*, of rational derivatives; but I shall proceed to show that such is not the case, but that only a line or chain of points will be thus obtained, usually infinite in extent, although for certain positions of the initial point coming to a stop, and in other cases winding round and round upon itself so as still to include only a finite number of distinct points. It will be shown subsequently that, in order to complete the theory of the chain for the purposes of this memoir, it will be necessary to take into account the rational derivatives not merely from a single arbitrary point, but from such points, *combined with a point of inflexion,* and that this additional element will not alter the surprising fact of the absence of reticulation or spread, but merely bring about the insertion into the chain of

* Spread, as a noun (scarcely to be found in the dictionaries), I employ in the sense in which it occurs in the phrase *spread of foliage.* On this continent the word *spread* is also used to denote a thick coverlet or padded woollen quilt, laid over the bedclothes in winter to keep out the cold; also on both continents as a familiar name for a college banquet.

points corresponding to missing numbers in it as first described, and to the duplication of the chain so completed, owing to every point in it having an opposite point also situated on the curve and collinear with it in respect to the given inflexion. This duplication will be of little importance in general to the arithmetical theory with which we shall be occupied, inasmuch as opposite points will correspond to the same arithmetical values, with merely a change of name between two out of the three variables which denote the co-ordinates of any point. First, let us consider the chain law of derivation when a point on the cubic curve alone is given. I shall call the original point 1, and its first and second tangentials 2 and 4 respectively, and in general use (m, n) to denote the point on a given cubic collinear with two points m, n also situated upon it*. Obviously, then, we shall have $(1, 1) = 2$ $(2, 2) = 4$, using $(1, 1)$ $(2, 2)$ to denote, in either case, two consecutive points upon the cubic. It is also obvious that if $(m, n) = p$ then $(m, p) = n$ and $(n, p) = m$, so that $(1, 2) = 1$ $(2, 4) = 2$.

Let us call $(1, 4) = 5$ $(2, 5) = 7$ $(1, 7) = 8$ $(2, 8) = 10$ $(1, 10) = 11$ $(2, 11) = 13$ and so on. It will be seen that no number which is a multiple of 3 is brought into existence by this process. Supposing a, b to be any two integers, neither of them divisible by 3, let us agree to signify by $a \ddagger b$ that of the two values $a + b$, $a - b$ which is not divisible by 3. The theorem to be established is that the point (m, n) collinear to m and n will have for its value $m \ddagger n$; as, for instance, $(4, 4)$, or the third tangential to 1, will have for its value 8, that is, will be identical with $(1, 7)$, that is to say, with $\{1, [2, (1, 4)]\}$, where 2 and 4 are the first and second tangentials to 1, which amounts to a rule for obtaining the third tangential, when a point on a cubic and its first and second tangentials are given, by collineation alone. The *theory of residuation*, in its simplest form (see Salmon's *Higher Plane Curves*, 3rd ed., p. 134)† teaches us that the rule of the older chemistry known by the name of double decomposition, namely that $\{(a, b), (c, d)\} = \{(a, c), (b, d)\}$ is applicable to the same symbols regarded as points on a cubic curve. This rule of double decomposition is all that is required to prove the theorem in question.

Thus, for example, in order to prove that $(1, 7) = (4, 4)$, I write $(1, 7) = \{(1, 2), (2, 5)\} = \{(2, 2), (1, 5)\} = (4, 4)$. Q. E. D.

So, to prove in general that $(r, s) = r \ddagger s$ I proceed as follows:

* Sometimes, however, it will be found more convenient to use $P_1, P_2 \dots P_n$; $P_1', P_2', \dots P_n'$ in lieu of $1, 2, \dots n$; $1', 2', \dots n'$.

† The theory of residuation was originally brought by me before the Mathematical Society of London, and subsequently, in the form of questions, in the *Educational Times*. Dr Salmon makes no allusion to the fact of my applying the theory to curves of all orders: in the case of the quartic, the residual becomes a system of three points; of a quintic, a system of six points, and so on. I understood Professor H. S. Smith to say that he made use of my theory for the quartic in his memoir which gained half the prize for the subject set by the Academy of Sciences of Berlin, but which I have never seen.

(1) Suppose $r = 3i + 1$; $s = 3j + 1$, where $j - i$ is positive. Then

$$(r, s) = \{(3i-1, 2), (3j+2, 1)\} = \{(3i-1, 1), (3j+2, 2)\}$$
$$= (3i-2, 3j+4) = (r-3, s+3).$$

Hence $(r, s) = (r - 3i, s + 3i) = (1, s + r - 1) = s + r$.

(2) Suppose $r = 3i - 1$; $s = 3j - 1$. Then $(r, s) = \{(3i-2, 1), (3j+1, 2)\}$

$$= \{(3i-2, 2), (3j+1, 1)\} = (3i-4, 3j+2) = (r-3, s+3),$$

as before. Hence $(r, s) = \{r - 3(i-1), s + 3(i-1)\} = (2, s + r - 2) = s + r$.

(3) Suppose $r = 3i - 1$; $s = 3j + 1$. Then $(r, s) = \{(3i-2, 1), (3j-1, 2)\}$

$$= \{(3i-2, 2), (3j-1, 1)\} = (3i-4, 3j-2) = (r-3, s-3).$$

Hence $(r, s) = (r - 3i + 3, s - 3i + 3) = (2, s - r + 2) = s - r$.

(4) Suppose $r = 3i + 1$; $s = 3j - 1$. Then $(r, s) = \{(3i-1, 2), (3j-2, 1)\}$

$$= \{(3i-1, 1), (3j-2, 2)\} = (3i-2, 3j-4) = (r-3, s-3).$$

Hence $(r, s) = (r - 3i, s - 3i) = (1, s - r + 1) = s - r$.

Collecting the four cases, it will be seen that I have proved, for all values of the points r, s in the chain, that $(r, s) = r \ddagger s$. Q.E.D.

The points 2^i correspond to tangentials of the ith order to the point 1. It is obvious from the above theorem that no process of continued collineation or tangentialization performed upon these points can lead to any points extraneous to the series of points 1, 2, 4, 5, 7, 8 ... which form a simple chain extending in general to infinity. Moreover, as it follows from the theory of residuation that any single point reached through the intervention of curves drawn through any number of points on a cubic can be reached by simple linear constructions, it follows that by no conceivable geometrical process can any rational point be reached not included in the numbered chain, and the inference becomes in the highest degree probable, and, as a matter of fact, is undoubtedly true (although the reasoning upon which it is here made to rest is not absolutely conclusive), that no rational deducts from a *general* point on a *general* cubic exist save those that belong to the numbered chain, the points upon which constitute what may properly be termed a self-contained *group*, infinite or finite (as the case may be) in regard to the number of terms which it contains. I shall presently determine the order of each successive derivative, meaning thereby the order in the co-ordinates of the initial point of any one of the three functions which express the co-ordinates of the derived one*.

* There is a further question, but which, as not material to the object of this memoir, I shall not discuss here, namely, the *degree* in the coefficients of each such derivative. For the tangential, the degree-order (being that of the minor determinants of the matrix made up of the differential derivatives of the function and its Hessian) we know to be 4, 4. If x, y, z, be the original co-ordinates, and X, Y, Z, those of the tangential, we know that $F(X, Y, Z)$ being zero when $F(x, y, z)$ (the given cubic) is zero, must be divisible by $F(x, y, z)$. The quotient will be of the degree-order 13, 12 – 1, 3, that is, 12, 9, and is in fact the skew covariant of F.

The case in which the chain forms a closed polygon, which can only happen when for some number i the ith tangential coincides with the initial point, has already been discussed in the note to the proem.

If the chain is an open but finite one, it is necessary that a tangential of some order shall fall upon a point of inflexion, in which case the succeeding tangentials remain fixed at that point, but otherwise continual new tangentials could be drawn. These are obviously necessary conditions of the chain being finite, whether it be an open chain or winding round upon itself; it remains to show that they are sufficient as well as necessary, but that will best appear after the theory of derivation from a general point combined with a point of inflexion has been discussed.

I shall begin with finding the co-ordinates X, Y, Z of a point on the cubic curve collinear with any two given points x, y, z; ξ, η, ζ. Let

$$X = \lambda x + \mu\xi,\ Y = \lambda y + \mu\eta,\ Z = \lambda z + \mu\zeta;$$

then

$$F(X, Y, Z) = \lambda^3 F(x, y, z) + \lambda^2\mu\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z)$$
$$+ \mu^3 F(\xi, \eta, \zeta) + \lambda\mu^2\left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta).$$

Hence X, Y, Z will be the collineal to (x, y, z), (ξ, η, ζ) if

$$\lambda : \bar{\mu} :: \left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta) : \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z).$$

If now we write $F(x, y, z)$ under its canonical form $x^3 + y^3 + z^3 + Kxyz$, it will be found, on substituting for λ and $\bar{\mu}$ the quantities to which they are proportional, that

$$X = (y^2\eta\xi - y\eta^2 x + z^2\zeta\xi - z\zeta^2 x) + K(yz\xi^2 - \eta\zeta x^2)$$
$$Y = (z^2\zeta\eta - z\zeta^2 y + x^2\xi\eta - x\xi^2 y) + K(zx\eta^2 - \zeta\xi y^2)$$
$$Z = (x^2\xi\zeta - x\xi^2 z + y^2\eta\zeta - y\eta^2 z) + K(xy\zeta^2 - \xi\eta z^2).$$

But these expressions admit of a surprising simplification, namely, we may neglect the terms not containing K, for it will be found that the quantities affected with the coefficient K are to each other in the same ratios as the other three corresponding groups in the values of X, Y, Z. Thus, for example

$$(yz\xi^2 - \eta\zeta x^2)(z^2\zeta\eta - z\zeta^2 y + x^2\xi\eta - x\xi^2 y)$$
$$- (zx\eta^2 - \zeta\xi y^2)(y^2\eta\xi - y\eta^2 x + z^2\zeta\xi - z\zeta^2 x)$$
$$= (\xi y - x\eta)\{\xi\eta\zeta(x^3 + y^3 + z^3) - xyz(\xi^3 + \eta^3 + \zeta^3)\}$$

hence $$X : Y : Z :: yz\xi^2 - \eta\zeta x^2 : zx\eta^2 - \zeta\xi y^2 : xy\zeta^2 - \xi\eta z^2.$$

We might, instead of these simple expressions, take for X, Y, Z the other three groups and (using $x_1 y_1 z_1$; $x_2 y_2 z_2$ instead of x, y, z; ξ, η, ζ and (pq) to

denote the determinant $p_1q_2-p_2q_1$) say that X, Y, Z are the minor determinants of

$$\begin{matrix} x_1 . x_2 & y_1 . y_2 & z_1 . z_2 \\ (yz) & (zx) & (xy), \end{matrix}$$

and these are actually the expressions found by Cauchy, and given by him in his *Exercices de Mathématiques*, Paris, 1826, p. 256, ll. 18—21, pp. 257—60. I take this reference from a loose page of an article by M. Lucas, but have not access either to that article or to Cauchy's.

It is remarkable that Cauchy should have given quadrinomial expressions for the collineal to two given points on a cubic curve, or their connective, as I shall hereafter term it, when, as shown above, binomial ones fulfil the same purpose. The correctness of these remarkable formulæ admits of easy verification, as follows:

For greater simplicity denote x^3, y^3, z^3, xyz by u, v, w, μ; and $\xi^3, \eta^3, \zeta^3, \xi\eta\zeta$ by u', v', w', μ' respectively. Then

$$\begin{aligned}\Sigma\,(yz\xi^2-\eta\zeta x^2)^3 &= \Sigma\,(vwu'^2-v'w'u^2)-3\mu\mu'\,\{(u'+v'+w')\,\mu-(u+v+w)\,\mu'\}\\ &= \Sigma\,(vwu'^2-v'w'u^2).\end{aligned}$$

Also

$$\begin{aligned}&K\,(yz\xi^2-\eta\zeta x^2)\,(zx\eta^2-\zeta\xi y^2)\,(xy\zeta^2-\xi\eta z^2)\\ &=-Kxyz\,(\xi^3\eta^3z^3+\eta^3\zeta^3x^3+\zeta^3\xi^3y^3)+K\xi\eta\zeta\,(x^3y^3\zeta^3+y^3z^3\xi^3+z^3x^3\eta^3)\\ &=(u+v+w)\,(uv'w'+vw'u'+wu'v')-(u'+v'+w')\,(u'vw+v'wu+w'uv)\\ &=\Sigma\,(u^2v'w'-u'^2vw).\end{aligned}$$

Hence, giving X, Y, Z the values indicated by the formula, we find

$$X^3+Y^3+Z^3+KXYZ=0,$$

which equation depends, as seen, and as we know *a priori* must be the case, on the pure algebraical fact that $X^3+Y^3+Z^3+KXYZ$ is a syzygetic function of $x^3+y^3+z^3+Kxyz$ and $\xi^3+\eta^3+\zeta^3+K\xi\eta\zeta$, taking no account of the function $\xi\eta\zeta\,(x^3+y^3+z^3)-xyz\,(\xi^3+\eta^3+\zeta^3)$, as that is itself a syzygetic function of the two others. If we call the syzygetic multipliers of those two Φ and F respectively, it will at once be seen from what precedes that

$$\Phi=3\xi^2\eta^2\zeta^2xyz-\xi^3\eta^3z^3-\eta^3\zeta^3x^3-\zeta^3\xi^3y^3$$
$$F=3x^2y^2z^2\xi\eta\zeta-x^3y^3\zeta^3-y^3z^3\xi^3-z^3x^3\eta^3\,*.$$

I now proceed to apply the foregoing results to the problem of determining the order in the co-ordinates of any derivative numbered j (where $j=3i\pm1$),

* Thus
$$F=-(yz\xi+zx\eta+xy\zeta)\,(yz\xi+\rho zx\eta+\rho^2xy\zeta)\,(yz\xi+\rho^2zx\eta+\rho xy\zeta)$$
$$\Phi=-(\eta\zeta x+\zeta\xi y+\xi\eta z)\,(\eta\zeta x+\rho\zeta\xi y+\rho^2\xi\eta z)\,(\eta\zeta x+\rho^2\zeta\xi y+\rho\xi\eta z),$$
and it is worthy of notice that we have incidentally solved with quantic values for F, Φ, U, V, W the simultaneous algebraico-diophantine equations
$$U^3+V^3+W^3=(a^3+b^3+c^3)\,\Phi-(\alpha^3+\beta^3+\gamma^3)\,F$$
$$UVW=abc\Phi-\alpha\beta\gamma F.$$

which may be called its index, and shall prove that *the order of any derivative is the square of its index**. It will also be shown that each of the derivatives above referred to will be of the form xU, yV, zW, where U, V, W are quantics in x^3, y^3, z^3 as variables, since these quantities satisfy the equation

$$(xU)^3 + (yV)^3 + (zW)^3 + KxyzUVW = 0,$$

where
$$Kxyz = -x^3 - y^3 - z^3.$$

From this it follows that, calling x^3, y^3, z^3; a, b, c respectively, the scheme of derivatives contains the various solutions of the algebraico-diophantine equation

$$aU^3 + bV^3 + cW^3 - (a+b+c)\,UVW = 0,$$

and that, supposing the law of the squares to be demonstrated, U, V, W will be of the order $\frac{1}{3}\{(3i \pm 1)^2 - 1\}$, that is, $3i^2 \pm 2i$ in a, b, c, where i is any integer. We thus see that the above equation admits of solutions in which U, V, W are of the orders 1, 5, 8, 16, 21, 33, 40 ... respectively. It will hereafter be shown, in like manner, that the missing derivatives, whose indices are multiples of 3 (belonging to the arbitrary point and point of inflexion combined), will satisfy the equation

$$U^3 + V^3 + abcW^3 - (a+b+c)\,UVW = 0,$$

where U, V, W will be necessarily of the orders $3i^2 \pm 2i, 3i^2 \pm 2i, (i \pm 1)(3i \pm 1)$ respectively, i, as before, representing any integer. Thus we see that, if $a + b + c = 0$, the equations

$$aU^3 + bV^3 + cW^3 = 0 \quad \text{and} \quad U^3 + V^3 + abcW^3 = 0$$

will admit of an infinite number of solutions in integers, when a, b, c are integer. This fact, as regards the latter equation, has been already pointed out by M. Lucas in this *Journal*, and previously by the Abbé Pépin in his memoir in *Liouville's Journal*, 2nd series, Tome XV.

Let us begin with applying the formulæ to obtaining the co-ordinates of the tangential.

Let
$$x^3 + y^3 + z^3 + 3kxyz = 0$$

be the equation to the cubic. If we take x, y, z; $x + \delta x$, $y + \delta y$, $z + \delta z$ two consecutive points, their connective will be the tangential.

Applying the formulæ just obtained, we shall obtain for its co-ordinates expressions each of the form $P\delta x + Q\delta y + R\delta z$ with only one relation between $\delta x, \delta y, \delta z$. Hence, if we write $\delta z = \lambda\delta x + \mu\delta y$ the resulting ratios must be

* The proof here supplied is sufficiently exact to dispel any reasonable doubt as to the truth of the law; but an exact proof which does not assume but demonstrates the non-existence of latent common measures to the reduced values of the co-ordinates of the connective to any two derivatives will be furnished under Title 5—one of the most surprising feats of demonstration which it has ever fallen to the author's lot to accomplish.

independent of λ and μ. Consequently we may make $\delta z = 0$. The two connectives then become

$$x, \; y, \; z$$

$$x + \delta x, \; y + \delta y, \; z,$$

and the co-ordinates of the tangential will therefore be proportional to

$$yz\,(x + \delta x)^2 - z\,(y + \delta y)\,x^2 : zx\,(y + \delta y)^2 - z\,(x + \delta x)\,y^2 : z^2\,\{xy - (x + \delta x)(y + \delta y)\}$$

that is, to $$x\,(2y\delta x - x\delta y) : y\,(2x\delta y - y\delta x) : z\,(x\delta y + y\delta x)$$

where $$\delta x : \delta y :: y^2 + kxz : x^2 + kyz.$$

Hence the co-ordinates required are as

$$x\,\{2y^3 + x^3 + 3kxyz\} : y\,\{-2x^3 - y^3 - 3kxyz\} : z\,(x^3 - y^3),$$

that is, as $$x\,(y^3 - z^3) : y\,(z^3 - x^3) : z\,(x^3 - y^3),$$

a result which appears to have been first found by Cauchy for the general form, but previously by Euler, and before him by Fermat, for the case $k = 0$.

If we write a, b, c, instead of x, y, z, and call the co-ordinates of the tangential x, y, z, we might find their values by virtue of the condition that the connective of a, b, c and x, y, z is a, b, c over again. This furnishes the equations

$$bcx^2 - a^2yz = am$$

$$cay^2 - b^2zx = bm$$

$$abz^2 - c^2xy = cm,$$

which may be satisfied by writing

$$x = a\,(b^3 - c^3)\,\rho\,; \quad y = b\,(c^3 - a^3)\,\rho\,; \quad z = c\,(a^3 - b^3)\,\rho\,;$$

$$(a^6 + b^6 + c^6 - a^3b^3 - b^3c^3 - a^3c^3)\,\rho^2 = m\,;$$

but whether or not the above is necessarily the only possible solution is not quite clear *a priori*, and *a posteriori* it looks as if the solutions might be manifold.

The co-ordinates of the point whose index is 4, that is, of the second tangential, will be those of the first tangential to the point

$$x\,(y^3 - z^3) : y\,(z^3 - x^3) : z\,(x^3 - y^3),$$

namely,

$$x\,(y^3 - z^3)\,\{y^3\,(x^3 - z^3)^3 + z^3\,(x^3 - y^3)^3\} : y\,(z^3 - x^3)\,\{z^3\,(y^3 - x^3)^3 + x^3\,(y^3 - z^3)^3\}$$

$$: z\,(x^3 - y^3)\,\{x^3\,(z^3 - y^3)^3 + y^3\,(z^3 - x^3)^3\},$$

and are of the order 16.

To find the co-ordinates of the point whose index is 5, we may take the connective of the one last found, and of x, y, z, that is, of 4 and 1. Let us

call them xU, yV, zW, and, for greater simplicity, denote x^3, y^3, z^3, by u, v, w. Then, omitting the common factor xyz,

$$U=(v-w)^2\{v(u-w)^3+w(u-v)^3\}^2$$
$$-(w-u)(u-v)\{w(v-u)^3+u(v-w)^3\}\{u(w-v)^3+v(w-u)^3\},$$

with similar quantities (*mutatis mutandis*) set against V and W.

These quantities will have the common measure

$$u^2+v^2+w^2-uv-uw-vw.$$

To prove this let either one of its factors, as $u+\rho v+\rho^2 w,=0$.

Then $$v-u=\rho^2(w-v) \text{ and } u-w=\rho(w-v),$$

and the representative of U above written becomes

$$\{(v-w)^2-(w-u)(u-v)\}(w-v)^8=(v^2+w^2+u^2-vw-uw-uv)(w-v)^8=0.$$

Hence the representative of U vanishes with, and therefore contains

$$u^2+v^2+w^2-uv-uw-vw$$

as a factor, and the same must evidently be true for the representatives of V and W; hence, U, V, W, will be of the order $10-2$ or 8, in u, v, w, and the co-ordinates xU, yV, zW, of the order $3.8+1$, that is, of the order 25 in xyz.

The preceding demonstration depends essentially on the fact that my simplified formulæ for the co-ordinates of the connective of two points on a cubic fail, that is to say, become illusory, for a particular relation between the two points, as is easily seen; for let x, y, z; $x, \rho y, \rho^2 z$ be two points on a cubic, then the formulæ for X, Y, Z, the connective's co-ordinates, become

$$(\rho y.\rho^2 z-yz)x^2;\ (\rho^2 z.x-xz\rho^2)y^2;\ (x.\rho y-xy\rho^4)z^2,$$

that is, all vanish, whereas it may be remarked that the general expressions given at page [354],

$$X=(y^2\eta\xi-y\eta^2x+z^2\zeta\xi-z\zeta^2x)+K(yz\xi^2-\eta\zeta x^2)$$
$$Y=(z^2\zeta\eta-z\zeta^2y+x^2\xi\eta-x\xi^2y)+K(zx\eta^2-\zeta\xi y^2)$$
$$Z=(x^2\xi\zeta-x\xi^2z+y^2\eta\zeta-y\eta^2z)+K(xy\zeta^2-\xi\eta z^2),$$

become the minors of

$$\begin{array}{ccc} x^2 & \rho y^2 & \rho^2 z^2 \\ (\rho^2-\rho)yz & (1-\rho^2)zx & (\rho-1)xy; \end{array}$$

that is, $(\rho^2-\rho)x(y^3-z^3)$, $(\rho-1)y(z^3-x^3)$, $(1-\rho^2)z(x^3-y^3)$,

which are the same as

$$x(y^3-z^3),\quad \rho^2 y(z^3-x^3),\quad \rho z(x^3-y^3),$$

and remain perfectly valid.

This law of the failing case enables me to prove very easily the *Law of Squares*, as follows:

Suppose it proved that for all indices inferior to $6i$ the order of the derivative is equal to the square of its index; then, to prove that the same law is true up to $6(i+1)$, it is only necessary to consider the cases of $6i+1$, $6i+5$, for, as regards the indices $6i+2$ and $6i+4$, the derivatives may be regarded as the tangentials of the derivatives to indices $3i+1$ and $3i+2$, and will consequently be of the orders $4(3i+1)^2$ and $4(3i+2)^2$, that is, $(6i+2)^2$ and $(6i+4)^2$ respectively.

Let us further suppose that for derivatives whose indices are inferior to $6i$ the co-ordinates are of the form xU, yV, zW; U, V, W being quantics in x^3, y^3, z^3; then, obviously, from the mode of forming the tangential, this will be true for derivatives whose indices are $6i+2$, $6i+4$: for the tangential to xU, yV, zW is

$$xU(y^3V^3 - z^3W^3), \quad yV(z^3W^3 - x^3U^3), \quad zW(x^3U^3 - y^3V^3).$$

Let us consider the point (1) whose co-ordinates x, y, z satisfy the equation

$$x^3 + \rho y^3 + \rho^2 z^3 = 0.$$

For such a point $\qquad y^3 - z^3 : z^3 - x^3 : x^3 - y^3 :: 1 : \rho : \rho^2,$

and the point (2) becomes x, ρy, $\rho^2 z$. Consequently the point (4) becomes $x(y^3 - z^3)$, $\rho y(z^3 - x^3)$, $\rho^2 z(x^3 - y^3)$, the same as x, $\rho^2 y$, ρz; hence the point (5), the connective of (1, 4), becomes $x(y^3 - z^3)$, $\rho y(z^3 - x^3)$, $\rho^2 z(x^3 - y^3)$, the same as x, $\rho^2 y$, ρz, so that, denoting the derivatives by their indices,

$$5 = 4 \qquad 7 = 1,\ 8 = 1,\ 1 = 2 \qquad 10 = 2,\ 8 = 2,\ 1 = 1$$
$$11 = 4,\ 7 = 4,\ 2 = 2 \qquad 13 = 2,\ 11 = 2,\ 2 = 4,\ \text{etc.}$$

We have, thus, for all values of the point i

$$9i \pm 1,\ 2,\ \pm 4 = 1,\ 2,\ 4,$$

when 1 is the point for which $x^3 + \rho y^3 + \rho^2 z^3 = 0$.

Hence, if p, p' be any two points for which $p - p' = 3$, then p, p' will be respectively identical with some two out of the three points 1, 2, 4. And it will at once be seen that the simplified formulæ for the connective of any two of these three points become illusory.

Now the point $6i+1$ is the connective of $3i-1$ and $3i+2$, and the point $6i+5$ is the connective of $3i+1$ and $3i+4$.

Hence, in each of these cases, the simplified formulæ become illusory, that is, the expressions for each of the co-ordinates vanish when

$$x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3$$

vanishes, and must therefore contain it as a common measure. Moreover, the simplified formulæ for the connective co-ordinates for the points xU, yV, zW;

xU', yV', zW' will contain x^2yz, y^2zx, z^2xy, and will therefore have the common measure xyz. Hence the values of the co-ordinates when freed from these common measures will be of the order in x, y, z, $2(3i-1)^2 + 2(3i+2)^2 - 9$ for the point $6i+1$, and $2(3i+1)^2 + 2(3i+4)^2 - 9$ for the point $6i+5$, that is $(6i+1)^2$ and $(6i+5)^2$ respectively, and will obviously continue to be quantics in x^3, y^3, z^3 multiplied by x, y, z respectively. Hence the theorem being true for index inferior to 6 is true universally.

It will be observed that any co-ordinate X of the point k must contain the X co-ordinate of the point k' where k' is any factor of k; for if $k = \delta k'$ the point k may be obtained by forming the point δ to the point k', and it has been shown that the δ derivative to any point has co-ordinates which contain respectively those of the initial point. Consequently the X co-ordinate to any point k may be resolved into factors containing a primitive part of the order τk (the totient of k) in the variables, and a non-primitive part containing the primitive part of each power of a prime contained in k, and with the exception of the single factor x all the others will be quantics in x^3, y^3, z^3; and, of course, the same remark applies to the other two co-ordinates Y and Z. We might obtain the point $m \ddagger n$ as the connective of m, n. In that case the simplified formulæ would give expressions of the order $2(m^2+n^2)$ in x, y, z; and as the actual order of the co-ordinates in those variables is $(m \ddagger n)^2$, it follows that when $m - n \equiv 0$, mod. 3, there will be a common measure (a symmetrical function of x, y, z) of the order $(m-n)^2$, and when $m + n \equiv 0$, mod. 3, of the order $(m+n)^2$ running through those expressions, and it might be desirable to ascertain its form; but without waiting to solve this problem*, which is irrelevant to the matter in hand, I shall proceed at once to consider the derivatives corresponding to indices which are multiples of the number 3, to obtain which it is only necessary, as will be seen immediately, to combine one given point of inflexion with one arbitrary point of the curve. But, before doing so, it may be well to notice, that while the preceding investigation serves to show that the abridged formulæ for the connective co-ordinates possess the common measure

$$xyz(x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3),$$

it does not demonstrate categorically that there is no other; or that some power of the second factor above written other than the first might not be a common measure. Consequently, what we have strictly proved, as will be evident on reviewing the argument, is that the order to a derivative of the index $3i \pm 1$ cannot *exceed* the square of that index; but before I come to an end of the discussion I trust to be able to establish with *Dirichletian* rigour that the order is actually equal to the square of the index†.

* It is completely solved in the corollary to Title 5.

† This anticipation (for it was only such when these words were written) will be found fully realised under Title 5.

Title 2.—*On the Completed or Continuously Numbered Scale of Rational Derivatives to an Arbitrary Point on a Cubic, of which one Point of Inflexion is given.*

Let I be the given point of inflexion, and let any point (or system of points) and another point (or system of points respectively) collinear with the former in respect to I be called opposites. It is obvious that $(I, I) = I$, or that the inflexion is its own opposite. It will be convenient to denote the opposite to any point by the same index, but accented.

We have, then, obviously,

$$(p', p) = I\,;\ (p')' = p \text{ and } (p', q)' = I,\ (p', q) = (I, I),\ (p', q)$$
$$= (I, p'),\ (I, q) = (p, q').$$

Let $(I', 2) = 3$; $(I', 5) = 6$; and in general $(I', 3i-1) = 3i$. This is matter of definition. Let, now, the infinite system 1, 2, 3, 4, 5, 6, 7 ... and its opposite be regarded as a single group. I say, (1), that this will be a closed group, in the sense that a straight line drawn through any two points (contiguous or apart) of this double chain will cut the cubic in a third point included in the group, (2), that the new points will be rational in respect to the co-ordinates of the initial point and the given point of inflexion, and, (3), that the order in the variables for every point, without regard to its relation to the modulus 3, will be, as before, the square of its index.

I proceed to show that the connective of any two points in the double chain may be expressed as a single point therein. Several cases present themselves according to the form of each of the two connected points in respect to the modulus 3, except when the indices are congruent in respect to that modulus.

When the residues (r, r'), in respect to that modulus, are dissimilar, the result will in general be different according as one of them (as r) belongs to the higher or lower index.

In what follows it is to be understood that $i \gtreqless j$.

Theorem 1. To prove that

$$3i+1,\ (3j+1)' = 3j - 3i$$

and

$$3i+2,\ (3j+2)' = (3j-3i)'.$$

[This will imply that

$$(3i+1)',\ 3j+1 = (3j-3i)'$$

and

$$(3i+2)',\ 3j+2 = 3j-3i.]$$

We have

$$3i+1,\ (3j+1)' = (3i-1, 2),\ [(3j-1)', 2'] = (2, 2'),\ [3i-1, (3j-1)']$$
$$= (3i-1)',\ 3j-1 = [(3i-2)', 1'],\ (3j-2, 1) = (1, 1'),\ [(3i-2)', 3j-2]$$
$$= 3i-2,\ (3j-2)'.$$

Hence, $3i+1,\ (3j+1)'=1,\ (3j-3i+1)'=(1,\ 2),\ [(3j-3i-1)',\ 2']$
$=(2,\ 2'),\ [1,\ (3j-3i-1)']=1',\ (3j-3i-1)=3j-3i$
and $3i-1,\ (3j-1)'=I,\ [3i-2,\ (3j-2)']=(3j-3i)'$.

Theorem 2. To prove that

$$3i+1,\ (3j-1)'=(3i+3j)'$$

and

$$3i-1,\ (3j+1)'=3i+3j.$$

[This will imply that

$$(3i+1)',\ 3j-1=3i+3j$$

and

$$(3i-1)',\ 3j+1=(3i+3j)'.]$$

We have $3i+1,\ (3j-1)'=3i-1,\ 2;\ (3j+1)',\ 2'=(3i-1)',\ 3j+1$
$=[(3i-2)',\ 1'],\ (3j+2,\ 1)=3i-2,\ (3j+2)'$.

Therefore, $3i+1,\ (3j-1)'=1,\ (3j+3i-1)'=(3i+3j)'$
and $3i-1,\ (3j+1)'=I,\ [(3i-1)',\ 3j+1]=3i+3j$.

Collecting the results of these two theorems, we see that

$$\left.\begin{aligned}&3i\pm 1,\ (3j+1)'=3j\mp 3i=(3i\mp 1)',\ 3j-1\\ \text{and}\quad &3i\pm 1,\ (3j-1)'=(3j\pm 3i)'=(3i\mp 1)',\ 3j+1\end{aligned}\right\}, \qquad \text{(A)}$$

so that, using $p \stackrel{\circ}{-} q$ (where neither p nor q contains 3), to denote that one of the two numbers $p+q,\ p \sim q$, which is divisible by 3, $(p,\ q')$ is always either $p \stackrel{\circ}{-} q$ or $(p \stackrel{\circ}{-} q)'$. Also

$$3i+1,\ (3j)'=(3i-1,\ 2),\ [1,\ (3j-1)']=(1,\ 2),\ [3i-1,\ (3j-1)']$$
$$=(3j-3i)',\ 1=(1',\ 3j-3i+1),\ (2,\ 1)=[(1',\ 1),\ (3j-3i+1,\ 2)]$$
$$=(3j-3i-1)';$$

again $3i,\ (3j+1)'=(3i-1,\ 1'),\ [(3j-1)',\ 2']=1',\ (3j-3i)'$
$=(1',\ 2'),\ [1,\ (3j-3i-1)']=(1,\ 1'),\ [(3j-3i-1)',\ 2']=3j+1-3i;$
and lastly $3i,\ (3i+1)'=(3i-1,\ 1'),\ [(3i-1)',\ 2']=I,\ 1'=1$.

Hence, collecting the results, $3\iota,\ (3i+1)'=(3i+1)\sim 3\iota$, whatever the relation of magnitude may be between i and ι.

Similarly,

$$3i-1,\ (3j)'=(3i+1,\ 2),\ [1,\ (3j-1)']=(1,\ 2),\ [3i+1,\ (3j-1)']$$
$$=1,\ (3i+3j)'=(3i+3j-1)';$$

$$(3i)',\ 3j-1=[(3i-1)',\ 1],\ (3j+1,\ 2)=1,\ (3i+3j)'=(3i+3j-1)';$$

and $(3i)',\ 3i-1=[(3i-1)',\ 1],\ (3i+1,\ 2)=1,\ (6i)'=(6i-1)'$.

Hence, collecting the results, $3i-1,\ (3\iota)'=(3i+3\iota-1)'$, and we have

$$\left.\begin{aligned}&3\iota,\ (3i+1)'=(3i+1)\sim 3\iota;\ \ (3\iota)',\ 3i+1=[(3i+1)\sim 3\iota]'\\ &3\iota,\ (3i-1)'=3i-1+3\iota;\ \ \ (3\iota)',\ 3i-1=(3i-1+3\iota)'.\end{aligned}\right\} \qquad \text{(B)}$$

Also,

$$\left.\begin{aligned} 3\iota,\ 3i-1&=(3\iota-1,\ 1'),\ (3i-2,\ 1)=(3\iota-1,\ 3i-2)'=[(3i-1)\sim 3\iota]' \\ 3\iota,\ 3i+1&=(3\iota-1,\ 1'),\ (3i+2,\ 1)=(3\iota-1,\ 3i+2)'=(3\iota+3i+1)'. \end{aligned}\right\} \quad (\mathrm{B}')$$

It remains only to determine the connectives of $3i$, 3ι and of $3i$, $(3j)'$ or $(3i)'$, $3j$, which is easily done, for

$$3i,\ 3\iota=(3i-1,\ 1'),\ (3\iota-1,\ 1')=(1',\ 1'),\ (3i-1,\ 3\iota-1)=2',\ 3i+3\iota-2.$$

Hence (by A) $3i$, $3j=(3i+3j)'$ and consequently $(3i)'$, $(3\iota)'=3i+3\iota$.

Again

$3i$, $(3j)'=(3i-1,\ 1')$, $[(3j-1)',\ 1]=(1,\ 1')$, $[3i-1,\ (3j-1)']=$ (by theorem A) I, $(3j-3i)'=3j-3i$. Hence also $3j$, $(3i)'=(3j-3i)'$.

These three results may be designated theorem C; and theorems A, B, B′, C collectively prove that the original scale 1, 2, 4, 5, 7, 8 ..., which formed a closed system (so to say "group"), remains still closed when we complete it by insertion of multiples of 3, provided that we join on to the completed system 1, 2, 3, 4, 5, 6, 7 ... the opposite system 1′, 2′, 3′, 4′, 5′, 6′, 7′ ....

In every case it will be observed the connective of two indices (disregarding the accent) is either their sum or their difference.

The double scale may be formed by alternate addition of 1 and 1′ in the manner following:

$$1,1=2 \quad 1',2=3 \quad 1,3=4' \quad 1',4'=5' \quad 1,5'=6' \quad 1',6'=7$$
$$1,7=8 \quad 1',8=9 \quad 1,9=10' \quad 1',10'=11' \quad 1,11'=12' \ldots$$

which gives the numbers 1, 2, 3, 4′, 5′, 6′, 7, 8, 9, 10′, 11′, 12′, etc.; and, in like manner, by interchanging 1, 1′, we may obtain 1′, 2′, 3′, 4, 5, 6, 7′, 8′, 9′, 10, 11, 12, etc.

The new points 3, 6, 9 ...; 3′, 6′, 9′ ... belong to the natural scales 1, 2, 5 ...; 1′, 2′, 5′ ... collectively and not respectively; and the accented and unaccented multiples of 3 might have had their significations interchanged without any impropriety. It is now necessary to extend the law of the order in the variables to these inserted points, and to prove that for them, as for the points in the natural scale, the order of any point, in the variables of the initial point, is the square of its index.

If the cubic be thrown into the canonical form $x^3+y^3+z^3+kxyz$, the point $x=1$, $y=-1$, $z=0$ may be taken to represent I, and if x, y, z be the initial point 1, the co-ordinates of 1′ (the connective of 1 and I) become by the general formula yz, zx, z^2, or, more simply, y, x, z.

To find 3, then, we have to take the connective of y, x, z and $x(y^3-z^3)$, $y(z^3-x^3)$, $z(x^3-y^3)$; its co-ordinates, accordingly, by the general formula, are

$$\begin{gathered} yz\,(z^3-x^3)\,(x^3-y^3)\,y^2-\;x^3z\;(y^3-z^3)^2 \\ xz\,(x^3-y^3)\,(y^3-z^3)\,x^2-\;y^3z\;(z^3-x^3)^2 \\ xy\,(y^3-z^3)\,(z^3-x^3)\,z^2-yxz^2\,(x^3-y^3)^2; \end{gathered}$$

or, neglecting the common factor z, the co-ordinates of 3 are

$$\begin{gathered} y^3\,(x^3-y^3)\,(x^3-z^3)+\;x^3\;(y^3-z^3)^2 \\ x^3\,(y^3-x^3)\,(y^3-z^3)+\;y^3\;(z^3-x^3)^2 \end{gathered}$$

and $$xyz\,(z^3-x^3)\,(z^3-y^3)+xyz\,(x^3-y^3)^2;$$

or $$\begin{gathered} y^3x^6+z^3y^6+x^3z^6-3x^3y^3z^3 \\ x^3y^6+z^3x^6+y^3z^6-3x^3y^3z^3 \end{gathered}$$

and $$xyz\,(z^6+y^6+x^6-x^3y^3-z^3x^3-y^3z^3).$$

In the particular case where $x^3+y^3+z^3=0$, these expressions (writing for greater brevity L, M, N for x^3, y^3, z^3) become

$$\begin{gathered} ML^2-(L+M)\,M^2+L\,(L+M)^2+3LM\,(L+M) \\ LM^2-(L+M)\,L^2+M\,(L+M)^2+3LM\,(L+M) \\ xyz\,[(L+M)^2+L^2+M^2-LM+(L+M)^2] \end{gathered}$$

or $$\begin{gathered} L^3+6L^2M+3LM^2-M^3 \\ M^3+6M^2L+3ML^2-L^3 \\ 3xyz\,(L^2+LM+M^2); \end{gathered}$$

which remain equally good, as co-ordinates of the point 3 to the initial point x, y, z, when the cubic is $x^3+y^3+Cz^3$, as is easily seen by writing $C^{\frac{1}{3}}z=\zeta$.

The point 3, it follows from what precedes, is of the order 9 in the variables x, y, z, and the same will be true for $3'$, which is obtained from 3 by the interchange of x and y; but in order that these points may be arithmetically as well as algebraically rational, it is of course necessary that the given cubic may admit of being expressed under the form

$$Ax^3+Ay^3+Cz^3+Kxyz,$$

where A, C and K are integers.

Again, since $6=3', 3'$, 6 is the 2 of $3'$, and similarly $6'$ is the 2 of 3; since $9=3', 6'$ and $6'$ is the 2 of 3, 9 is the 3 of 3. So again, since $12=3', 9'$ and $9'$ is the 3 of $3'$, 12 is the (1, 3) of $3'$, that is, the $4'$ of $3'$ or 4 of 3; and similarly $12'$ is the 4 of $3'$. So again,

$15=(3', 12')=(1, 4)$ of $3'=5$ of $3'$, and $15'=5$ of 3

$18=(3', 15')=(1, 5')$ of $3'=6'$ of $3'=6$ of 3, and $18'=6$ of $3'$

$21=(3', 18')=(1, 6)$ of $3'=7'$ of $3'=7$ of 3, and $21'=7$ of $3'$

$24=(3', 21')=(1, 7)$ of $3'=8$ of $3'$, and $24'=8$ of 3;

$27=(3', 24')=(1, 8')$ of $3'=9'$ of $3'=9$ of 3 ....

Hence, in general,

$$9i+3=(3i+1) \text{ of } 3;\ 9i+6=(3i+2)' \text{ of } 3;\ \text{and } 9i=3i \text{ of } 3.$$

Consequently

$$3^q(3i+1)=(3i+1) \text{ of } 3 \text{ of } 3 \text{ of } 3 \ldots (q \text{ times repeated}),$$

and $$3^q(3i+2)=(3i+2)' \text{ of } 3 \text{ of } 3 \text{ of } 3 \ldots (q \text{ times repeated}).$$

From this it follows, obviously, that $3^q(3i\pm 1)$ and $[3^q(3i\pm 1)]'$ are each of the order $[3^q(3i\pm 1)]^2$ in the variables, and thus the law of the squares extends to all points alike in the completed scale.

Title 3.—*On Compound Derivation.*

The object of what follows is to show that any derivative of a derivative has for its index (due regard being paid to the accents) the product of the numerical values of the indices of the operator and operand derivatives, that is to say, the $i^?$ of $j^?=ij^?$; the mark of interrogation denoting either a blank or an accent, as the case may be. Thus, while connection involves addition or subtraction, composition involves a process of multiplication.

(1) Let us consider the i of j when neither i nor j contains 3. Then

$3k+1$ of $j=(2$ of $j)$, $(3k-1$ of $j)$ and $3k+2$ of $j=(1$ of $j)$, $(3k+1$ of $j)$.

Suppose the theorem proved up to $3k-1$. Then

$$3k+1 \text{ of } j=2j,\ 3kj-j=(3k+1)j$$
$$3k+2 \text{ of } j=j,\ 3kj+j=(3k+2)j.$$

Hence it is true up to $3(k+1)-1$, and, being true when $k=1$ (since 1 of $j=j$ and 2 of $j=j, j=2j$), it is true universally.

In like manner, since 1 of $j'=j'$ and 2 of $j'=j', j'=I, (j, j)=(2j)'$, it may be shown that i of $j'=(ij)'$. Moreover

$$1' \text{ of } j=j', \text{ and therefore } 2' \text{ of } j=(1' \text{ of } j), (1' \text{ of } j)=j', j'=2j'$$

and
$$(3k+1)' \text{ of } j=(2' \text{ of } j), [(3k-1)' \text{ of } j]$$
$$(3k+2)' \text{ of } j=(1' \text{ of } j), [(3k+1)' \text{ of } j];$$

so that, if the equation i' of $j=(ij)'$ holds good up to $i=3k-1$,

$$(3k+1)' \text{ of } j=[(3k+1)j]', \text{ and } (3k+2)' \text{ of } j=[(3k+2)j]';$$

so that the equation i' of $j=(ij)'$ will hold good up to $3(k+1)-1$, and, being true for $k=1$, is true universally.

In like manner, since $1'$ of $j'=j$, it will follow that i' of $j'=ij$.

It remains to obtain the corresponding equations when i, j are one or both of them multiples of 3.

Since 3 of $3^q = (3^q, 3^q), (3^q)' = (2 . 3^q)', (3^q)' = 3^{q+1}$,

$$9 \text{ of } 3^q = 3 \text{ of } 3 \text{ of } 3^q = 3 \text{ of } 3^{q+1} = 3^{q+2},$$

$$27 \text{ of } 3^q = 3 \text{ of } 9 \text{ of } 3^q = 3 \text{ of } 3^{q+2} = 3^{q+3}, \text{ and so on.}$$

Hence $$3^p \text{ of } 3^q = 3^{p+q}.$$

Again, $$3 \text{ of } 3j+1 = (3j+1, 3j+1), (3j+1)'$$
$$= 6j+2, (3j+1)' = 9j+3 \text{ by A.}$$

Hence 3^2 of $3j+1 = 3$ of $9j+3 = (18j+6)', (9j+3)' = 27j+9$ by C,

3^3 of $3j+1 = 3$ of $27j+9 = (54j+18)', (27j+9)' = 81j+27$ by C,

and so on. Hence $$3^p \text{ of } 3j+1 = 3^p(3j+1).$$

Again, $$3 \text{ of } 3j+2 = (3j+2, 3j+2), (3j+2)'$$
$$= 6j+4, (3j+2)' = (9j+6)' \text{ by A.}$$

Hence 3^2 of $3j+2 = 3$ of $(9j+6)' = 18j+12, 9j+6 = (27j+18)'$ by C,

and so on. Hence $$3^p \text{ of } 3j+2 = [3^p(3j+2)]'.$$

Again, $3j+1$ of $3^p = (2 \text{ of } 3^p), (3j-1 \text{ of } 3^p) = (3^p, 3^p), (3j-1 \text{ of } 3^p)$
$$= (2 . 3^p)', (3j-1 \text{ of } 3^p)$$

and $$3j-1 \text{ of } 3^p = (1 \text{ of } 3^p), (3j-2 \text{ of } 3^p).$$

Suppose it true that $3j-2$ of $3^p = (3j-2)\,3^p$ for a certain value of j.

Then $$3j-1 \text{ of } 3^p = 3^p, (3j-2)\,3^p = [(3j-1)\,3^p]'$$

and $$3j+1 \text{ of } 3^p = (2 . 3^p)', [(3j-1)\,3^p]' = (3j+1)\,3^p.$$

But 1 of $3^p = 1 . 3^p$; hence, for all values of j,

$$3j+1 \text{ of } 3^p = (3j+1)\,3^p = 3^p \text{ of } 3j+1$$

$$3j-1 \text{ of } 3^p = [(3j-1)\,3^p]' = 3^p \text{ of } 3j-1.$$

Hence, by the well-known method of successive transformation, we obtain the following results:

When neither m nor n contains 3, when both contain 3, and when one of them contains 3 and the other is of the form $3j+1$, we have

$$m \text{ of } n = n \text{ of } m = m' \text{ of } n' = n' \text{ of } m' = mn$$

$$m \text{ of } n' = n' \text{ of } m = m' \text{ of } n = n \text{ of } m' = (mn)'.$$

In the remaining case (namely when of m and n, one contains 3 and the other is of the form $3j-1$), we have

$$m \text{ of } n = n \text{ of } m = m' \text{ of } n' = n' \text{ of } m' = (mn)'$$

$$m \text{ of } n' = n' \text{ of } m = m' \text{ of } n = n \text{ of } m' = mn.$$

This completes the algorithm of rational derivation.

Title 4.—On Pertactile or Periodic Points on a Cubic Curve.

A pertactile point, or point of pluperfect tactility, on a general cubic is a point at which the cubic admits of a higher order of contact with another curve than is in general possible. Thus the points of inflexion are pertactile points, because a tangent at one of them will meet the curve in three consecutive points. The same is the case with Plücker's twenty-seven points, because at each of them a conic of closest contact will pass through six consecutive points, the sixth point in which any conic passed through five consecutive points cuts the curve coinciding, in this case, with the point of contact. So, in general, a curve of the ith order can only be made to pass through $3i-1$ consecutive points situated at P; but if the ith derivative of P is a point of inflexion, then the $3i$th point common to all curves of the ith order passing through $3i-1$ consecutive points at P will coincide with P, so that such curves will pass through $3i$ consecutive points, and P may accordingly be termed a point of pluperfect tactility, or more briefly, a pertactile point.

To prove that this is the case, it is necessary, in the first place, to prove that, at a general point P in the cubic, the $3i$th point in which all curves of the ith order passing through $3i-1$ consecutive points at P intersect the cubic, is the $(3i-1)$th derivative of P, which may be done inductively as follows:

Suppose P_{3i-1} is the residual of $3i-1$ consecutive points at P. To find the residual of $3i+2$ consecutive points there, we may combine $3i-1$ giving the residual P_{3i-1}, two more of them giving the residual P_2, and one giving Q, R, any two points collinear with P. We then combine (P_{3i-1}, P_2), (Q, R) and obtain P_{3i+1}, P_1 which gives P_{3i+2} as the required residual. Hence the theorem, being true for P_2 (the residual of two consecutive points at P) and true for $P_{3(i+1)-1}$ if true for P_{3i-1}, is true universally.

If, now, the residual of $3i-1$ points at P is to fall at P we must have $P_1 = P_{3i-1}$.

(1) Suppose $i = 3k-1$, then $P_1, P_{i-1} = P_{i-1}, P_{3i-1}$, that is $P_i = P_{2i}$.

Hence P_i is a point of inflexion I, or, as we may express it, P is an ith sub-derivative of such point, or $P = I_{\frac{1}{i}}$.

(2) Suppose $i = 3k+1$, then $P_1, P_2 = P_2, P_{3i-1}$, that is $P_1 = P_{3i+1}$.

Hence $P_1, P_{i+1} = P_{i+1}, P_{3i+1}$, that is $P_i = P_{2i}$, and, as before, $P = I_{\frac{1}{i}}$.

(3) Suppose $i = 3k$.

Then $1, (i-1)' = (i-1)', 3i-1$, that is $i' = 2i = i', i'$. Consequently i', and therefore also i, is a point of inflexion.

Hence, as in the other two cases, P is an ith sub-derivative of a point of inflexion*, which may either be the point used to form the scale, or any of the eight other inflexions†.

It may be well to notice here that whilst P_i, when i does not contain 3, is, as already shown, of the form xU, yV, zW, it follows from the law of compound derivation, since P_3 is of the form R, S, $xyz\Theta$ (where R, S, Θ, like U, V, W, are quantics in x^3, y^3, z^3) that P_i, when i is a multiple of 3 or any power of 3, will be of the form M, N, $xyz\Omega$ (where M, N, Ω are still quantics in x^3, y^3, z^3).

Calling X, Y, Z any ith derivative to $x^3+y^3+z^3+kxyz=0$, we must have $X^3+Y^3+Z^3+kXYZ=0$; and, in order for such derivative to be a point of inflexion, it is necessary and sufficient that $X=0$ or $Y=0$ or $Z=0$; combining these equations respectively with the given cubic, we shall obtain, in all, 3 times $3i^2$ or $9i^2$ points, sub-derivatives of the ith grade to one or other of the inflexions; but out of these, whether i be or be not divisible by 3, nine will correspond to $x=0$, $y=0$, or $z=0$ combined with the curve, that is, will be the points of inflexion themselves. Moreover, unless i be a prime number, it follows from the law of compound derivation, combined with the fact that x, y, z enter distributively or collectively into the derived co-ordinates X, Y, Z, that, if i' be any factor of i, and X', Y', Z' the co-ordinates of the i'th derivative, Z will contain Z' and X, Y or Y, X, will contain X', Y' respectively. There will thus be a *primitive* part to X, Y, Z which results from driving out all the factors corresponding to any factor of i (unity included), and, if we suppose $i=a^\alpha . b^\beta . c^\gamma \ldots$, the order of this primitive part in the variables x, y, z, it is easy to see, will be

$$a^{2(\alpha-1)} . b^{2(\beta-1)} . c^{2(\gamma-1)} \ldots \{(a^2-1)(b^2-1)(c^2-1)\ldots\},$$

which may be called the quadri-totient to i, and is the product of two factors, one the totient of i and the other what that totient becomes when $+1$ is substituted throughout for -1 in its expression, and which, if a name were needed for it, might be called the contra-totient.

The number of proper, or primitive, ith sub-derivatives of any point of inflexion will thus be the quadri-totient of i (just as the number of primitive ith roots of unity is the totient), and the total number of pertactile points of the ith grade, 9 times the quadri-totient of i.

It is easy to see that the points corresponding to the non-primitive factors of X, Y, Z satisfy, but in an *improper* manner, the conditions of the question. For, if i' is any sub-multiple of i $\left(\text{say } i'=\dfrac{i}{\delta}\right)$ and P' is an i'th sub-derivative

* A sub-derivative of an inflexion may conveniently be termed a sub-inflexion.

† The above formulæ show that i, $i'=3i=3i'$; hence $3i$ and $3i'$ coincide with the original point of inflexion, whereas i, i', $2i$, $2i'$ need not coincide with the original point of inflexion.

of a point of inflexion, through P' may be drawn δ curves each of the order i' (constituting an improper curve of the order i), each passing through $3i'$ consecutive points, and consequently their *ensemble* passes through $\delta . 3i'$ or $3i$ consecutive points. We have now obtained the generalization of the theorem of which the enumeration of the points of inflexion and Plücker's points constitute the two first steps, and it is very easy to calculate the number of pertactile points N of any given grade i. Thus for

$$i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\ldots$$
$$\frac{N}{9} = 1, 3, 8, 12, 24, 24, 48, 48, 72, 72, 120, 96\ldots$$

The calculation is facilitated by the remark that if i, j are prime to each other, the number of (ij)th sub-derivatives to any one point of inflexion is the product of the number of ith by the number of jth sub-derivatives; the quadritotient obeying the same law as the totient in this particular.

If i is the grade of the pertactile point P, so that $P_1 = P_{3i-1}$, then P_i is an inflexion, and P_{3i} is I, the original inflexion. Moreover

$$P_1 = P_1,\ P_2 = P_{3i-1},\ P_2 = P_{3i+1}$$

$$P_2 = P_1,\ P_1 = P_1,\ P_{3i-1} = P_{3i-2} \text{ and also} = P_2,\ P_4 = P_{3i-2},\ P_4 = P_{3i+2}$$

$$P_4 = P_2,\ P_2 = P_2,\ P_{3i-2} = P_{3i-4} \text{ and also} = P_2,\ P_{3i+2} = P_{3i+4}, \text{ and so on.}$$

And again, $\quad P'_3 = P_3,\ I = P_3,\ P_{3i} = P'_{3i+3}$, and therefore $P_3 = P_{3i+3}$;

and $\quad P_{3i-3} = P'_{3i+3},\ P_6 = P'_3,\ P_6 = P'_3$ whence $P_3 = P'_{3i-1}$;

$$P_6 = P'_3,\ P'_3 = P'_{3i+3},\ P'_{3i+3} = P_{6i+6} \text{ and also} = P_{3i-3},\ P_{3i-3} = P'_{6i-6}.$$

Thus in general, $\quad P_{3r+1} = P_{3i\pm(3r+1)}$; $P_{3r-1} = P_{3r\pm(3r-1)}$

and $$P_{3r} = P_{3i+3r} = P'_{3i-3r}.$$

Thus the natural scale $\quad P_1P_2P_4P_5\ldots$

and the completed scale $\quad \begin{cases} P_1P_2P_3P_4P_5P_6\ldots \\ P'_1P'_2P'_3P'_4P'_5P'_6\ldots \end{cases}$

are each of them periodic, the period of the indices being $3i$. We may, accordingly, describe pertactile by the simpler name of periodic points. Every complete set of periodic points forms a closed system. By a complete set is to be understood the $9i^2$ sub-derivatives of the 9 points of inflexion, and by a closed system is to be understood one such that every connective and tangential of the points which it contains is itself a point of the system. According to what law such closed system may be resolved into partial closed systems must form the subject of further inquiry. When $i = 2$, the complete closed system of 36 points we know is resoluble into nine closed systems, each containing one point of inflexion and its three collinear anti-tangentials, and also, in four different ways, into three closed systems, each containing a collinear set of inflexions and their three sets of anti-tangentials.

We are now in a position to solve the problem of in-and-exscribed k-laterals.

Suppose $k=3$, then $2^3+1=3i$ where $i=3$, and the point P_1 will coincide with the point P_8, provided P_3 is a point of inflexion. So that the apices of the in-and-exscribed triangles are the 81 points which satisfy the equation $P_3=P'_6$, of which 9 will correspond to the points of inflexion and 72 remaining over will give 24 finite triangles. If we denote by p, p', p'' three consecutive points in a straight line at any point of inflexion, pp', $p'p''$, $p''p$ form an infinitesimal triangle degenerating into a straight line, and this furnishes an improper solution of the question.

Calling M, N, $xyz\Omega$ the co-ordinates of P_3 when $P_1=x, y, z$, the 72 points are given by combining the equation $MN\Omega=0$ with the equation to the curve.

If $k=4$, we make $2^4-1=3i$ where $i=5$, and if $P_1=P_{3i-1}$, we have also $P_1=P_{3i+1}$; and the apices of the quadrilateral are found by making P_i, that is P_5, a point of inflexion.

The general form of P_5 being xU, yV, zW, the proper sub-derivatives P_5 result from $UVW=0$ combined with the equation to the cubic, and there result $\dfrac{9(25-1)}{4}$, that is 54 in-and-exscribed quadrilaterals.

Each point of inflexion may still be regarded as yielding an improper solution of the question, since pp', $p'p''$, $p''p'$, $p'p$ may be viewed as a degenerate infinitesimal quadrilateral.

So when $k=5$, making $2^5+1=3i$, $i=11$; and there will result $\dfrac{9(11^2-1)}{5}=216$ in-and-exscribed pentagons.

Likewise, since $\dfrac{2^7+1}{3}=43$, there result $9\,\dfrac{43^2-1}{7}$, that is 9.264 or 2376 in-and-exscribed heptagons.

Let us now consider a case of k a composite number, and to fix the ideas, suppose $k=15$. Make $\dfrac{2^{15}+1}{3}=i$, then $i=10923$. $\dfrac{2^{15}+1}{3}$, *by virtue of its form*, contains the factors $\dfrac{2^3+1}{3}$ and $\dfrac{2^5+1}{3}$, that is 3 and 11, and is in fact equal to $3.11.331$. P_i will therefore be of the form $xU_3U_{11}\mathbf{U}$, $yV_3V_{11}\mathbf{V}$ $zW_3W_{11}\mathbf{W}$ (xU_3, yV_3, zW_3 corresponding to P_3, and xU_{11}, yV_{11}, zW_{11} to P_{11}).

Accordingly $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ will each be of the degree $(3.11.331)^2-3^2-11^2+1$, and the equation $\mathbf{UVW}=0$, combined with the equation to the curve, will give the apices of the in-and-exscribed quindecagons, not including the improper solutions due to the points of inflexion, nor those due to the apices of the in-and-exscribed triangles or pentagons, which, in a certain but improper

sense, each belong to the case of quindecagons. The number of apices of the proper quindecagons will therefore be $9[(3.11.331)^2 - 3^2 - 11^2 + 1]$, comprising sub-inflexions of several grades, as follows: $9(331^2 - 1)$ of the 331th grade, $9(3^2 - 1)(11^2 - 1)$ of the 33rd grade, $9(3^2 - 1)(331^2 - 1)$ of the 993rd grade, $9(11^2 - 1)(331^2 - 1)$ of the 3641th grade, and $9(3^2 - 1)(11^2 - 1)(331^2 - 1)$ of the 10923rd grade*. The above number of apices may be written $9[11^2 . 3^2(331^2 - 1) + (3^2 - 1)(11^2 - 1)]$, so that the number of quindecagons is $9[11^2 . 3^2 . 22 . 332 + 8^2]$.

It may be noticed that the primitive algebraical factor of $2^{15} + 1$, namely 331, is a prime number. But the primitive part of $2^k - 1$ (k being even) or $2^k + 1$ (k being odd), that is $2^k - 1$ or $2^k + 1$ stripped of its obligatory factors dependent algebraically on the prime factors of k, may be a composite number.

Thus, let us suppose $k = 9$, the problem being that of finding the nature and number of the in-and-exscribed nonagons. Here $i = \frac{2^9 + 1}{3} = 171$, $2^9 + 1$ having, besides the obligatory factor $2^3 + 1$ due to its algebraical form, the *two* factors 3 and 19.

Taking each divisor of 171, namely 3, 9, 19, 57, 171, we see that the 3rd, 9th, 19th, 57th, and 171th sub-derivatives of the nine points of inflexion will each of them be an apex of an in-and-exscribed nonagon. Of these, the 3rd sub-derivatives, and they only, give improper solutions of the problem, they being the apices of the in-and-exscribed triangles. Hence the aggregate of proper apices and the corresponding nonagons separate into four distinct groups, corresponding to the primitive sub-derivatives of the 9th, 19th, 57th, and 171th grades respectively, of the inflexions. The number of the nonagons belonging to the several groups will be the quadritotients of 9, 19, 57, 171, that is $9^2 - 9$, $19^2 - 1$, $(19^2 - 1)(3^2 - 1)$, $(9^2 - 9)(19^2 - 1)$ respectively, that is $171^2 - 9$, exactly the same as if 57 had been a prime number N, in which case the $(3N)^2$ sub-derivatives of an inflexion of the grade $3N$ would be subject to the deduction of $9 - 1$ for in-and-exscribed triangles, and 1 for the point itself.

To make more clear the distinct solutions of which the problem of in-and-exscription of a k-lateral in general admits, consider the case of $k = 8$. Here

$$i = \frac{2^8 - 1}{3} = \frac{2^4 - 1}{3}(2^4 + 1) = 85.$$

The first factor (the one algebraically contained in i) is 5 and the primitive algebraical factor is 17. The total number of octagonal apices

* It is obvious that any derivative of an inflexion is itself an inflexion. For instance, if J is an inflexion, J_2 is the same as J, and J_3 (namely, J', J_2) is either J, J_2, that is J, or (I, J), J_2, that is, (I, J), J, that is, I (I being some other point of inflexion). Hence if $P_{\frac{i}{\delta}}$ is an inflexion, P_i is also an inflexion.

will be $9(85^2-5^2)$, the number 5^2 corresponding to the points of inflexion and the in-and-exscribed quadrilaterals. These 255^2-15^2 apices will consist of points of the form $I_{\frac{1}{17}}$ and $I_{\frac{1}{85}}$, the number of the former being $9(17^2-1)$ and of the latter $9(17^2-1)(15^2-1)$.

It is easily seen that, in general, the number of apices of in-and-exscribed k-laterals is nine times the *functional totient* of $\left(\frac{2^k-\bar{1}^k}{3}\right)^2$, or, what is the same thing the number of apices is the functional totient of $(2^k-\bar{1}^k)^2$, as previously stated in Note to Proem in the last number of the *Journal**; the number of k-laterals is, of course, the number of apices divided by k. For instance, we thus have for the number of apices of quindecagons, nonagons, and octagons, respectively,

$$(2^{15}+1)^2-(2^3+1)^2-(2^5+1)^2+(2^1+1)^2,$$
$$(2^9+1)^2-(2^3+1)^2,\ (2^8-1)^2-(2^4-1)^2,$$

as found above.

Since i is odd, every divisor of i will necessarily be so too. Conversely, it is easy to prove that every odd sub-derivative of a point of inflexion is an apex of an in-and-exscribed polygon, and to determine the number of its sides. For let i, any odd number, be given, and let k be the least number which will satisfy the condition that $2^k-\bar{1}^k$ shall be a multiple of $3i$, then the sub-inflexions of the ith grade will be the apices of an in-and-exscribed k-lateral. I give, in the annexed table, the values of k corresponding to a given value of i, which, of course, are unique; whereas to a given value of k, in general, several values of i will correspond.

i	3	5	7	9	11	13	15	17	19	21	23	25	27
k	3	4	6	9	5	12	12	8	9	6	22	20	27

to which may be subjoined the reciprocal table

$k=3$	$i=3$
$k=4$	$i=5$
$k=5$	$i=11$
$k=6$	$i=7, 21$
$k=7$	$i=43$
$k=8$	$i=17, 85$
$k=9$	$i=9, 19, 57, 171$
$k=10$	$i=31, 341$
$k=11$	$i=683$
$k=12$	$i=13, 15, 35, 39, 65, 91, 195, 273, 455, 1365.$

[* See p. 345 above.]

To illustrate the way in which this table is formed, take the case of $k=12$; then $\frac{2^{12}-1}{3}=3.5.7\times 13$ where 3 belongs to $k=3$, 5 to $k=4$, 7 to $k=6$; the values of i are found by taking the divisors of 1365, except those which are found set against $k=3$, $k=4$, $k=6$, that is 3, 5, 7, 21.

The successive tangentials of any even-graded inflexional sub-derivative as $2^q i$, where i is odd, will evidently consist of a chain of q points attached to the ring formed by the apices of an in-and-exscribed polygon of k sides, where k is the least number which makes $2^k \pm 1$ divisible by $3i$.

In all cases (since k is to have the minimum value which makes $\frac{2^k \pm 1}{3}$ contain i) $2k$ must be $\tau(3i)$ or a submultiple of it, so that, if $i=3^q j$, k is either $3^q \tau j$ or a submultiple of it; when $i=3^q$, since the cyclotomic functions of the first species $\chi_3\, 2$, $\chi_{3^2}\, 2, \ldots \chi_{3^q}\, 2$ can only contain the first power of the intrinsic divisor 3, it follows that $k=3^q=i$, as is seen in the table to be the case for $i=3$, 9, 27; or, in other words, a 3^qth sub-derivative of a point of inflexion is an apex of an in-and-exscribed polygon of 3^q sides.

It may be as well to mention again here, by way of a remind, that the number of in-and-exscribed k-laterals whose apices are ith sub-derivatives of the inflexions, is always the kth part of nine times the quadritotient of i; when $i=3^q$ this number will be $\frac{1}{3^{q-2}}\{3^{2q}-3^{2q-2}\}$, that is $3^{q+2}-3^q$, being thus 24, 72, 216, etc., for triangles, nonagons, eikosiheptagons, etc.

Title 5.—*An Exact Proof of the Scalar Law of Squares*.*

I will now give an exact proof of the law that the order in the variables of P_n is n^2 in regard to the co-ordinates of P, and furthermore that the co-ordinates when $i=3m\pm 1$ are of the form xU, yV, zW, and when $i=3m$ are of the form M, N, $xyz\Omega$; x, y, z being the co-ordinates of the primitive P_1 and U, V, W, M, N, Ω quantics in x^3, y^3, z^3. Of course the order of a point means the order of its system of co-ordinates *expressed in its lowest terms*, that is to say when the values of the three co-ordinates have no common measure, and consequently the co-ordinates of any *two* of them are relatively prime in an algebraical sense, as follows from the equation

$$X^3+Y^3+Z^3+kXYZ=0.$$

The law to be established comprises, it will be seen, two elements,—one numerical, the *rule of squares;* the other formal, containing two rules, one regarding the *distribution* of x, y, z between the co-ordinates, the other the quanticity of the parts not multiplied by x, y, z or xyz in respect to x^3, y^3, z^3.

Let us suppose that the law is true up to n inclusive. I shall show that it is true up to $2n$ inclusive.

[* See below, p. 385.]

(1) For the case of $2i$ where $i \gtreqless n$.

Let X, Y, Z be the system of co-ordinates to P_i in its lowest terms; then, by the law of compound derivation, P_{2i} is

$$X(Y^3 - Z^3),\ Y(Z^3 - X^3),\ Z(X^3 - Y^3).$$

If these regarded as functions of X, Y, Z had any common measure X, Y or $X, Z^3 - X^3$ would have a common measure. Hence X, Y, Z would all have a common measure. Nor can they have any common factor F, a function of x, y, z. For in that case, when $F = 0$, we should have

$$Y^3 - Z^3 = 0,\ Z^3 - X^3 = 0 \text{ or } X^3 = Y^3 = Z^3,$$

and the arbitrary parameter k would be $-3 \,.\, 1^{\frac{1}{3}}$, so that the cubic would become a triplet of straight lines, a supposition which falls outside the pale of the question.

Hence P_{2i} will be of four times the order of P_i, and therefore, by hypothesis, of the order $4i^2$, that is, $(2i)^2$. Also, obviously, the form xU, yV, zW or $M, N, xyz\Omega$ (as the case may be) which exists for i is maintained for $2i$, which is or is not divisible by 3 according as i is or is not so divisible.

(2) Let the index be any odd number less than $2n$.

I shall first establish a Lemma concerning the co-ordinates given by my formulæ for the connectives of P, Q and P', Q, where P' is the opposite to P in respect to a given point of inflexion (say $x = 1$, $y = -1$), and

$$x^3 + y^3 + z^3 + kxyz = 0$$

is the equation to the cubic.

The connectives of $\begin{matrix}(u, v, w)\\(u', v', w')\end{matrix}$ and of $\begin{matrix}(v, u, w)\\(u', v', w')\end{matrix}$

are represented respectively by

$$\left.\begin{matrix} vwu'^2 - v'w'u^2 \\ wuv'^2 - w'u'v^2 \\ uvw'^2 - u'v'w^2 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} uwu'^2 - v'w'v^2 \\ wvv'^2 - w'u'u^2 \\ vuw'^2 - u'v'w^2, \end{matrix}\right.$$

the 3rd co-ordinate being the same in both systems, which, of course, remain to be reduced to their simplest terms, being at present each of the order $2i^2 + 2j^2$.

I say that the same quantity F cannot divide each of the two sets of quantities when u, v, w; u', v', w' are derivatives, one of an even, the other of an odd grade of the same point on the cubic.

For, if so, let $F = 0$; then each quantity in the two systems becomes zero.

Call $\dfrac{u}{w}, \dfrac{v}{w}; \dfrac{u'}{w'}, \dfrac{v'}{w'}$, $r, s; r', s'$ respectively.

Then

$$(1)\ldots\ldots sr'^2 - s'r^2 = 0 \qquad rr'^2 - s's^2 = 0\ldots\ldots(3)$$
$$(2)\ldots\ldots rs'^2 - r's^2 = 0 \qquad r'r^2 - ss'^2 = 0\ldots\ldots(4)$$
$$(5)\ldots\ldots rs = r's'.$$

Writing $r^3 = R$, $s^3 = S$, $r'^3 = R'$, $s'^3 = S'$; 5, (3, 4), (1, 2) respectively give $RS = R'S'$, $RR' = SS'$, $R'S = RS'$. The second and third of these combined give $R^2 = S^2$, $R'^2 = S'^2$ and the first and second combined give $R'^2 = S^2$. Hence, $R^2 = R'^2 = S^2 = S'^2$, and consequently the original equations (1), (2), (3) give $S = S'$, $R = R'$, $R = S'$ or $r^3 = s^3 = r'^3 = s'^3$.

Let $r = \alpha s$, $r' = \beta s'$, $s = \gamma s'$. Then $\alpha^3 = \beta^3 = \gamma^3 = 1$, and all the equations (1), (2), (3), (4), (5) will easily be found to be satisfied when (and only when) $\alpha = \beta\gamma$.

The equations $r^3 = s^3$, $r'^3 = s'^3$, that is, $u^3 = v^3$, $u'^3 = v'^3$, imply that the points P, Q are two either distinct or identical anti-tangentials to the same point of inflexion $x = 1$, $y = -1$. I say that this is impossible when P, Q are derivatives of the degrees i, j of the same point U on the curve, if $i + j$ is an odd number. It must be noticed that P and Q (two Plückerian points belonging to the same point of inflexion I) are identical with P' and Q' respectively.

Any even-degreed derivative of P or Q is I, and any odd-degreed derivative is the same point P or Q over again.

Let now $i\mu - j\nu = 1$. Then $U = U_{i\mu - j\nu}$ will be (without regard to the modulus 3) the connective of $U_{i\mu}$ and $U_{j\nu}$, because we may substitute at will U_i' for U_i and U_j' for U_j. But $U_{i\mu}$ and $U_{j\nu}$, if μ, ν be both odd, will be U_i and U_j over again, or if μ, ν be one odd and the other even, will be I and one of the two Plückerian points.

Hence U is the connective of I and a Plückerian point, or else of two Plückerians which are identical, or of two Plückerians (both appurtenant to I) which are distinct.

In the 1st and 3rd cases, then, U is a Plückerian, in the 2nd case a point of inflexion. But every derivative of a point of inflexion is a point of inflexion, and every even-degreed derivative of a Plückerian is also a point of inflexion; but by hypothesis (since one of the two numbers i, j is even) an even-degreed derivative of U is a Plückerian, which is self-contradictory. Hence, it follows that the expressions given by *my* formulæ for the connectives of P_i, P_j and P_i', P_j when $i + j$ is odd, say P, Q, R; P', Q', R', cannot have a common factor; so that if M is a common measure of P, Q, R and M' of P', Q', R', M is relatively prime to M'.

Let ϕ, ψ, ω be always understood to mean $\phi(x^3, y^3, z^3)$, $\psi(x^3, y^3, z^3)$, $\omega(x^3, y^3, z^3)$; let (μ), (ν) be understood to mean the prime systems of coordinates u, v, w; u', v', w' which represent μ, ν (μ and ν being numbers,

accented or unaccented, representing derivatives to the indices μ and ν); let $[\mu, \nu]$ represent the unreduced system of the co-ordinates of the connective of μ, ν, namely, $v'w'u^2 - vwu'^2$, $w'u'v^2 - wuv'^2$, $u'v'w^2 - uvw'^2$; (μ, ν) the above system *reduced* by elimination of the greatest common measure of its terms.

If (μ), (ν) are each of the form $x\phi$, $y\psi$, $z\omega$, $[\mu, \nu]$ is of the form $x^2yz\phi_1$, $xy^2z\psi_1$, $xyz^2\omega_1$, but $[\mu', \nu]$, that is, the unreduced connective of $y\psi$, $x\phi$, $z\omega$; $x\phi'$, $y\psi'$, $z\omega'$, is of the form $z\phi_1$, $z\psi_1$, $xyz^2\omega_1$.

Again, if (μ) is of the form $x\phi$, $y\psi$, $z\omega$ and (ν) of the form ϕ_1, ψ_1, $xyz\omega_1$, $[\mu', \nu]$, the unreduced connective of the systems $y\psi$, $x\phi$, $z\omega$ and ϕ_1, ψ_1, $xyz\omega_1$, is easily seen to be of the form $zx\Phi$, $zy\Psi$, $z^2\Omega$.

Furthermore, the order in the variables of (p') is obviously the same as that of (p).

Now it has been shown under Title 2 that

$$6i - 1 = (3i-1)', \; 3i \qquad 6i - 5 = (3i-3)', \; (3i-2)' \qquad 6i - 3 = (3i-2)', \; 3i - 1.$$

If, then, $(3i)$ and $(3i-3)$* are of the form ϕ, ψ, $xyz\omega$, and $(3i-2)$, $(3i-1)$ each of the form $x\phi$, $y\psi$, $z\omega$, it follows that $[6i-1]$ and $[6i-5]$ will be of the form $zx\phi$, $zy\psi$, $z^2\omega$, and $[6i-3]$ of the form $z\phi$, $z\psi$, $xyz^2\omega$.

The above inference suffices to show that, if, for all values of $3\mu \pm 1$ and 3μ up to n inclusive, it be true that $(3\mu \pm 1)$ is of the form $x\phi$, $y\psi$, $z\omega$ and of the order $(3\mu \pm 1)^2$, and (3μ) is of the form ϕ, ψ, $xyz\omega$ and of the order $(3\mu)^2$; then the same will be true up to $2n$ inclusive.

That this is true for even values not exceeding $2n$ appears from what has been already shown. Confining, then, our attention to odd numbers less than $2n$; these must be representable by $6i-5$, $6i-3$ or $6i-1$, and by hypothesis the form of each of the systems $(3i)$, $(3i-1)$, $(3i-2)$, $(3i-3)$ fulfils the conditions of the last paragraph but one; consequently the form of $[6i-5]$, $[6i-3]$, $[6i-1]$ will be $zx\phi$, $zy\psi$, $z^2\omega$; $z\phi$, $z\psi$, $xyz^2\omega$; $zx\phi$, $zy\psi$, $z^2\omega$, namely, in every case the factor z will be contained in each term of the system $[(\iota-1)', \iota^?]$, which represents an unreduced system of co-ordinates of the point $2\iota - 1$, the mark of interrogation signifying a blank or an accent as the case may be.

But either the point 1 or the point 1′ will, in every case, correspond to the connective obtained by changing $(\iota-1)'$ into $\iota - 1$†; moreover, the unreduced system of co-ordinates to that connective will have the third term, say π, in common with the unreduced system to $2\iota - 1$ above mentioned.

This contrary system we know must have the common factor $\frac{\pi}{z}$ because 1

* $(3i-3)'$ will obviously be of the same form as $3i-3$.

† For, on consulting Title 2, it will be found that *in every case*, if the arithmetical value of the index of P_i, P_j is $i \pm j$, that of P_i', P_j is $(\iota \mp j)^?$.

and $1'$ are denoted by x, y, z; y, x, z respectively. Hence the unreduced system for $2\iota - 1$ can have no other common factor except z, which they have been shown to have; since, were it otherwise, the *two* contrary systems would have some quantity contained in $\frac{\pi}{z}$ for a joint common measure, which has been proved to be impossible.

Hence, the form of $(2\iota - 1)$ is $x\phi$, $y\psi$, $z\omega$ or ϕ, ψ, $xyz\omega$ according as $2\iota - 1$ is not or is divisible by 3, and its order is in all cases $2(\iota - 1)^2 + 2\iota^2 - 1$, that is, $(2\iota - 1)^2$.

Hence the form-law of distribution of the simple powers of the variables x, y, z and of the quanticity in x^3, y^3, z^3 of the multipliers of x, y, z or of 1, 1, xyz, as well as the numerical law that the order of any derivative is the square of its index, will be true up to $2n$ inclusive if true up to n inclusive; and being true for $n = 1$, is true universally.

As a corollary we may now do away with the restriction of $i + j$ being odd, and affirm that in all cases (the futile one of $i = j$ alone excepted), if the reduced system of co-ordinates to the connective of P_i, P_j be F, G, H, and to that of P_i', P_j be F', G', H', then the unreduced system expressing those connectives given by my formulæ of connection will be $H'F$, $H'G$, $H'H$; HF', HG', HH', respectively; for the two systems of unreduced co-ordinates (each of the order $2i^2 + 2j^2$) contain, one of them a common factor of the order $(2i^2 + 2j^2) - (i - j)^2$, that is, $(i + j)^2$, the other a common factor of the order $(2i^2 + 2j^2) - (i + j)^2$, that is, $(i - j)^2$, and these two factors being prime to each other, their product must be contained in the term common to the two systems, and being of the same order $(i + j)^2 + (i - j)^2$ as that common term, must be equal to it.

Hence, if π be the common unreduced term, and H, H' the two reduced terms, we must have $\pi = \frac{\pi}{H} \cdot \frac{\pi}{H'}$ or $\pi = HH'$, as was to be shown.

As a matter rather of curiosity than of real importance I will state the analogous law when the connective and cross-connective between two derivatives is expressed by Cauchy's formulæ instead of my own. These formulæ, it will be remembered, give for the co-ordinates of the connective of u, v, w; u_1, v_1, w_1 the minor determinants of the matrix

$$\begin{vmatrix} vw_1 - v_1w\,; & w_1u - wu_1\,; & uv_1 - u_1v \\ uu_1 \quad ; & vv_1 \quad ; & ww_1 \end{vmatrix}$$

If, now, the prime system of co-ordinates to the connectives of P_i, P_j; P_i', P_j be denoted as before by F, G, H; F', G', H', I find by calculation that the Cauchian formulæ will present these two systems under the unreduced forms

$$(F' + G')F, \quad (F' + G')G, \quad (F' + G')H$$
$$(F + G)F', \quad (F + G)G', \quad (F + G)H',$$

between which there is no common term; and consequently, had I not discovered my own simpler formulæ, the method of proof of the Law of Squares which I have employed would have been inapplicable, and it is not easy to see what other strict method of proof could have taken its place.

I have thus accomplished the very difficult task of proving a negative, in this instance the non-existence of *latent* common factors to the co-ordinates of the connective of any two given derivatives. I might have founded a much easier proof of the Law of Squares upon Mr Franklin's geometrical solution of the problem of finding the number of in-and-exscribed k-laterals to a cubic (if one could feel quite assured *à priori* of the strict logic of the process*) as follows: He has virtually found (*vide* last number of the *Journal*) that the number of apices of the in-and-exscribed k-laterals of *every kind* [and not excluding the points of inflexion] is $(2^k - \bar{1}^k)^2$. If, then, $2^k - \bar{1}^k = 3i$, it follows from what has been shown in the preceding pages, that the order of P_i in the co-ordinates of P is $\frac{1}{9}(3i)^2$, that is, i^2.

Let now i' be any number whatever, and τ the totient of $3i'$; then τ is even, and, by Fermat's Theorem, $2^\tau - \bar{1}^\tau = 3i'i''$.

Hence, if μ', μ'' are the orders of $P_{i'}$, $P_{i''}$ respectively, the law of compound derivation will suffice to lead to the conclusion that $\mu'\mu''$ will be the order of $P_{i'i''}$, and accordingly $\mu'\mu'' = i'^2 i''^2$; but $\frac{\mu'}{i'^2}$, $\frac{\mu''}{i''^2}$, it has been proved under a preceding Title, are neither of them greater than unity: hence each of them is equal to unity, and i'^2 is the order of $P_{i'}$, as was to be shown.

Addendum on the Degorder of the Derivatives to a Point on a Cubic in the Natural Scale.

Let n be any number not divisible by 3. The nth derivative, it has been proved, is of the order n^2 in the variables. It remains to determine its *degree in the coefficients*.

When $n = 2$ we know that the degorder is [4; 4], each new co-ordinate being one of the minors of the rectangular matrix

$$\left|\begin{matrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dH}{dx} & \frac{dH}{dy} & \frac{dH}{dz} \end{matrix}\right|,$$

where U is the cubic and H its Hessian.

* In that solution the apices are found as the intersections of the cubic with another curve. Certain of these intersections are seen from geometrical considerations to count twice, and others three times; but while we have no reason to suppose any further cause of reduction, the non-existence of such cause is not proved.—F. F.

Suppose ν to be the degree in the coefficients of the nth derivative. Then the degree of the $(2n)$th derivative regarded as the second of the nth will be $4\nu + 4$, and regarded as the nth of the second will be $n^2 . 4 + \nu$, and these two must be equal. Hence $3\nu = (n^2 - 1)\, 4$ or $\nu = \frac{4}{3}(n^2 - 1)$.

Hence the degorder of any nth derivative in the natural scale is $\left[\dfrac{4n^2 - 4}{3};\ n^2\right]$. If we substitute the co-ordinates of this derivative in the given cubic U, the result must be of the form $U . R$ and will be of the degorder $[1 + 4n^2 - 4;\ 3n^2]$. Hence R is of the degorder $[4n^2 - 4;\ 3n^2 - 3]$. If the well-known covariant of the degorder $[12;\ 9]$ be called J, R is of the same degorder as $J^{\frac{n^2-1}{3}}$, and possibly may be found to be identical with it. To corroborate the validity of the determination of the degorder of the nth derivative, we may proceed as follows:

Imagine, at first, the cubic to be reduced to the canonical form $x^3 + y^3 + z^3 - 3kxyz$. The connective of P_1, P_2 in its reduced form is x, y, z; but in its unreduced form and prior to all simplification, will, by virtue of the theory (Titles 1 and 5), be of the form Mx, My, Mz where

$$M = x^3y^6 + y^3z^6 + z^3x^6 + x^6y^3 + y^6z^3 + z^6x^3 - 6x^3y^3z^3$$
$$+ kxyz\,(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)^{*};$$

consequently M expressed (as I shall hereafter suppose) in terms of the original coefficients and variables, will be of the degorder $[9;\ 9]$: for Mx, My, Mz are of the degorder $[1 + 2 . 4;\ 2(1 + 4)]$, that is, $[9;\ 10]$†. Also the degorder of P_4 will be $[4 + 4 . 4;\ 16]$, that is, $[20;\ 16]$.

Suppose now we wish to find the degorder of P_5.

The unreduced connective of P_1, P_4 will be of the form MX, MY, MZ, where X, Y, Z are the reduced co-ordinates and M is exactly the same thing as before. The degorder of the unreduced co-ordinates will be $[1 + 2 . 20;\ 2(1 + 16)]$, that is, $[41;\ 34]$; and consequently, subtracting $[9;\ 9]$, the degorder of X, Y, Z will be $[32;\ 25]$, that is, $\left[4\,\dfrac{5^2 - 1}{3};\ 5^2\right]$.

So, again, to find P_7 we may regard it as the connective of P_2, P_5. The unreduced degorder of P_7 will thus be seen to be $[1 + 2(4 + 32);\ 2(4 + 25)]$, that is, $[73;\ 58]$, and subtracting, as before, $[9;\ 9]$, the degorder of the

* It is worthy of remark that, if we make $U = 0$, so that $3kxyz$ becomes equal to $x^3 + y^3 + z^3$, the expression in the text for M gives $3M$ equal to the norm of $x + 1^{\frac{1}{3}}y + 1^{\frac{1}{3}}z$, namely,

$$(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3.$$

† In fact, M, as may easily be shown, is the covariant $\left[\Sigma\left(\dfrac{dU}{dy} \cdot \dfrac{dH}{dz} - \dfrac{dU}{dz} \cdot \dfrac{dH}{dy}\right)\dfrac{d}{dx}\right]^2 U$, in other words the symmetrical determinant of the 5th order formed by double-bordering the Hessian matrix with the differential derivatives of the Hessian and of the original cubic.

reduced co-ordinates of P_7 becomes [64; 49], that is, $\left[4\,\frac{7^2-1}{3};\ 7^2\right]$, agreeable to what has been previously found; and so, in general, supposing the degrees of P_μ and $P_{\mu+3}$ in the coefficients to be $4\,\frac{\mu^2-1}{3}$ and $4\,\frac{(\mu+3)^2-1}{3}$, the unreduced degree of $P_{2\mu+3}$ will be $1+8\left\{\frac{\mu^2-1}{3}+\frac{(\mu+3)^2-1}{3}\right\}$, from which subtracting 9, the reduced degree becomes $8\left\{\frac{2\mu^2+6\mu+4}{3}\right\}$, which is the same thing as $4\left\{\frac{(2\mu+3)^2-1}{3}\right\}$, as ought to be the case. There is, therefore, no loophole for doubt left open as regards the degorder of any *natural* derivative to the index k (a number necessarily of the form $3i \pm 1$) being $[\frac{4}{3}(k^2-1);\ k^2]$, a notable result!

We are now in possession of a method for finding any natural derivative to the index n. If n is even, it may be derived immediately from the derivative to the index $\frac{n}{2}$. If n is odd, it must be of the form $2\mu+3$ where μ is not divisible by 3.

Taking P as the initial point, P_μ and $P_{\mu+3}$ may be considered as known. Calling their co-ordinates X, Y, Z; X_1, Y_1, Z_1 respectively, and substituting $\lambda X+\mu X_1$, $\lambda Y+\mu Y_1$, $\lambda Z+\mu Z_1$ in the equation to the cubic, we shall obtain an equation of the form $\lambda^2\mu B+\lambda\mu^2 C=0$. The unreduced co-ordinates of $P_{2\mu+3}$ will then be $CX-BX_1$, $CY-BY_1$, $CZ-BZ_1$, which will contain a common measure M of the degorder [9; 9], and $\frac{CX-BX_1}{M}$, $\frac{CY-BY_1}{M}$, $\frac{CZ-BZ_1}{M}$ will be the expression for the point $P_{2\mu+3}$ in its simplest terms.

More generally, if $n=2\mu+3i$, we may obtain, in like manner as above, the unreduced co-ordinates of the connective to P_μ, $P_{\mu+3i}$, and, by an easy calculation, it will be found that the new common measure will be of the degorder $[12i^2-3;\ 9i^2]$, and will be constant, that is, independent of μ for any given value of i, and identical with the common measure to the unreduced co-ordinates of P_{3i+2} regarded as the connective to P and P_{3i+1}.

It is well worthy of remark that if X, Y, Z be the co-ordinates of any derivative, and ξ, η, ζ contragredient to x, y, z, $X\xi+Y\eta+Z\zeta$ will be an invariantive concomitant to the given cubic. This gives rise to a new series of reflexions, the development of which must be deferred to a more convenient occasion*.

* It is obviously a step towards the attainment of the desideratum of finding the general expression for any derivative in an explicit form, or, at all events, by explicit processes and without the necessity for division of the unreduced co-ordinates by a common measure. This latter, it should be observed however, by virtue of what is stated above, is always known *à priori*.

CHAPTER I.

EXCURSUS C.—ON THE TRISECTION AND QUARTISECTION OF THE ROOTS OF UNITY TO A PRIME-NUMBER INDEX.

What follows, so far as it relates to the trisection of the primitive roots of unity, may be regarded as auxiliary to Postscriptum 2, [p. 345, above], inasmuch as it establishes the equation in ω which, when $x = \frac{\omega - 1}{3}$, becomes the equation there assumed. The rest is episodical, except so far as it may be regarded as correlative to the subject matter of Titles 1 and 2 of Excursus A* [pp. 317 ff.].

It will be seen that the equations to a system of three and four periods, usually obtained by long and tedious processes, may, with the aid of one simple and well-known principle, be deduced by processes almost elementary in their character, and into which enter no algebraical calculations except of the very easiest kind.

A sketch of the method was laid by me before the Scientific Congress held at Rheims in the month of August last [p. 438, below].

The index p of the roots is, as usual, supposed to be a prime number; e is the number of the periods, f the number of roots whose sum forms a period, so that $ef = p - 1$; the periods themselves will be called η, namely, $\eta_1, \eta_2, \dots \eta_e$.

Preliminaries.

1. I say, in the first place, that the sum of the ith powers of the periods will be congruous to $-f^{i-1}$ in respect to the modulus p.

For, were it not that in the development of the ith power of any one of the η's some of the combinations of the powers of the roots were unity, it is obvious that we should have $\Sigma\eta^i = -ef^i \div (p-1)$, that is, $-f^{i-1}$, and that we might regard every term in such development as equivalent to $-\frac{1}{p-1}$, without affecting this result. The existence of terms equal to unity will render it necessary to substitute for any such term 1 instead of $-\frac{1}{p-1}$, in order to obtain a correct result, and if there be N of them, the correction to be introduced will be $N\left(1 + \frac{1}{p-1}\right)$, that is, $\frac{N}{p-1} \cdot p$; but as it is obvious that the result must be an integer, it follows that N must be double by

* In any future redistribution of the contents of the entire memoir, it would be proper to incorporate the matter contained in Postscriptum 2, pp. [345—347], with this Excursus.

$(p-1)$, and consequently the value of $\Sigma\eta^i$ *to modulus* p will be $-f^{i-1}$, that is, $-\left(\frac{p-1}{e}\right)^{i-1}$, as was to be shown.

2. From the above it follows that *to modulus* p,

$$\Sigma(e\eta+1)^i \equiv (-1)^i + e(-1)^{i-1} + e\frac{e-1}{2}(-1)^{i-2} + \text{etc.}, \equiv (-1+1)^e \equiv 0,$$

or, in other words, $\Sigma(e\eta+1)^i$ is divisible by p.

But, if s_i and σ_i represent, respectively, the sum of the $i$ary combinations and $i$ary powers of the roots of an equation, we know that $(-)^i s_i =$ coefficient of x^i in $e^{-\sigma_1 x - \frac{\sigma_2}{2}x^2 - \frac{\sigma_3}{3}x^3 \ldots}$, so that s_i multiplied by numbers none exceeding i, is expressible as the sum of integer multiples of $\sigma_\lambda \sigma_\mu \sigma_\nu \ldots$ where

$$\lambda + \mu + \nu + \ldots = i.$$

3. Consequently, s_i multiplied by integers none greater than i, when the roots in question are the e values of $e\eta+1$ and $i>0$, will be divisible by p, and consequently, since e is less than p, all the coefficients of the equation to which those roots appertain will be divisible by p, the first, of course (which is unity), excepted.

Since $\Sigma(e\eta+1) = e\Sigma\eta + e = 0$, the equation whose roots are $\omega_1, \omega_2, \ldots \omega_e$ where $\omega = e\eta+1$ will be of the form $\omega^e + P\omega^{e-2} + Q\omega^{e-3} +$ etc., where P, Q, etc., each contain p; and I may remark, incidentally (although the fact is immaterial to the object in view), that, as may easily be seen, $\Sigma\omega^i$ will be divisible not only by p but also by e, and that consequently the coefficient of ω^{e-i}, in the above equation, will contain the greatest common divisor to e and i.

4. The coefficient P has one or the other of two determinate algebraical values according as f, that is, $\frac{p-1}{e}$, is even or odd.

In the former case, the congruence $x^e + 1 \equiv 0$ [mod p] is soluble, and in the latter, insoluble. Accordingly, in the latter case, we shall have $\Sigma\eta^2 = -f$, and in the former $\Sigma\eta^2 = p - f$, and in each case $\Sigma\eta^2$ will be an odd number. Also, when f is odd (which involves the necessity of e being even)

$$\Sigma\omega^2 = \Sigma(e\eta+1)^2 = -e^2\frac{p-1}{e} - 2e + e = -ep,$$

and when f is even $\Sigma\omega^2$ will be this result augmented by e^2p, that is, $(e^2-e)p$.

Consequently, $P = \frac{e}{2}p$, or $= -\frac{e^2-e}{2}p$, according as f is odd or even.

Thus, when $e=3$, f being necessarily even, $P=-3p$, and when $e=4$, $P=-6p$, or $=2p$, according as $\frac{p-1}{4}$ is even or odd*.

5. With regard to what immediately follows it will also be necessary to determine the *form* of Q in respect to certain moduli for the cases of e equal to 3 and e equal to 4. In the former case

$$\Sigma\omega^3 = \Sigma(e\eta+1)^3 = \Sigma(e^3\eta^3+3e^2\eta^2+3e\eta+1) \equiv 3 \bmod 9,$$

and consequently, since $Q=-\frac{1}{3}\Sigma\omega^3$, $-3Q\equiv 3 \bmod 9$ and $-Q\equiv 1 \bmod 3$.

In the latter case, that is, when $e=4$, since $\Sigma\eta^2$ is always odd $\Sigma\omega^3$ [to mod 32] $\equiv 16-12+4$, that is, $\equiv 8$, and, consequently, $-3Q\equiv 8$ to that modulus.

These *preliminaries* being established, I will now proceed to state the principle referred to in the exordium.

Principle.

A rational integer function of any set of periods of the roots of unity whose coefficients are all whole numbers, which does not change its value for a circular substitution executed upon the periods, it is well-known, must be an integer number; but to this I add that if such function, without changing its arithmetical value, undergoes a change of sign when such a substitution is made, it must necessarily be an integer number multiplied by the difference of the two periods into which the entire sum of the roots may be divided, that is to say, will be a multiple of $\sqrt{p}$, when p is of the form $4K+1$ and of $\sqrt{(-p)}$, when p is of the form $4K-1$†.

As an example, the product of the differences of the roots of the equation in η will be an integer number when e, the number of the periods, is odd, and an integer number multiple of $\sqrt{p}$ or $\sqrt{(-p)}$ (according as $\frac{p-1}{2}$ is even or odd), when the number of periods is even. As another example, if $e=2\epsilon$, the function

$$(\eta_0-\eta_\epsilon)(\eta_1-\eta_{\epsilon+1})(\eta_2-\eta_{\epsilon+2})\dots(\eta_{\epsilon-1}-\eta_{2\epsilon-1}),$$

which changes its sign but not its quantitative value, when $0, 1, 2, 3, \dots (2\epsilon-1)$ are replaced by $1, 2, 3, \dots -1, 0$ will be an integer multiple of $\sqrt{p}$, or of $\sqrt{(-p)}$, according as ϵ is even or odd.

* When $e=2$, $P=p$ or $-p$ according as f is odd or even, so that the equation in ω takes the known form $\omega^2\pm p=0$.

† To put the matter more clearly, call the alternating function F and the difference spoken of Δ. Then ΔF is invariable in sign as well as in magnitude for the circular substitutions in question. Hence $F=\frac{\text{An Integer}}{\sqrt{(\pm p)}}$ but F^2 is an Integer; therefore $F=\text{An Integer}\sqrt{(\pm p)}$. Q.E.D.

We are now in a position to obtain without difficulty the well-known equivalent to the equation corresponding to $e = 3$, given at p. [345], and the corresponding pair of equations for the case of $e = 4$.

A. *Case of* $e = 3$.

The equation in ω, from what has been shown in the preliminaries, must be of the form $\omega^3 - 3px + pq = 0$, and it only remains to determine q.

The discriminant of the above equation being $q^2p^2 - 4p^3$, it follows that the product of the differences of its roots will be $27(4p^3 - q^2p^2)$. But this product is 3^6 into $(\eta_0 - \eta_1)^2(\eta_0 - \eta_2)^2(\eta_1 - \eta_2)^2$, which latter, by the *principle*, is of the form M^2. We have, therefore,

$$4p^3 - q^2p^2 = 27M^2 = 27m^2p^2.$$

Hence, $$4p = q^2 + 27m^2,$$

which serves to determine the value of q^2 absolutely.

To find the value of q, it follows from the preliminaries that $qp \equiv -1 \bmod 3$, and, consequently, since $p \equiv 1 \bmod 3$, $q \equiv -1 \bmod 3$, so that q is perfectly determined.

B. *Case of* $e = 4$.

$\omega^2 - 2\sqrt{p}\omega + R = 0$, $\omega^2 + 2\sqrt{p}\omega + R' = 0$, will be the form of the equations containing, respectively, the pairs of roots ω_0, ω_2 and ω_1, ω_3; for

$$\omega_0 + \omega_2 = (4\eta_0 + 1) + (4\eta_2 + 1) = 2\{2(\eta_0 + \eta_2) + 1\} = 2(2\delta_0 + 1),$$

and, similarly,

$$\omega_1 + \omega_3 = 2\{2(\eta_1 + \eta_3) + 1\} = 2(2\delta_1 + 1)$$

where δ_0 and δ_1 are the two periods which make up together the sum of all the roots, so that $2\delta_0 + 1$ and $2\delta_1 + 1$ are the roots of the equation $\Omega^2 - p = 0$, the sign of the last term being fixed from the fact of $\frac{p-1}{2}$ being by hypothesis even.

Furthermore, R, R' must be of the form $Ap + B\sqrt{p}$, $Ap - B\sqrt{p}$; for $(R - R')\sqrt{p}$, being integer, requires that R, R' shall be of the form $A_1 + B\sqrt{p}$, $A_1 - B\sqrt{p}$, and then RR' being an integer multiple of p involves the necessity of A_1^2, and therefore of A_1 containing p.

The product $(\eta_0 - \eta_2)(\eta_1 - \eta_3)$ consequently becomes

$$\{(A-1)p + B\sqrt{p}\}\{(A-1)p - B\sqrt{p}\},$$

which by the principle must be of the form m^2p, and consequently,

$$(A-1)^2p - B^2 = C^2 \text{ or } (A-1)^2p = B^2 + C^2.$$

The coefficient of ω^2 becomes $-4p+2Ap$ which, by the preliminaries, when $\frac{p-1}{4}$ is even must be equal to $-6p$, so that $A=-1$, and when $\frac{p-1}{4}$ is odd must be equal to $2p$, so that $A=3$.

In each case, therefore, $(A-1)^2=4$ and $4p=B^2+C^2$; consequently, if $p=g^2+h^2$, $4g^2=B^2$, and $4h^2=C$, and the complete equation in ω containing the roots ω_0, ω_1, ω_2, ω_3, becomes $(\omega^2-p)^2-4p(\omega+g)^2=0$ when $\frac{p-1}{4}$ is even and $(\omega^2+3p)^2-4p(\omega+g)^2=0$ when $\frac{p-1}{4}$ is odd. In either case g^2 is given, but the sign of g requires to be determined; alike, however, for one case as for the other, $-8pg$ being the 3rd coefficient after the first, we must have, as shown in the preliminaries, $24pg\equiv 8$ [mod 32], and consequently, since p is of the form $4K+1$, $24g\equiv 8$ [mod 32]. Hence, $3g\equiv 1$ [mod 4], that is, $g\equiv -1$ [mod 4], which gives the required complete determination of g.

The quartisecting equations thus naturally arrived at are expressed in the form in which, according to Bachmann (*Kreistheilung*, p. 230), they were first presented by Lebesgue; the method here given for finding the equations for the trisection and quartisection of the roots of unity will be found on examination to be incomparably simpler, shorter, and more direct than any in common use, and as removing a serious stumbling-block from the path of the student, and, occurring, so far as regards trisection, in the natural course of the development of my subject, I have thought entitled to a place in this memoir. Why I require the trisecting equation is, as will be remembered, to enable me to obtain the conditions of 2 and of 3 being cubic residues to a given index. The condition for 2 being such, strange to say, is nowhere to be found in Bachmann's *Kreistheilung*, although the cubic character of 3 is there duly and fully made out.

The conditions of the one and of the other being cubic residues were, I am informed by M. Lucas, given for the first time in a letter from Gauss to Mlle. Sophie Germain.

Excursus B.

Title 5 (*bis*).—*On the Law of Squares.*

There being errors and inaccuracies not a few in the matter printed under this title, owing to my absence abroad as it went through the press, I have thought it desirable to rewrite it, rectifying the errors, and supplying some steps which were wanting in the demonstrations*. I shall, in what follows,

* In the postscript [p. 378 above] which was thought out on board the transatlantic steamer, the *Bothnia*, and written out, as far as I can recollect, at a single sitting a day or two before

use throughout P_i to denote the ith derivative of P, and x_i, y_i, z_i to signify the reduced coordinates of P_i, so that P_1, x_1, y_1, z_1 will mean the same as P, x, y, z respectively. $x+y=0$, $z=0$ will be taken as the auxiliary point of inflexion, serving to complete the scale, and will be called I. In the natural scale it is easy to see that any derived co-ordinate, as z_i, must contain

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posting it at Queenstown, I have not been able to detect any inaccuracy in the results, although some additional steps and explanations might advantageously have been supplied.

There is, perhaps, one slight exception to be made to this statement as regards the very important theorem, stated but not proved [p. 380], concerning the nature of the form $X\xi+Y\eta+Z\zeta$, where the coefficients of ξ, η, ζ are supposed to be the reduced co-ordinates of any derivative to x, y, z. If $U=0$ is the equation to the cubic in its general form, obviously X, Y, Z are indeterminate, as each may be augmented by an arbitrary multiple of U of suitable degree and order. Consequently, the theorem ought to have been stated in the following form. The co-ordinates X, Y, Z of any such derivative *may be* so expressed that $X\xi+Y\eta+Z\zeta$ shall be a mixed concomitant to U. The fundamental invariantive concomitants to a ternary cubic involving not more than one system of cogredients and a single linear system of contragredients are eleven in number and of the types underwritten:

4 . 0 . 0	4 . 4 . 1
6 . 0 . 0	5 . 4 . 1
1 . 3 . 0	7 . 4 . 1
3 . 3 . 0	9 . 7 . 1
8 . 6 . 0	11 . 7 . 1
12 . 9 . 0	

Hence the co-ordinates of every rational derivative in the natural scale to a point on a cubic curve may be expressed as the coefficients of the contragredient variables in a rational integer function of the above eleven quantities, linear in the latter five, and such that its degree and orders for the nth grade are $\frac{4(n^2-1)}{3}$; n^2, 1.

The particular forms of X, Y, Z which appertain to the concomitant $X\xi+Y\eta+Z\zeta$, and which may be called the *normal* forms, it may be added, are those which actually arise from the processes of *colligation* and *reduction* described in the excursus. By *colligation* I mean the determination of the *general* analytical connective of x, y, z; x', y', z' by the same method as that applied at pages [354, 355] to the canonical quadrinomial form of the cubic. The co-ordinates of such connective are absolutely determinate, inasmuch as the equation which each set of co-ordinates must satisfy is of the order 3, whereas the co-ordinates in question are of the second order only in each set of variables (and of course of the first degree in the coefficients of the cubic). By *reduction* I mean that when in the co-ordinates of the general connective for x, y, z; x', y', z' are substituted the *normal* forms of the co-ordinates for derivatives of the grades μ, $\mu+3i$, their common factor of the degorder $(12i^2-3, 9i^2)$ is to be cast out.

This common factor, it may be noticed, is *always* a covariant of the cubic. When $i=1$, it is seen *a posteriori* that this is the case, for its value is expressible (see footnote, p. [379]) under the form of a known covariant, say Θ (which was obtained by means of using the canonical form of the cubic); that it must be true for all values of i may be deduced from the general algebraical theorem that if in a covariant to any given form, in place of the variables x, y, z be substituted $\frac{d\Omega}{d\xi}$, $\frac{d\Omega}{d\eta}$, $\frac{d\Omega}{d\zeta}$, where Ω is any invariantive concomitant to such form, and ξ, η, ζ are contragredient to x, y, z, the resulting expression will be itself an invariantive concomitant. To obtain now the reducing factor for the connective to P_μ, $P_{\mu+3i}$ (p. [380]) it is only necessary to substitute in Θ x_i, y_i, z_i (the normal co-ordinates of the ith derivative) in lieu of x, y, z where $x_i\xi+y_i\eta+z_i\zeta$ is known to be an invariantive concomitant to the cubic. Hence, by the algebraical theorem above stated, the corresponding reducing factor (not containing ξ, η, ζ) is necessarily a covariant to the cubic, as was to be shown.

the original one, as z. For when $z=0$, P will be a point of inflexion and P_i identical with P, hence (x_i, y_i, z_i) will express the same point of inflexion, and consequently $z_i=0$; hence z_i must contain z. When we leave the rational scale, so that i is a multiple of 3, z must contain xyz. For when $z=0$, the ith derivative P will be one of the three points I, I', I'', expressed by $z=0$, $x^3+y^3=0$. If P is I, P_3 is obviously I; if P is I', P_2 is I', and P_3 will be the connective of P_2 and I''; consequently P_3 is I and $z=0$, and the same will be the case if P is I''; hence z_3' contains z.

Again, if $y=0$, P will be some inflexion J, and the connective to I, J being called K, P_3 will be the connective of J, K, that is I, as before; hence z_3 will contain y, and in like manner it will contain x. Also, since in each case P_3 is I, every derivative of P_3 will be I; hence, when $xyz=0$, $z_{3\nu}$ becomes 0; consequently z_i (if i is a multiple of 3) contains xyz.

Again, if x_i, y_i, z_i are the reduced co-ordinates of P_i, I say that $x_i(y_i^3-z_i^3)$; $y_i(z_i^3-x_i^3)$; $z_i(x_i^3-y_i^3)$ will be the *reduced* co-ordinates of x_{2i}, y_{2i}, z_{2i}.

For, if possible, let two of the above co-ordinates have a common factor F; then, since x_i, y_i, z_i have no common factor, $x_i^3-y_i^3$, $y_i^3-z_i^3$ have a common factor, and when $F=0$, $x_i^3=y_i^3=z_i^3$; but $x_i^3+y_i^3+z_i^3+Kx_iy_iz_i=0$. Hence, unless $x_i^3=y_i^3=z_i^3=0$, we must have $3+\sqrt[3]{1}K=0$, but K is arbitrary. Hence, F must be contained in x_i, y_i, z_i contrary to hypothesis.

Although it is a consequence of a general law* that z_i cannot contain z^2, for present purposes it will be sufficient to establish that z_i cannot, for each of two consecutive values of i, contain z^2. Thus, suppose z_{2i-1} and z_{2i} each contained z^2, then, because z_{2i} contains z^2, z_i must do so too; since, otherwise, $x_i^3-y_i^3$ must contain z. If that is possible, let $z=0$; then $x_i^3-y_i^3=0$; but P, and therefore P_i, becomes an inflexion, whereas $x_i^3=y_i^3$ is the necessary and sufficient condition that P_i is a Plückerian point, which is self-contradictory. But since z_i contains z^2, z_{i-1} must also contain z^2, for z_{2i-1} will be contained (see p. [374]) in $\frac{1}{z}(x_iy_iz^2_{i-1}-x_{i-1}y_{i-1}z_i^2)$, and therefore, if z_{i-1} does not contain z^2, z must be contained in x_i or y_i, which is impossible. In like manner, if z^2 is contained in z_{2i}, z_{2i+1}, it will be contained also in z_i and z_{i+1}. Hence it would be contained eventually in z, which is absurd.

Again, it may be shown that z will be the only common measure to z_{i-1} and z_i. For, if possible, let them have any other common measure F, and let F become zero. Then P_{i-1} and P_i both become points of inflexion belonging to the system previously designated as I, I', I'', and by a collineation process

* The law is that $x^i \,.\, y^i \,.\, z^i x_iy_iz_i$, cannot for any value of i contain a square algebraical factor, just as, and *en dernière analyse* for the same general kind of reason, the binomial exponential (a^i+b^i) can contain no such factor.

performed on these points alone or combined with I, P may be obtained. Hence P belongs to the same system of inflexions, that is $z = 0$. Hence F would be contained in a power of z, contrary to hypothesis.

I will now show that if the two systems of unreduced co-ordinates obtained by the colligation of

$$\left.\begin{matrix} x_{i-1}, & y_{i-1}, & z_{i-1} \\ x_i\;\;, & y_i\;\;, & z_i \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} y_{i-1}, & x_{i-1}, & z_{i-1} \\ x_i\;\;, & y_i\;\;, & z_i \end{matrix}\right.$$

be called F, G, H; F', G', H; respectively, the terms F, G, H; F', G', H can have no other measure common to all four than z, or, in other and more precise terms, z is the greatest common measure to the greatest common measures of F, G, H and of F', G', H. For brevity call the two sets of co-ordinates of P_{i-1} and P_i, u, v, w; u', v', w' respectively. Then the unreduced co-ordinates in question will be (p. [374])

$$\left.\begin{aligned} F &= vwu'^2 - v'w'u^2 \\ G &= wuv'^2 - w'u'v^2 \\ H &= uvw'^2 - u'v'w^2 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} uwu'^2 - u'w'v^2 &= F' \\ wvv'^2 - w'v'u^2 &= G' \\ vuw'^2 - v'u'w^2 &= H \end{aligned}\right.$$

into each of which z necessarily enters as a factor, because w, w' have been proved each to contain z.

(u, v, it will be observed, cannot have a common factor, for then u, v, w would have a common factor contrary to hypothesis; and, in like manner, u', v' can have no common factor.)

I say, in the first place, that no indecomposable function of x, y, z, say M, not contained either in w or in w', can be common to F, G, F', G'. For, if so, let F vanish; then, calling $\frac{u}{w}$, $\frac{v}{w}$; $\frac{u'}{w'}$, $\frac{v'}{w'}$, r, s; r', s' respectively, we have

$$(1)\quad sr'^2 - s'r^2 = 0, \qquad\qquad (3)\quad rr'^2 - s's^2 = 0,$$
$$(2)\quad rs'^2 - r's^2 = 0, \qquad\qquad (4)\quad r'r^2 - ss'^2 = 0.$$

Now, none of the terms r, s, r', s' can vanish: for example r cannot vanish, for, if so, from (1) it would follow that $s = 0$, or $r' = 0$, and from (3) that $s = 0$, or $s' = 0$, so that either $r = 0$ and $s = 0$, or $r' = 0$ and $s' = 0$, that is the general values of u and v or of u' and v' must have a common factor M, which is impossible. Hence, combining (1) and (2) or (3) and (4), we derive $rs = r's'$ (5), as might also be obtained immediately by equating to zero the term common to the two systems above.

From (5), from (3) and (4), and from (1) and (2) we obtain respectively

$$r^3s^3 = r'^3s'^3, \quad r^3r'^3 = s^3s'^3, \quad r'^3s^3 = r^3s'^3,$$

the second and third of which are equivalent to $r^6 = s^6$, $r'^6 = s'^6$, and the first and second combined give $r'^6 = s^6$. Hence $r^6 = r'^6 = s^6 = s'^6$, and consequently the original equations (1), (2), (3) give $r^3 = s^3 = r'^3 = s'^3$.

The equations $r^3 = s^3$, $r'^3 = s'^3$ imply that P_{i-1}, P_i are each of them distinct or identical antitangentials to one of the points of inflexion corresponding to $z = 0$, that is are each of them a Plückerian point on the cubic, and P or (P, I) will be a residual either to P_{i-1}, P_i or to (P_{i-1}, I), P_i where I is the auxiliary inflexion used to complete the scale. Hence P is either a Plückerian or an inflexion point, and in either case P_2 will necessarily be an inflexion. Hence one at least of the derivatives P_{i-1}, P_i is an inflexion, but each is a Plückerian, which is absurd.

Thus M (an irresoluble factor common to F, G, F', G') must be contained either in w or in w'. Suppose it is not z and is contained in w, then it cannot be contained in w', for w, w' have no common measure except z, and consequently when $M = 0$, $v'u^2 = 0$, $u'v^2 = 0$, $v'v^2 = 0$, and $u'u^2 = 0$, and either u and v or u' and v' each become zero, which is impossible seeing that neither the general values of u, v nor those of u', v' can have any common factor. In like manner, it follows that M cannot be contained in w'. Consequently, the two systems F, G, H; F', G', H can have no other common measure, except some power of z.

Finally, I say that the only common measure in question is z itself. (1) Suppose it were possible (which it is not) that one of the two terms w or w' (say w) contains z^2, then it has been proved that the other (w') cannot contain z^2. Hence, if $wvv'^2 - w'u'u^2$ contains z^2, u or u' must contain z, and in like manner, if w' and not w contained z, v or v' must contain z, none of which suppositions are admissible. (2) Suppose that neither w nor w' contains z^2. Then writing $w = \omega z$, $w' = \omega' z$, and writing for $\frac{u}{\omega}$, $\frac{v}{\omega}$; $\frac{u'}{\omega'}$, $\frac{v'}{\omega'}$, r, s; r', s' respectively, we shall obtain over again, as before, $r^3 = s^3$, $r'^3 = s'^3$, indicating as before that P_{i-1} and P_i are each of them Plückerian points when $z = 0$, that is, when P is a point of inflexion, which is doubly absurd. Hence it follows that the common measures of F, G, H and of F', G', H have the common measure z, and no other.

We are now in a position to prove the *law of squares*. Suppose it is true for P_{i-1} and P_i, I say it will be true for P_{2i-1}. For consider the connectives of

$$\left.\begin{matrix} x_{i-1}, & y_{i-1}, & z_{i-1} \\ x_i\ \ , & y_i\ \ , & z_i \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} y_{i-1}, & x_{i-1}, & z_{i-1} \\ x_i\ \ , & y_i\ \ , & z_i \end{matrix}\right.$$

as expressed by the formulas above employed. Let $z^2\Omega$ be the third term common to the unreduced systems of co-ordinates.

Allowing (as is the fact) that Ω does not contain z, the reducing factor common to the unreduced co-ordinates of P (or it may be its opposite in

respect to I) must be $z\Omega$, and consequently to the other system corresponding to P_{2i-1} or its opposite, can only be z or z^2; but the latter is impossible, for then z_{2i-1} would not contain z.

Again, if Ω could be conceived equal to $z^q\Omega_1$, the reducing factor for P or its opposite would be $z^{1+q}\Omega_1$, and consequently that for P_{2i-1} or its opposite could not be z^2 and would be z as before. Hence the order of P_{2i-1} in the variables is necessarily $2(i-1)^2+2i^2-1$, that is, $4i^2-4i+1$ or $(2i-1)^2$.

Moreover, it has been shown that if x_i, y_i, z_i are the reduced co-ordinates for P_i, $x_i(y_i^3-z_i^3)$, $y_i(z_i^3-x_i^3)$, $z_i(x_i^3-y_i^3)$ are such for P_{2i}, and consequently, if the law is true for i, it is true for $2i$. Hence, being true for 1, it is true for 2, and therefore for 3, and therefore for 4 and 5 and 6, and therefore for $3+4$, that is, 7, and for 2.4, that is, 8, and for $4+5$, that is, 9, and for 2.5, that is, 10, and so on for every number, as was to be proved*. Thus, this negative proposition, as I have termed it (p. [356]), is completely established. There remains to prove the important proposition contained (but incorrectly proved) on p. [377], to wit, that the unreduced systems of co-ordinates arising from the colligation of

$$\left.\begin{matrix}(x_i,\, y_i,\, z_i)\\(x_j,\, y_j,\, z_j)\end{matrix}\right\} \text{ and of } \left\{\begin{matrix}(y_i,\, x_i,\, z_i)\\(x_j,\, y_j,\, z_j)\end{matrix}\right.$$

will be of the forms LN', MN', NN'; $L'N$, $M'N$, $N'N$, where L, M, N; L', M', N' are the reduced systems of the co-ordinates of the connectives of P_i, P_j, and P_i', P_j respectively.

To illustrate this proposition by an example, consider the connectives of P', P_3, that is, P_2 and of P, P_3, that is, P_4.

z_2 is $z(y^3-x^3)$ and z_4 is of the form $z(y^3-x^3)\,\Omega$, where Ω is of the order 12 in the variables.

Call X_4, Y_4, Z_4 the unreduced co-ordinates arising from the colligation of P, P_3. Suppose x^3-y^3 to become zero, then P becomes a Plückerian, and P_3 will be also such, namely, one of the nine appertaining to the inflexions given by $z=0$†. Hence $x_3^3-y_3^3$ becomes zero. Now X_4, Y_4 represent $yzx_3^2-y_3z_3x^2$,

* In other words, if the theorem is true up to i inclusive, any number between $i+1$ and $2i$ inclusive is either of the form $2j$ or $2j-1$, where j does not exceed i; and being true for j, it is true for $2j$, and being true for $j-1$ and j, it is true for $2j-1$. Hence, if true up to i it is true up to $2i$, but it is true for $i=1$ and therefore for all values of i. Q.E.D.

† The nine points of inflexion on a cubic curve form a closed group, but so also do any three of them which lie in a right line, and also any single one. In like manner, the nine inflexions with their antitangentials, any three of these lying in a right line with their antitangentials, and any one with its antitangentials, form closed groups containing 36, 12, and 4 points respectively. The ornamental-gardening problem of *alignement*, anglice *allineation*, which consists in so disposing a number of points on a plane as to obtain the maximum number or all the various possible numbers of right lines each containing three of the points, finds its systematic solution in the theory of groups of inflexional and sub-inflexional points of various grades.

$xzy_3^2 - x_3z_3y^2$ respectively, and since

$$yzx_3^2 \cdot x_3z_3y^2 - y_3z_3x^2 \cdot xzy_3^2 = zz_3(x_3^3y_3^3 - y_3^3x_3^3) = 0,$$

$X_4 : Y_4 :: yx_3^2 : xy_3^2$, and consequently $X_4^3 - Y_4^3 = 0$; but P_4 is a point of inflexion and not a Plückerian; hence X_4, Y_4 must each contain the factor $x^3 - y^3$, and Z_4 must be of the form $z^2(x^3 - y^3)^2\,\Omega$, for after division by $z(x^3 - y^3)$ it must still contain that factor. Also X_4, Y_4, Z_4 can have no other common measure except $z(x^3 - y^3)$, for after throwing out that factor the quotient is of the order 16, the order of z_4 given by the law of squares. Thus we see that the third unreduced coefficient common to (P, P_3) and (P', P_3) is equal to $z_2 \cdot z_4$, as it ought to be according to the proposition in question.

In some very old numbers of the *Educational Times* will be found questions of the kind proposed by me (not reproduced in the Reprint), of which the solution depends on this order of considerations. In certain cases that had been studied, I ascertained the possible existence of a larger number of collineations than had previously been imagined by other writers on the subject, among whom Mr S. B. Woolhouse deserves special mention for the ingenuity of his constructions. As far as I am aware, the theory of allineation has never been treated by other writers than myself, except by empirical methods, and its dependence on the theory of the general cubic curve was not even suspected.

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TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS FOR SIMULTANEOUS BINARY QUANTICS OF THE FIRST FOUR ORDERS, TAKEN TWO AND TWO TOGETHER.

[*American Journal of Mathematics,* II. (1879), pp. 293—306, 324—329.]

IN the Generating Functions given below, the exponents of the letters a, b, c, d, refer to degree in the coefficients of the quantics of the 1st, 2nd, 3rd and 4th orders respectively; the exponents of the letter x to order in the variables. Where the system consists of two quantics of the same order, the Latin letter and the corresponding Greek letter have been used. In the tabulated numerators, the *minus* sign has been placed *over* the number which it affects.

In each of the systems considered in this paper, with the exception of that consisting of a cubic and a quartic, it is found that there is never more than one groundform of any given type (that is, of a given order in the variables and given degrees in the coefficients of the quantics); where, therefore, in the enumeration of the groundforms, the *type* alone is given, the *number* of groundforms of the type is to be understood to be 1. The symbol (λ, μ) is used to indicate a form of the degrees λ and μ in the coefficients of the two quantics, the number placed first always relating to the quantic of lower order, when the orders are different. In the last three cases, the numbers, as well as the types, of the groundforms are given in tables, which require no explanation.

SYSTEM OF TWO LINEARS*.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-\alpha)(1-a\alpha)}$.

G. F. for covariants, $\dfrac{1}{(1-a\alpha)(1-ax)(1-\alpha x)}$.

Groundforms :

Of order 0 .... (1, 1).

„ „ 1 .... (0, 1), (1, 0).

* "Linear" is here used as a noun, in conformity with the use of the words quadric, cubic, &c.

SYSTEM OF LINEAR AND QUADRIC.

G. F. for differentiants, $\dfrac{1+ab}{(1-a)(1-b)(1-b^2)(1-a^2b)}$.

G. F. for covariants, $\dfrac{1+abx}{(1-b^2)(1-a^2b)(1-ax)(1-bx^2)}$.

Groundforms:

Of order 0 .... (0, 2), (2, 1).
" " 1 .... (1, 0), (1, 1).
" " 2 .... (0, 1).

SYSTEM OF LINEAR AND CUBIC.

G. F. for differentiants, $\dfrac{1+a^2c+(a-a^3)c^2+(1-a^2)c^3-ac^4-a^3c^5}{(1-a)(1-c)(1-c^2)(1-c^4)(1-ac)(1-a^3c)}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-ac)(1-a^3c)(1-ax)(1-cx)(1-cx^3)$.

Numerator: $1-ac+a^2c^2+\{(-1+a^2)c+(2a-a^3)c^2-a^2c^3\}x$
$+\{ac+(1-2a^2)c^2+(-a+a^3)c^3\}x^2+\{-ac^2+a^2c^3-a^3c^4\}x^3$.

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-a^3c)(1-a^2c^2)(1-ax)(1-c^2x^2)(1-cx^3)$.

Numerator: $1+a^3c^3+\{a^2c+ac^2+(a^2-a^4)c^3\}x+\{ac+(a-a^3)c^3-a^3c^5\}x^2$
$+\{(1-a^2)c^3-a^3c^4-a^2c^5\}x^3+\{-ac^3-a^4c^6\}x^4$.

Groundforms:

Of order 0 .... (0, 4), (2, 2), (3, 1), (3, 3).
" " 1 .... (1, 0), (1, 2), (2, 1), (2, 3).
" " 2 .... (0, 2), (1, 1), (1, 3).
" " 3 .... (0, 1), (0, 3).

SYSTEM OF LINEAR AND QUARTIC.

G. F. for differentiants,

$$\frac{1+(a+a^3)d+(a+a^2-a^5)d^2+(1-a^3-a^4)d^3+(-a^2-a^4)d^4-a^5d^5}{(1-a)(1-d)(1-d^2)^2(1-d^3)(1-a^2d)(1-a^4d)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-a^2d)(1-a^4d)(1-ax)(1-dx^2)(1-dx^4)$.

Numerator: $1-a^2d+a^4d^2+\{a^3d+(a^3-a^5)d^2\}x+\{(-1+a^2)d$
$+(2a^2-a^4)d^2-a^4d^3\}x^2+\{ad+(a-2a^3)d^2+(-a^3+a^5)d^3\}x^3$
$+\{(1-a^2)d^2-a^2d^3\}x^4+\{-ad^2+a^3d^3-a^5d^4\}x^5$.

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-a^4d)(1-a^4d^2)(1-ax)(1-dx^4)(1-d^2x^4)$.

Numerator: $1+a^6d^3+\{a^3d+a^3d^2+(a^5-a^7)d^3\}x+\{a^2d+a^2d^2+(a^4-a^6)d^3\}x^2$
$+\{ad+ad^2+(a^3-a^5)d^3\}x^3+\{(a^2-a^4)d^3-a^6d^4-a^6d^5\}x^4$
$+\{(a-a^3)d^3-a^5d^4-a^5d^5\}x^5+\{(1-a^2)d^3-a^4d^4-a^4d^5\}x^6$
$+\{-ad^3-a^7d^6\}x^7.$

Groundforms:

Of order 0 .... (0, 2), (0, 3), (4, 1), (4, 2), (6, 3).
" " 1 .... (1, 0), (3, 1), (3, 2), (5, 3).
" " 2 .... (2, 1), (2, 2), (4, 3).
" " 3 .... (1, 1), (1, 2), (3, 3).
" " 4 .... (0, 1), (0, 2), (2, 3).
" " 5 .... (1, 3).
" " 6 .... (0, 3).

System of Two Quadrics.

G. F. for differentiants, $\dfrac{1+b\beta}{(1-b)(1-b^2)(1-\beta)(1-\beta^2)(1-\beta b)}.$

G. F. for covariants, $\dfrac{1+b\beta x^2}{(1-b^2)(1-\beta^2)(1-b\beta)(1-bx^2)(1-\beta x^2)}.$

Groundforms:

Of order 0 .... (0, 2), (1, 1), (2, 0).
" " 2 .... (0, 1), (1, 0), (1, 1).

System of Quadric and Cubic.

G. F. for differentiants,

$$\frac{1+(2b+b^2)c+(b+b^2+b^3)c^2+c^3-b^4c^4+(-b-b^2-b^3)c^5+(-b^2-2b^3)c^6-b^4c^7}{(1-b)(1-b^2)(1-c)(1-c^2)(1-c^4)(1-bc^2)(1-b^3c^2)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-cx)(1-cx^3).$

Numerator: $1+b^3c^4+\{(-1+b+b^2)c+(b+b^2)c^3-b^3c^5\}x$
$+\{(1+b^3)c^2+(-b-b^4)c^4\}x^2+\{bc+(-b^2-b^3)c^3$
$+(-b^2-b^3+b^4)c^5\}x^3+\{-bc^2-b^4c^6\}x^4.$

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-c^2x^2)(1-cx^3).$

Numerator: $1+b^3c^4+\{(b+b^2)c+(b+b^2)c^3\}x+\{(b+b^2+b^3)c^2$
$+(b^2-b^4)c^4-b^3c^6\}x^2+\{bc+(1-b^2)c^3+(-b-b^2-b^3)c^5\}x^3$
$+\{(-b^2-b^3)c^4+(-b^2-b^3)c^6\}x^4+\{-bc^3-b^4c^7\}x^5.$

Groundforms:

Of order 0 .... (0, 4), (1, 2), (2, 0), (3, 2), (3, 4).
" " 1 .... (1, 1), (1, 3), (2, 1), (2, 3).
" " 2 .... (0, 2), (1, 0), (1, 2).
" " 3 .... (0, 1), (0, 3), (1, 1).

SYSTEM OF QUADRIC AND QUARTIC.

G. F. for differentiants,

$$\frac{1+(b+b^2)\,d+(2b-b^3)\,d^2+(1-2b^2)\,d^3+(-b-b^2)\,d^4-b^3d^5}{(1-b)(1-b^2)(1-d)(1-d^2)^2(1-d^3)(1-bd)(1-b^2d)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-bd)(1-b^2d)(1-bx^2)(1-dx^2)$
$(1-dx^4)$.

Numerator: $1-bd+b^2d^2+\{(-1+b+b^2)\,d+(2b-b^3)\,d^2-b^2d^3\}\,x^2$
$+\{bd+(1-2b^2)\,d^2+(-b-b^2+b^3)\,d^3\}\,x^4$
$+\{-bd^2+b^2d^3-b^3d^4\}\,x^6$.

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-b^2d)(1-b^2d^2)(1-bx^2)(1-dx^4)$
$(1-d^2x^4)$.

Numerator: $1+b^3d^3+\{(b+b^2)\,d+(b+b^2)\,d^2+(b^2-b^4)\,d^3\}\,x^2+\{bd+bd^2$
$+(b-b^3)\,d^3-b^3d^4-b^3d^5\}\,x^4+\{(1-b^2)\,d^3+(-b^2-b^3)\,d^4$
$+(-b^2-b^3)\,d^5\}\,x^6+\{-bd^3-b^4d^6\}\,x^8$.

Groundforms:

Of order 0..............................(0, 2), (0, 3), (2, 0), (2, 1), (2, 2), (3, 3).
„ „ 2..............................(1, 0), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3).
„ „ 4..............................(0, 1), (0, 2), (1, 1), (1, 2), (1, 3).
„ „ 6..............................(0, 3).

SYSTEM OF TWO CUBICS.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-\gamma)(1-\gamma^2)(1-\gamma^4)(1-c\gamma)$
$(1-c^3\gamma)(1-c\gamma^3)$.

Numerator: $1+c^3+(2c+2c^2-c^5-c^6)\,\gamma+(2c+2c^2-c^4-c^5-c^6-c^7)\,\gamma^2$
$+(1+2c^3-c^4-2c^5-c^6-c^7)\,\gamma^3+(-c^2-c^3-c^5-c^6)\,\gamma^4$
$+(-c-c^2-2c^3-c^4+2c^5+c^8)\,\gamma^5+(-c-c^2-c^3-c^4+2c^6+2c^7)\,\gamma^6$
$+(-c^2-c^3+2c^6+2c^7)\,\gamma^7+(c^5+c^8)\,\gamma^8$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-cx)(1-cx^3)$
$(1-\gamma x)(1-\gamma x^3)$.

Numerator:

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^0	c^0	1						
	c^2			1				
	c^3				1			
	c^5						1	
x^1	c^0		$\bar{1}$					
	c^1	$\bar{1}$		1		1		
	c^2		1					
	c^4		1					
	c^5							$\bar{1}$
	c^6						$\bar{1}$	
x^2	c^0			1				
	c^1		2				$\bar{1}$	
	c^2	1		$\bar{1}$		$\bar{1}$		
	c^4			$\bar{1}$				
	c^5		$\bar{1}$					
	c^6							1

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^6	c^2		1					
	c^4				1			
	c^5					1		
	c^7							1
x^5	c^1		$\bar{1}$					
	c^2	$\bar{1}$						
	c^3						1	
	c^5						1	
	c^6			1		1		$\bar{1}$
	c^7						$\bar{1}$	
x^4	c^1	1						
	c^2						$\bar{1}$	
	c^3					$\bar{1}$		
	c^5			$\bar{1}$		$\bar{1}$		1
	c^6		$\bar{1}$				2	
	c^7					1		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^3	c^1		$\bar{1}$		$\bar{1}$		
	c^2	$\bar{1}$				1	
	c^3				$\bar{1}$		$\bar{1}$
	c^4	$\bar{1}$		$\bar{1}$			
	c^5		1				$\bar{1}$
	c^6			$\bar{1}$		$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-c^2x^2)(1-cx^3)$
$(1-\gamma^2x^2)(1-\gamma x^3)$.

Numerator :

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^0	c^0	1							
	c^2			1					
	c^3				1				
	c^5						1		
x^1	c^1			1		1			
	c^2		1		1				
	c^3			1		1			
	c^4		1		1				
x^2	c^1		1		1				
	c^2			1					
	c^3		1		1				
	c^4					1			
	c^5								$\bar{1}$
	c^7						$\bar{1}$		
x^3	c^0				1				
	c^1			1		$\bar{1}$		$\bar{1}$	
	c^2		1						
	c^3	1				$\bar{2}$		$\bar{1}$	
	c^4		$\bar{1}$		$\bar{2}$				
	c^5							$\bar{1}$	
	c^6		$\bar{1}$		$\bar{1}$		$\bar{1}$		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7	γ^8
x^8	c^3			1					
	c^5					1			
	c^6						1		
	c^8								1
x^7	c^4					1		1	
	c^5				1		1		
	c^6					1		1	
	c^7				1		1		
x^6	c^1			$\bar{1}$					
	c^3	$\bar{1}$							
	c^4				1				
	c^5					1		1	
	c^6						1		
	c^7					1		1	
x^5	c^2			$\bar{1}$		$\bar{1}$		$\bar{1}$	
	c^3		$\bar{1}$						
	c^4					$\bar{2}$		$\bar{1}$	
	c^5		$\bar{1}$		$\bar{2}$				1
	c^6							1	
	c^7		$\bar{1}$		$\bar{1}$		1		
	c^8					1			

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^4	c^1	1				$\bar{1}$		
	c^2				$\bar{1}$		$\bar{1}$	
	c^3			$\bar{1}$		$\bar{2}$		$\bar{1}$
	c^4		$\bar{1}$		$\bar{2}$		$\bar{1}$	
	c^5	$\bar{1}$		$\bar{2}$		$\bar{1}$		
	c^6		$\bar{1}$		$\bar{1}$			
	c^7			$\bar{1}$				1

*Table of Groundforms**.

Order in the Variables.	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3	4
0	0					1
	1		1		1	
	2			1		
	3		1		1	
	4	1				
1	1			1		1
	2		1		1	
	3			1		
	4		1			

Order in the Variables	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3
2	0			1	
	1		1		1
	2	1		1	
	3		1		
3	0		1		1
	1	1		1	
	2		1		
	3	1			
4	1		1		

System of Cubic and Quartic.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-d)(1-d^2)^2(1-d^3)(1-c^2d)$
$(1-c^4d)(1-c^2d^3)(1-c^4d^3)$.

Numerator:
$$\begin{aligned}
&1 + c^3 + (3c + 2c^2 + 2c^3 + c^4 - 2c^5 - c^6 - c^7)\,d\\
&+ (3c + 5c^2 + 2c^3 + 2c^4 - 3c^5 - 4c^6 - 2c^7 - 2c^8 + c^9)\,d^2\\
&+ (1 + 3c^2 + 3c^3 + c^4 - c^5 - 6c^6 - 5c^7 - 4c^8 + 2c^{10})\,d^3\\
&+ (-c^2 + c^3 - c^4 - 2c^5 - 5c^6 - 6c^7 - 3c^8 - c^9 + 3c^{10} + 2c^{11} + c^{12})\,d^4\\
&+ (-2c^2 - 3c^3 - 3c^4 - 3c^5 - 2c^6 - 2c^7 - c^8 + 2c^{10} + 4c^{11} + 3c^{12} + c^{13})\,d^5\\
&+ (-c^2 - 3c^3 - 4c^4 - 2c^5 + c^7 + 2c^8 + 2c^9 + 3c^{10} + 3c^{11} + 3c^{12} + 2c^{13})\,d^6\\
&+ (-c^3 - 2c^4 - 3c^5 + c^6 + 3c^7 + 6c^8 + 5c^9 + 2c^{10} + c^{11} - c^{12} + c^{13})\,d^7\\
&+ (-2c^5 + 4c^7 + 5c^8 + 6c^9 + c^{10} - c^{11} - 3c^{12} - 3c^{13} - c^{15})\,d^8\\
&+ (-c^6 + 2c^7 + 2c^8 + 4c^9 + 3c^{10} - 2c^{11} - 2c^{12} - 5c^{13} - 3c^{14})\,d^9\\
&+ (c^8 + c^9 + 2c^{10} - c^{11} - 2c^{12} - 2c^{13} - 3c^{14})\,d^{10} + (-c^{12} - c^{15})\,d^{11}.
\end{aligned}$$

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^2d)(1-c^4d)(1-c^2d^3)(1-c^4d^3)$
$(1-cx)(1-cx^3)(1-dx^2)(1-dx^4)$.

* The forms of ord. 1, deg. 3, 4 and of ord. 1, deg. 4, 3 given by Clebsch and Gordan, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist. [See page 409, below.]

Numerator :

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^0	c^0	1									
	c^2		$\bar{1}$								
	c^4			2	2	2	1				
	c^6			1	1		$\bar{1}$	$\bar{1}$			
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$			
	c^{10}								1		
	c^{12}									$\bar{1}$	
x^1	c^1	$\bar{1}$		1							
	c^3		3	2	1	1					
	c^5		1	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1			
	c^7			$\bar{2}$	$\bar{1}$	$\bar{1}$		1	$\bar{1}$		
	c^9				1	1	1		$\bar{2}$		
	c^{11}							$\bar{1}$	$\bar{2}$		
	c^{13}									1	
x^2	c^0		$\bar{1}$								
	c^2	1	1	3	2	1					
	c^4		$\bar{1}$		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^6		$\bar{1}$		$\bar{2}$	$\bar{2}$			1		
	c^8			1		1	1		2		
	c^{10}						$\bar{1}$	$\bar{1}$		$\bar{2}$	
	c^{12}							1	1		1
x^3	c^1		2								
	c^3			$\bar{3}$							
	c^5		$\bar{1}$	$\bar{2}$	1		1		$\bar{1}$		
	c^7					$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	1	
	c^9						$\bar{1}$			1	
	c^{11}					1	1	2	1	2	
	c^{13}									1	$\bar{1}$

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
x^8	c^2		1								
	c^4			$\bar{1}$							
	c^6				2	2	2	1			
	c^8				1	1		$\bar{1}$	$\bar{1}$		
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{12}									1	
	c^{14}										$\bar{1}$
x^7	c^1		$\bar{1}$								
	c^3			2	1						
	c^5			2		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^7			1	$\bar{1}$		1	1	2		
	c^9				$\bar{1}$	1	1	1	2	$\bar{1}$	
	c^{11}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	
	c^{13}								$\bar{1}$	$\bar{1}$	1
x^6	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$						
	c^4		2		1	1					
	c^6			$\bar{2}$		$\bar{1}$	$\bar{1}$		$\bar{1}$		
	c^8			$\bar{1}$			2	2		1	
	c^{10}				1	2	2	1		1	
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{14}									1	
x^5	c^1	1	$\bar{1}$								
	c^3		$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$				
	c^5		$\bar{1}$			1					
	c^7		$\bar{1}$	1	2	1	1				
	c^9			1		$\bar{1}$		$\bar{1}$	2	1	
	c^{11}								3		
	c^{13}									$\bar{2}$	

Numerator—(*Continued.*)

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^4	c^0		1							
	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$	$\bar{1}$				
	c^4	$\bar{1}$		$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^6		1	$\bar{2}$	$\bar{1}$			1		
	c^8			$\bar{1}$			1	2	$\bar{1}$	
	c^{10}					1	2	2		1
	c^{12}					1	1	1		1
	c^{14}								$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^4d)(1-c^4d^2)(1-c^2d^3)(1-c^4d^3)$
$(1-cx^3)(1-c^2x^2)(1-dx^4)(1-d^2x^4)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^0	c^0	1											
	c^4			1	2	2	1						
	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{14}										$\bar{1}$		
x^1	c^1		1	1									
	c^3		2	3	2	1							
	c^5		1	2	3	2	1	1					
	c^9				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
	c^{11}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{13}								$\bar{1}$	$\bar{1}$			

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^{11}	c^3			1									
	c^7					1	2	2	1				
	c^9					1	3	2	1				
	c^{11}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{13}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{17}												$\bar{1}$
x^{10}	c^4				1	1							
	c^6				2	3	2	1					
	c^8				1	2	3	2	1	1			
	c^{12}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	c^{14}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^{16}										$\bar{1}$	$\bar{1}$	

Numerator—(*Continued.*)

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^2	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^8			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}										1		
x^3	c^1		1	1	1								
	c^3	1	1	3	5	3	1						
	c^5		$\bar{1}$		2	3	2						
	c^7		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^9					$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{1}$			
	c^{11}				1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$		
	c^{13}						1	1	2	1	$\bar{1}$		
	c^{15}								1	1	1		
x^4	c^2		1	2	3	1							
	c^4		$\bar{1}$		2	1	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}					$\bar{1}$	$\bar{1}$	1	1	1			
	c^{12}					1	3	4	4	2			
	c^{14}						1	2	3	2	1	1	1
x^5	c^1		1	1									
	c^3				$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^5		$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^7			$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^9					$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	1	
	c^{11}					1	2	2	4	5	4	2	
	c^{13}						1	3	4	4	4	1	
	c^{15}							1	1	1	1		
	c^{17}										$\bar{1}$		

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^9	c^1			$\bar{1}$									
	c^5			1	2	2		$\bar{1}$	$\bar{1}$				
	c^7				2	3	2	1					
	c^9					1	2	3	3	3	1		
	c^{13}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{15}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
x^8	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4			1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$					
	c^6			1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^8				1	3	3	4	2				
	c^{10}					1	8	3	3	4	2	1	
	c^{12}							$\bar{2}$	$\bar{3}$	$\bar{2}$		1	
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{16}									$\bar{1}$	$\bar{1}$	$\bar{1}$	
x^7	c^3	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^5				$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^7				$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1				
	c^9			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{11}					1	3	4	4	4	2	1	
	c^{13}					1	2	1	$\bar{1}$	$\bar{2}$		1	
	c^{15}								$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
x^6	c^0			1									
	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4		$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$					
	c^6		$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1				
	c^{10}				1	2	2	4	5	4	2		
	c^{12}				1	2	2	3	4	5	3	1	
	c^{14}						1	1	1	1			
	c^{16}										$\bar{1}$	$\bar{1}$	

*Table of Groundforms**.

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	1		1	1			
	3		2	3	2	1	
	5		1	2	2		

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3
2	2	1	2	2	1
	4		2	2	
3	1	1	1	1	1
	3	1	1	1	1
4	0		1	1	
	2		1	1	1
5	1		1	1	
6	0				1

System of Two Quartics.

G. F. for differentiants,

Denominator: $(1-d)(1-d^2)^2(1-d^3)(1-\delta)(1-\delta^2)^2(1-\delta^3)(1-d\delta)(1-d^2\delta)$
$(1-d\delta^2)$.

Numerator: $1+d^3+(3d+3d^2-d^4-d^5)\delta+(3d+4d^2-d^3-3d^4-2d^5-d^6)\delta^2$
$+(1-d^2-2d^4-3d^5-d^6)\delta^3+(-d-3d^2-2d^3-d^5+d^7)\delta^4$
$+(-d-2d^2-3d^3-d^4+4d^5+3d^6)\delta^5+(-d^2-d^3+3d^5+3d^6)\delta^6$
$+(d^4+d^7)\delta^7$.

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^2)(1-dx^4)(1-\delta x^2)(1-\delta x^4)$.

* The form of ord. 1, deg. 5, 4, and the two forms of ord. 2, deg. 4, 3, given by Gundelfinger, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist. [See below, p. 409.]

Numerator :

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5
x^0	d^0	1					
	d^2			1			
	d^4					1	
x^2	d^0		$\bar{1}$				
	d^1	$\bar{1}$	1	1	1		
	d^2		1	1			
	d^3		1		1		
	d^4						$\bar{1}$
	d^5					$\bar{1}$	
x^4	d^0			1			
	d^1		2		$\bar{1}$	$\bar{1}$	
	d^2	1		$\bar{1}$	$\bar{2}$		
	d^3		$\bar{1}$	$\bar{2}$			
	d^4		$\bar{1}$				$\bar{1}$
	d^5					$\bar{1}$	1

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^{10}	d^2		1				
	d^4				1		
	d^6						1
x^8	d^1		$\bar{1}$				
	d^2	$\bar{1}$					
	d^3			1		1	
	d^4				1	1	
	d^5			1	1	1	$\bar{1}$
	d^6					$\bar{1}$	
x^6	d^1	1	$\bar{1}$				
	d^2	$\bar{1}$				$\bar{1}$	
	d^3				$\bar{2}$	$\bar{1}$	
	d^4			$\bar{2}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{1}$		2	
	d^6				1		

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^4)(1-d^2x^4)(1-\delta x^4)(1-\delta^2 x^4)$.

Numerator:

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^0	d^0	1						
	d^2			1				
	d^4					1		
x^2	d^1		1	1	1			
	d^2		1	1	1			
	d^3		1	1	1			
x^4	d^1		1	1				
	d^2		1	1				
	d^3				1	1		
	d^4				1		$\bar{1}$	$\bar{1}$
	d^5					$\bar{1}$		
	d^6					$\bar{1}$		
x^6	d^0				1			
	d^1		1	1		$\bar{1}$	$\bar{1}$	
	d^2		1	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	
	d^3	1		$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	d^4		$\bar{1}$	$\bar{2}$	$\bar{2}$			
	d^5		$\bar{1}$	$\bar{1}$	$\bar{1}$			

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7
x^{14}	d^3			1				
	d^5					1		
	d^7							1
x^{12}	d^4				1	1	1	
	d^5				1	1	1	
	d^6				1	1	1	
x^{10}	d^1			$\bar{1}$				
	d^2			$\bar{1}$				
	d^3	$\bar{1}$	$\bar{1}$		1			
	d^4			1	1			
	d^5					1	1	
	d^6					1	1	
x^8	d^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	d^3				$\bar{2}$	$\bar{2}$	$\bar{1}$	
	d^4		$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{2}$	$\bar{1}$	1	1	
	d^6		$\bar{1}$	$\bar{1}$		1	1	
	d^7				1			

Table of Groundforms *.

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
2	1		1	1	1
	2		1	1	1
	3		1	1	

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
4	0		1	1	
	1	1	1	1	
	2	1	1		
6	0				1
	1		1	1	
	2		1		
	3	1			

The following table exhibits the total numbers of groundforms; the quantics themselves and the absolute constant are included in the numbers†.

Order of Quantic.	Order of Quantic. 0	1	2	3	4
0	1	2	3	5	6
1		4	6	14	21
2			7	16	19
3				27	62
4					29

* The forms of ord. 4, deg. 2, 2, and of ord. 6, deg. 2, 2, given by Gordan, do not appear in this table, and have been proved by the author to be compound forms. [See below, p. 409.]

† Some remarks on the preceding tables (to save delay in going to press) have been made the subject of a separate article in this number. [p. 406, below.]

REMARKS ON THE TABLES FOR BINARY QUANTICS.

The valuable idea of using different roman letters, a, b, c, d, to correspond to the coefficients of quantics of different orders, is due to Mr Franklin. Had it occurred previously I should have employed it in the tables of the generating functions and groundforms of single quantics. The nth letter of the alphabet, say θ, will in this way symbolize the $(n+1)$ coefficients $\theta_0, \theta_1, \theta_2, \ldots \theta_n$ and so x regarded as a new point of departure in the alphabet will symbolize x_0, x_1.

I pass on to a remark of greater importance referring to the separation of the Parallelopiped which may be imagined to represent the complete tabulation of the representative G.F. to a system of two simultaneous quantics, and its use in simplifying the process of tamisage.

To fix the ideas, let us take the case of a Cubic and Quartic. Then, to represent the collected signification of the rectangles at pp. [400, 401, above]*, we may suppose a parallelopiped 12 inches in length, 17 in breadth, and 11 in depth, 12, 17, 11 being the highest exponents which appear in such rectangles of d, c, x, respectively, and confine our attention to the sign proper to each of the 12.17.11 cubical spaces (inch cubes) which may be either $+$ or $-$ or vacancy, if sign that may be called where sign is none. We may, if we please, imagine these cubes or cells to be filled with positive, negative or neutral electricity.

According to the chorographical law (foot-note, p. [310, above]), it ought to and would be found that the occupied portions of this parallelopiped would separate into a certain number of distinct blocks of positive and negative signs. Let us limit our attention to the first of these blocks†. The tamisage, according to the principle laid down in the remarks at the end of the preceding paper, may be limited to this block, although, as a matter of fact (and for greater assurance) in deducing the tables of

* The vacant lines and columns suppressed in the rectangular tables referred to, are supposed to be supplied.

† Planes passing through that angle of the parallelopiped at which is situated the absolute constant, may be termed the planes of reference.

In order to determine whether or not a given space or cell (as we may term it) belongs to the first block, the following is the rule to be observed: (1) If its sign is negative, it is to be rejected. (2) If three lines be drawn through its centre parallel to the edges of the parallelopiped *towards* the planes of reference, and any of these passes through a negative cell, it is to be rejected. (3) In every other case, the cell (or term which occupies it) forms a part of the primary block. So to obtain the second block required for determining the syzygants of the first species, (and notice that under a general point of view groundforms may be regarded as syzygants of species zero or on the other hand and preferably syzygants of the ith may be regarded as groundforms of the $(i+1)$th species) we may take any negative cell such that the three lines drawn through it parallel to the edges and towards the plane of reference shall not pass through any positive one. The *ensemble* of such constitute the second block. Then for the third block we may take the

groundforms, it was actually applied to all the positive terms in the 11 rectangles.

An inspection of the rectangle affected with x^7 and x^8, p. [401, above] will show that they may be omitted as forming no part of the first positive block. In the rectangle affected with x^9, it will be found that the only terms subject to examination, that is, the only terms with positive coefficients which are not *preceded* vertically or horizontally by terms with negative coefficients, are

$$\begin{array}{ccc} 2c^5d^4x^9 & 2c^5d^5x^9 & \\ 2c^7d^4x^9 & 3c^7d^5x^9 & 2c^7d^6x^9 \\ & c^9d^5x^9 & 2c^9d^6x^9 \end{array}$$

Calling any one of these terms $kc^\lambda d^\mu x^9$, it will be found, on examining the preceding rectangles, that $c^\lambda d^\mu$ occurs in one or more of them affected with a negative numerical coefficient. Consequently, these terms do not belong to the primary block, and, in like manner, it will be found that the rectangles subsequent to x^9 form no part of it.

The tamisage may therefore be confined to the rectangles belonging to x^0, x^1, x^2, x^3, x^4, x^5, x^6 and the only terms to be retained will be seen to be those exhibited in the following table:

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ensemble of positive cells not included in the first block and such that the lines through any one of them drawn as before shall not pass through a negative cell, and so on until all the cells are distributed into their respective blocks.

It may not be out of place to observe here that groundforms and syzygants may be regarded as existences and privations of existence, and the Fundamental Postulate so often previously quoted (on which the legitimacy of tamisage depends) is analogous to the asertion that free electricities of the two kinds cannot coexist at the same time at the same point of a body. Are there not some phenomena in electricity (certain visible effects at the poles of an electrical machine or at the extremities of the electric arc) which seem to indicate that the two electricities, although mutually quelling, are not absolutely antithetical in the sense that they might be reversed throughout an environment without any change of effect of any kind resulting? Unless this is true the analogy of the relation of Groundforms and Syzygants to Positive and Negative Electricity halts on one foot. But if it be true we may perhaps see foreshadowed in the constitution of the generating function, the possibility of physical research hereafter bringing to light residual phenomena in which freer and rarer kinds of positive and negative electricity in succession will make their appearance.

Their supposed possible prototypes as yet, play no part in any developed algebraical theory, and indeed the consciousness of only a few algebraists is as yet fully awakened to a sense of their existence. If to any one the idea of physical being foreshadowed in algebraical laws should appear extravagant and visionary, let him reflect on the certain fact that the conception of chemical units as molecules composed of atoms and of the new theory of atomicity or *valence* in each essential particular might have been safely inferred as a possible hypothesis, from the ascertained laws of the constitution and mutual actions upon one another of invariantive forms. If we only allow that the so-called laws of nature have their origin in reason and are not merely arbitrary or *fiat* laws, we can very well understand how an unfailing parallelism should exist between the phenomena of the outer world and those phenomena of the pure intelligence with which algebraical science is concerned.

	c^4d^2	$2c^4d^3$	$2c^4d^4$	c^4d^5		
	c^6d^2	$3c^6d^3$	$2c^6d^4$	c^6d^5		
	cdx	cd^2x				
	$2c^3dx$	$3c^3d^2x$	$2c^3d^3x$	c^3d^4x		
	c^5dx	$2c^5d^2x$	$3c^5d^3x$	$2c^5d^4x$	c^5d^5x	c^5d^6x
	$2c^2dx^2$	$3c^2d^2x^2$	$2c^2d^3x^2$	$c^2d^4x^2$		
	$2c^4dx^2$	$4c^4d^2x^2$	$5c^4d^3x^2$	$3c^4d^4x^2$	$c^4d^5x^2$	
	cdx^3	cd^2x^3	cd^3x^3			
c^3x^3	c^3dx^3	$3c^3d^2x^3$	$5c^3d^3x^3$	$3c^3d^4x^3$	$c^3d^5x^3$	
	c^2dx^4	$2c^2d^2x^4$	$3c^2d^3x^4$	$c^2d^4x^4$		
	cdx^5	cd^2x^5				
	d^3x^6					

Thus, it is evident at a glance that the highest order in the variables, the highest degrees in the cubic and quartic coefficients respectively, of any groundform, are 6, 4 and 5 respectively. Prior to all tamisage, 6, 4, 5 are seen to be superior limits to such order and degrees, because no powers of x, d, c figure among the above terms higher than 6, 4, 5, and a slight examination shows that some terms, containing x^6, d^4, c^5, survive the operation of the tamisage.

The number of types submitted to tamisage, it will be seen, is 45, as previously stated.

The number of forms contained under these types is 83.

The number of types absolutely abolished by the operation is 10, bringing down the number to 35; and the reduction in the total number of forms is 33, bringing down the number to 50*.

These remarks have reference *solely to the groundforms* represented by the *numerator* of the Generating Function. The denominator yields 11 groundforms, thus raising the total number to 61, which is the right number when the absolute constant is not counted *in* as the representative of an invariant†.

Possibly, when I may be again able to secure the services of Mr Franklin, without whose intelligent cooperation I believe it would have been impracticable for me to have calculated the tables contained in this and the preceding

* There is every reason to believe that a calculating machine might be constructed without difficulty for performing mechanically the process of *tamisage* whether simple (involving only a single variable) as for invariants of single forms or compound (involving several variables) as for covariants or invariants of systems.

† It should be noticed that some of the entries in the Table of Groundforms, p. [402], are made up partly from the numerator and partly from the denominator, as for example the number 3 in the column headed 3 and in the line marked 4 for the order 0, is made up partly of the 2 in the surviving term $2d^3c^4$ of the numerator and partly of a unit taken from the term $1 - d^3c^4$ of the

number of the *Journal,* I shall be able to extend the limit to the order of the combined quantics. At all events, the labour of forming the tables of the combinations of 1, 2, 3, 4, 5, 6 with 6, would probably not exceed the amount which has been incurred in calculating the groundforms of a single quantic of the 9th order. The references to the *Comptes Rendus* made in the foot-notes are to Vol. LXXXIV. 1ier semestre for 1877, p. 1285, for the disproof of the existence of the *two* forms given in the accepted tables belonging to a system of two binary quartics*; to Vol. LXXXVII. 2me semestre for 1878, p. 445, and again p. 477, for the disproof of the existence of the *three* accepted superfluous forms for a system of a binary cubic and quartic†, and to Vol. LXXXIX. 2me semestre for 1879, p. 828, for the disproof of the existence of the *two* superfluous accepted forms belonging to the system of two binary cubics‡. The proof of the Fundamental Theorem is given as a Postscriptum in a paper in *Borchardt's Journal* "Sur les actions mutuelles des formes invariantives," 1878 [p. 232, above], and in a paper entitled "Proof of the hitherto undemonstrated fundamental theorem for Invariants," in the *Philosophical Magazine* for the same year, 1878 [p. 117, above].

The term *Reduced Generating Function* being apt to lead to the erroneous impression that it is obtained by reducing the representative one, whereas the representative is in fact obtained from the reduced G. F. by multiplication of its numerator and denominator by a common factor, it may be well to explain that I use the appellation *reduced* with reference to the *crude form of the generating function,* the former representing that branch, or the totality of those branches, in the development of the crude form which contain no negative powers of x.

I add a few words respecting differentiants which are simply such symmetrical functions of the roots as are complete functions of the differences of the roots of the form or system of forms to which the several tables refer.

In the G. F. for differentiants for a single quantic, the coefficient of a^j represents the total number of linearly independent differentiants of the degree j belonging to a quantic of the order i; that is, the total number of covariants of the degree j in the coefficients and of *all* orders in the variables, belonging to that quantic. The G. F. for differentiants can therefore be obtained from the G. F. for covariants (although not in its simplest form) by putting $x = 1$ in the latter. In like manner, for a system of quantics, the

denominator. It is an erroneous and misleading expression into which invariantists (myself included) have fallen of speaking of a definite number, say ν, of groundforms of a certain type. The true idea is that of an unique form of that type with ν parameters. It is, so to say, a single *form* of the νth degree of plasticity or deformability or of ν dimensions in the sense in which we speak of the dimensions of space. I mean that an elastic string, an india-rubber disk and an india-rubber ball may be regarded as symbols of a groundform with one, two and three parameters respectively.

[* p. 63, above.] [† pp. 132, 136, above.] [‡ p. 258, above.]

G. F. for differentiants (or to speak more precisely, its algebraical equivalent) can be obtained from the G. F. for covariants by putting $x = 1$.

To obtain the G. F. for differentiants for a single form without previously having the G. F. for covariants, we may make use of the fact that the sum of the quantities

$$(w:\ i, j) - (w - 1:\ i, j)^*$$

for *all* admissible values of w is equal to the value of $(w:\ i, j)$ for the *highest* admissible value of w. Now the *order* corresponding to the highest weight is 0 or 1†; hence the number of differentiants of the degree j belonging to a quantic of the order i is the coefficient of $a^j x^0$ or of $a^j x^1$ (according as ij is even or odd) in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i+2})(1-ax^{-i})}.$$

The generating function for differentiants is therefore the sum of the multipliers of x^0 and x^1 in the development of the above fraction. (When the quantic is of even order, x^1 does not appear in the development, and the G. F. for differentiants is simply the part independent of x in the development.)

In like manner, for a system of two quantics, the G. F. for differentiants is the sum of the multipliers of x^0 and x^1 in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i})(1-ax^{i'})(1-ax^{i'-2})\dots(1-ax^{-i'})}.$$

And we may proceed in an analogous manner when a system of forms is in question. I need hardly add that a differentiant in respect to either variable, say x, is only another name for any rational integral function of the coefficients of a quantic which, when the coefficient of the highest power of the selected variable (x) in the quantic is made equal to unity, becomes a function of the differences of its $\frac{x}{y}$ roots. Gordan's and Jordan's results concerning symbolical determinants are correlative and coextensive with theorems concerning root-differences, so that the method of differentiants when fully developed would lead to the substitution of actual differences or determinants for symbolical determinants in the Gordan theory, it being borne in mind that to determine the ground-covariants of a quantic or quantic system is the same question as that of determining its ground-differentiants, inasmuch as to every covariant corresponds a single differentiant, and *vice versâ*.

* w is the weight of any covariant, j its degree in the coefficients and i the order of the quantic in the variables; and $(w:\ i, j)$ denotes the number of modes of composing w with j of the elements $0, 1, 2, 3, \dots i$ or *vice versâ* with i of the elements $0, 1, 2, 3, \dots j$ each any number of times repeated.

† If e is the order of the covariant in the variables $2w = ij - e$.

41.

NOTE SUR UNE PROPRIÉTÉ DES ÉQUATIONS DONT TOUTES LES RACINES SONT RÉELLES.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVII. (1879), pp. 217—219.]

(1) SOIT f une forme binaire $(a, b, c, \ldots l \between x, y)^i$, ϕ un covariant de f de l'ordre ϵ et $F(a, b, c, \ldots l)$ le coefficient de x^ϵ dans ϕ. Supposons que si dans la forme f on remplace y par $y - \dfrac{Y}{X}x$, les coefficients $a, b, c, \ldots l$ se changent en $a', b', c', \ldots l'$. Cela posé, si dans le covariant ϕ on remplace y, x par Y, X, on sait que ϕ se change en $X^\epsilon F(a', b', c', \ldots l')$.

(2) Soit $(a_0, a_1, \ldots a_{2\epsilon} \between x, y)^{2\epsilon}$ une forme binaire qui a toutes ses racines $\alpha_1, \alpha_2, \ldots \alpha_{2\epsilon}$ (c. à d. les valeurs de $\dfrac{y}{x}$, qui font évanouir la forme) réelles, et soit

$$(-1)^\epsilon \left\{a_0 a_{2\epsilon} - 2\epsilon \,.\, a_1 a_{2\epsilon-1} + \frac{2\epsilon(2\epsilon-1)}{2} a_2 a_{2\epsilon-2} - \ldots\right\} \qquad (1)$$

son invariant quadratique dont le signe est fixé de sorte que son dernier terme proportionnel à a_ϵ^2 ait le signe positif. Cet invariant divisé par le carré de $a_{2\epsilon}$ peut d'ailleurs, comme on sait, être présenté (à un facteur numérique près) sous la forme d'une somme de produits tels que

$$(\alpha_1 - \alpha_2)^2 (\alpha_3 - \alpha_4)^2 \ldots (\alpha_{2\epsilon-1} - \alpha_{2\epsilon})^2,$$

par conséquent cet invariant est positif pour les formes à racines réelles.

Considérons à present la forme binaire $(a_0, a_1, \ldots a_{2\epsilon} \ldots a_{2\epsilon+\eta} \between x, y)^{2\epsilon+\eta}$ de l'ordre $2\epsilon + \eta$, qui ait également toutes ses racines réelles, alors l'expression (1) formée par rapport aux coefficients de la nouvelle forme gardera son signe positif, car en différentiant η fois de suite la nouvelle forme on retombe sur la forme binaire de l'ordre 2ϵ d'où l'on est parti.

(3) Remplaçons dans f la variable y par $y - \frac{Y}{X}x$ et supposons que Y, X soient des quantités réelles. Cette substitution ne changera en rien le caractère de la forme f relatif à la réalité de ses racines. Donc en combinant les deux observations précédentes on en conclut le résultat suivant:

Soit f une forme binaire qui a toutes ses racines réelles et ϕ un de ses covariants du second degré dans les coefficients, ϕ sera d'un signe invariable, c. à d. *si toutes les racines de f sont réelles, toutes les racines de tous les quadricovariants* (c. à d. des covariants du second degré) *de f sont imaginaires.*

POSTSCRIPTUM.

M. Schramm, dans un mémoire inséré dans les *Annali di Matematica*, année 1867, avait déjà remarqué la propriété démontrée plus haut pour le cas du Hessien en se servant des fonctions covariantives en x, y, qu'il a démontré pouvoir remplacer les fonctions de Sturm en ce qui regarde la détermination du *nombre total* des racines réelles d'une équation.

M. Schramm a obtenu ces formules par une certaine transformation opérée sur celles qui portent mon nom ; mais on peut les obtenir immédiatement en se servant de la loi d'inertie pour les formes quadratiques.

En supposant que $f(x) = 0$ est une équation algébrique dont les racines sont $e_1, e_2, \ldots e_n$, en écrivant

$$\Phi = \sum_{i=n}^{i=1} (\phi_1 e_i u_0 + \phi_2 e_i u_1 + \ldots + \phi_n e_i u_n)^2$$

où $\phi_1, \phi_2, \ldots \phi_n$ sont des fonctions rationnelles quelconques, on voit très facilement que l'inertie de Φ est égale à $n - 2\nu$, ν étant le nombre de *paires* de racines imaginaires. Si l'on pose

$$\phi_1 e = 1, \ \phi_2 e = e, \ \ldots \ \phi_n e = e^{n-1},$$

la fonction quadratique Φ aura pour déterminant l'expression

$$\Delta = \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{n-1} \\ s_1 & s_2 & s_3 & \ldots & s_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \ldots & s_{2n-2} \end{vmatrix}.$$

De plus en considérant les mineurs successifs

$$s_0 \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \quad \ldots \quad \Delta$$

on sait que le nombre de permanences de signes dans cette série exprimera l'inertie de ϕ, c. à d. sera égal à $n - 2\nu$.

Comme on a d'ailleurs

$$s_0 = n, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = \Sigma (e_1 - e_2)^2, \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \Sigma (e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2, \quad \ldots$$

on déduit immédiatement de là la règle de Sturm pour le nombre total des racines réelles et imaginaires.

Si au lieu de poser $\phi_i e = e^{i-1}$ on posait $\phi_i e = \frac{1}{(\lambda - e)^{i-1}}$, λ étant une constante réelle quelconque, on obtiendrait de même une série de termes où le nombre de permanences serait encore égal à $n - 2\nu$.

En multipliant ces termes respectivement par des puissances paires d'un degré convenable de $(\lambda - e_1)(\lambda - e_2) \ldots (\lambda - e_n)$, c. à d. par des quantités réelles et positives, on obtient les fonctions de Schramm avec cette seule différence que x et y s'y trouvent remplacées par λ et 1. Mais on fera disparaître cette différence en posant $\frac{x}{y}$ à la place de λ et en multipliant par la puissance paire de y qui rend l'expression entière.

42.

ON THE THEOREM CONNECTED WITH NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

[*Messenger of Mathematics*, IX. (1880), pp. 71—84.]

To save needless repetition in what follows I beg to refer the reader to Mr Todhunter's section 26, p. 236, in the third edition of his *Treatise on the Theory of Equations.* It will there be seen that in order to provide against any loss of double permanences consequent upon any of the f's changing sign $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_{n-1}$ must all be positive; and in order to provide against the same thing happening consequent upon any of the G's changing sign we must have, from $i=2$ to $i=n-1$ inclusive, $2-\gamma_i=\frac{1}{\gamma_{i+1}}$; and, moreover, $2-\gamma_{n-1}$ [denoted by $\frac{1}{\gamma_n}$, although strictly there is no γ_n, since G_n is simply a positive absolute], as well as $\gamma_1, \gamma_2, \ldots \gamma_{n-1}$, must be positive.

The solution of the equation $2-\gamma_i=\frac{1}{\gamma_{i+1}}$ is $\gamma_i=\frac{C+i-1}{C+i}$; and, in order that $\gamma_1, \gamma_2, \ldots \gamma_{n-1}, \gamma_n$ may all be *positive,* it is necessary that C shall be either positive or, if negative, of greater absolute value than n.

If we put $C=0$, $\gamma_1=0$; if we put $C=-n$, $\gamma_n=\infty$, so that, the condition of γ_i being *positive,* from 1 to n, will not in either case be complied with, the signs of zero and of infinity being ambiguous. It is well known, however, that we may put $C=-n$; in fact, $-n$ is the value ordinarily attributed to C, for the corresponding value of γ_i, namely, $\frac{n-i+1}{n-i}$, it is which leads to that form of the theorem in which, when we put $\mu=\infty$ and $\lambda=0$, or $\mu=0$ and $\lambda=-\infty$ in the equation $pP(\mu)-pP(\lambda)=$ (the number of roots between λ and μ) $+2i$, gives Newton's rule as stated by Newton himself. Equally, we shall find it is lawful to put $C=0$, but each of these two suppositions requires to be subjected to a special examination before its validity can be admitted. Take the much more important case first, that where $C=-n$, we

have then $\gamma_{n-1} = 2$, and the only object of $2 - \gamma_{n-1}$ being positive is to prevent mischief in the event of G_{n-1}, that is, $(f_{n-1}x)^2 - 2f_{n-2}xf_nx$, changing its sign. But in this case $\frac{dG_{n-1}}{dx} = 0$ by simple differentiation from $\frac{df_nx}{dx} = 0$: in other words, G_{n-1} *is a constant* and never can change its sign. Thus, then, all necessity for $2 - \gamma_{n-1}$ being positive is abolished by the very fact of its being zero.

It is worth noticing that this critical value of C, which makes $\gamma_i = \frac{n-i+1}{n-i}$, has the effect of lowering the degree of each G by two units; for if $\lambda = n-i+1$, we may write $f_{i-1} = px^\lambda + qx^{\lambda-1} + \ldots$, and then

$$G_i = f_i^2 - \frac{\lambda}{\lambda-1} f_{i-1}f_{i+1} = \{p\lambda x^{\lambda-1} + q(\lambda-1)x^{\lambda-2} + \ldots\}^2$$

$$+ \frac{\lambda}{\lambda-1}(px^\lambda + qx^{\lambda-1} + \ldots)\{p\lambda(\lambda-1)x^{\lambda-2} + q(\lambda-1)(\lambda-2)x^{\lambda-3} + \ldots\};$$

so that the coefficient of $x^{2\lambda-2}$ becomes

$$p^2\left\{\lambda^2 - \frac{\lambda}{\lambda-1}(\lambda^2-\lambda)\right\} = 0,$$

and that of $x^{2\lambda-3}$ becomes

$$pq\left\{2\lambda(\lambda-1) + \frac{\lambda}{\lambda-1}\lambda(\lambda-1) + (\lambda-1)(\lambda-2)\right\} = 0.$$

So again it will be found that C may be taken at the other extremity of the chasm or gap, which it is not permitted to enter; for if $C = 0$ so that $g_1 = 0$, $G_1 = (f'x)^2$.

Consider now the first three terms of the double series

$$\begin{array}{ccc} fx, & f'x, & f''x, \\ I, & I, & G_2x, \end{array}$$

where the two I's denote absolute positive quantities; at the moment of $f'x$ becoming zero, G_2x becomes positive, so that the succession of double permanences of sign for this double series is the same as of single permanences for $fx, f'x, f''x$, and consequently no double permanences can be lost by $f'x$ changing its sign. Since, then, we have shown that values of C giving rise to no negative but to an ambiguous sign, either of γ_1 or of γ_n, are not prohibited, it might for a moment be imagined that any negative integer value of C, say $-\omega$, lying in the gap between 0 and $-n$ might also be admissible, seeing that such value would also *not introduce any negative value of* γ, but only two values of ambiguous signs, namely, for γ_ω and $\gamma_{\omega+1}$, ∞ and 0 respectively; all the other γ's will be positive. But it will be seen that this is inadmissible, for the course of the demonstration shows that every γ_i and $2 - \gamma_i$ must both be positive, which conditions cannot be fulfilled for γ_ω, whether we consider it equal to plus or minus infinity.

As I have referred to Mr Todhunter's treatise, I may notice the omission therein of the equation

$$\nu P\lambda - \nu P\mu = (\mu, \lambda) + 2i',$$

where i' is any positive integer and (μ, λ) the number of real roots between λ and μ. This may be deduced *pari passu,* and in precisely the same way as the parallel equation

$$pP\mu - pP\lambda = (\mu, \lambda) + 2i,$$

or either of these may be deduced from the other as follows. Let $fx = \phi(-x)$, and using the same parameter γ_i for the G's belonging to f and for those belonging to ϕ, let f_i, G_i for f become ϕ_i, T_i for ϕ. Then obviously

$$T_i(-c) = G_i c \text{ and } \phi_i(-c) = (-)^{n-1} f_i(c).$$

Hence, using π, Π in regard to ϕ in the same sense as p, P in regard to f, $\pi\Pi(-c) = \nu Pc$; also $(-\lambda, -\mu)$ in regard to ϕ is the same as (μ, λ) in regard to f. But remembering that if μ is greater than λ, then $-\lambda$ is greater than $-\mu$, the second equation above written applied to ϕ becomes

$$\pi\Pi(-\lambda) - \pi\Pi(-\mu) = (-\lambda, -\mu) \text{ in regard to } \phi + 2i.$$

Hence $\qquad \nu P(\lambda) - \nu P(\mu) = (\mu, \lambda)$ in regard to $f + 2i$,

as was to be shown*.

One other point deserves mentioning. If any G, say G_i, becomes incapable of changing its sign (of which G_1 becoming f_1^2 when $C = 0$, offers a particular example), the necessity for the equation $2 - \gamma_i = \frac{1}{\gamma_{i+1}}$ is done away with for that value of i, so that γ_{i+1} becomes arbitrary (within limits), and we may start with a new definition of the values of the γ's lying beyond γ_i, namely, $\gamma_{i+i'} = \frac{C' - 1 + i}{C' + i'}$ and so on, *toties quoties,* whenever in passing from G_1 to G_{n-1}, any of the G's becomes incapable of changing its sign†.

* This equation is stated in the original memoir in the *Proceedings of the Mathematical Society of London*‡. Dr Julius Petersen, of Copenhagen, in his treatise on Algebraical Equations, not having had the opportunity, as he has since informed me, of consulting this, and taking Mr Todhunter's chapter on the subject as his authority, was led to lay the fault of the omission at my door.

† Thus we see that in the expression $\gamma_r = \frac{C-1+r}{C+r}$, C is not absolutely prohibited from entering the gap comprised between 0 and $-n$, but that C may be $-i$ where i is an integer, or any quantity between $-i$ and $-\infty$, provided that G_{i-1}, that is, $f^2_{i-1} - \gamma_{i-1} f_{i-2} f_i$ is incapable of changing its sign. If $C = -i$, $\gamma_{i-1} = 2$.

As an application of the same principle we may make the γ series begin with G_2, that is, make G a positive absolute so as to have two positive absolutes instead of one positive absolute at the beginning of the series of "the Quadratic elements," that is, we may make $\gamma_1 = 0$ and $\gamma_{1+r} = \frac{C-1+r}{C+r}$, and continuing this process, $1+k$ (any number) of the initial G's may be converted into positive absolutes; that is to say, we may make $\gamma_1 = 0$, $\gamma_2 = 0$, ... $\gamma_k = 0$, $\gamma_{k+r} = \frac{C-1+r}{C+r}$.

[‡ Vol. II. of this Reprint, p. 501, *footnote.*]

It will have been noticed in what precedes, that I have made no allusion to special forms of an equation, whether absolute or having reference to the assumed arbitrary parameter in G, but have confined myself to the general case where only one term in the double series can vanish for any given value of x. Nor is it necessary to do more than this in treating the theory; for (1) if f contains no equal roots, we may, by infinitesimal or infinitely small variations attributed to the coefficients, cause those relations between them to subsist which are necessary in order that two or more of the terms may vanish simultaneously, and cannot thereby alter the character of the roots, which can only make the passage from real to imaginary, or *vice versâ*, after one or more pairs of them have passed through the state of equality; (2) if f contains equal roots, we may vary the coefficients in such a manner as not to disturb the equalities which subsist between them, and shall have independent relations enough to spare to abolish as before the relations implied in the fact of the simultaneous evanescence above referred to.

Thus it seems to me that we need trouble ourselves with the discussion of the consequences of such simultaneous evanescence only if we wish to know what inferences to draw if we are unfortunate enough to find that event occurring at one or the other of the actual limits λ, μ we may be dealing with, and for no other purpose.

Postscript.

As I was on the point of despatching what precedes by post to England, it occurred to me, in consequence of the previously unnoticed depression of the degrees of the terms in the G series, to examine more closely their constitution for the critical case, that namely where $\gamma_i = \frac{n-i+1}{n-i}$, and I have had the satisfaction of finding that every such G is proportional to the

If we make $k=n$, all the G's become positive absolutes, and *the theorem passes into Fourier's.* In connexion with this fact, it should be noticed that my theorem in its form as hitherto given does not logically contain Fourier's as a consequence; for it is possible that for certain values of λ and μ, $pP(\mu) - pP(\lambda)$ may be greater than $p(\mu) - p(\lambda)$, so that Fourier's theorem may indicate the passage of a smaller number of roots than the seemingly more stringent one; hence in applying my theorem, Fourier's should always be employed simultaneously with it, a practical direction which has hitherto been overlooked. Of course when the question concerns the total number of roots, Descartes' rule *is* logically contained in Newton's, or my generalisation of it as previously given.

It may be well to mention here, that a more general form of my theorem introducing a second arbitrary parameter will be found in some far back number of the *Educational Times* as the solution of a question proposed in a previous number. It is founded, if I recollect right, on the principle that if for the equation of the nth degree in x, say $fx=0$, we substitute $\epsilon x^{n+\nu} + fx = 0$, where ν is any positive integer (ϵ being an infinitesimal), no new real root is introduced if ν is even, provided ϵ be taken with the right sign, and only one (of infinite value) if ν is odd. [See below: Solutions contributed by the Author to the *Educational Times*.]

Hessian of the f antecedent to it, regarded as a homogeneous function of x and 1, being that Hessian multiplied by a negative number.

To prove this I have to show that if $F(x, y)$ is of the order λ, then

$$\lambda F \frac{d^2F}{dx^2} - (\lambda - 1)\left(\frac{dF}{dx}\right)^2$$

is a positive multiple of y^2 multiplied by the Hessian of F in regard to x, y.

Now
$$\lambda F = x\frac{dF}{dx} + y\frac{dF}{dy},$$

and
$$(\lambda - 1)\frac{dF}{dy} = x\frac{d}{dx}\frac{dF}{dy} + y\frac{d^2F}{dy^2}.$$

Hence
$$y\frac{dF}{dy} = \lambda F - x\frac{dF}{dx},$$

$$y\frac{d^2F}{dx\,dy} = (\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2},$$

and
$$y^2\frac{d^2F}{dy^2} = (\lambda - 1)\left(\lambda F - x\frac{dF}{dx}\right) - x\frac{d}{dx}\left(\lambda F - x\frac{dF}{dx}\right)$$

$$= (\lambda^2 - \lambda)F - (2\lambda - 2)x\frac{dF}{dx} + x^2\frac{d^2F}{dx^2}.$$

Hence $y^2\left\{\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \left(\frac{d^2F}{dx\,dy}\right)^2\right\}$, that is, $-y^2H(F)$,

$$= (\lambda^2 - \lambda)\frac{d^2F}{dx^2}F - (2\lambda - 2)x\frac{d^2F}{dx^2}F + x^2\left(\frac{d^2F}{dx^2}\right)^2 - \left\{(\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2}\right\}^2$$

$$= (\lambda - 1)\left\{\lambda\frac{d^2F}{dx^2}F - (\lambda - 1)\left(\frac{dF}{dx}\right)^2\right\},$$

where the least value of λ is 2 so that $\lambda - 1$ is always positive.

Thus the f and G series may be put under the following form, where f_i of course means $\frac{d^if}{dx^i}$ and $H\phi x$ signifies the Hessian of ϕ regarded as a quantic in x and 1,

$$\begin{array}{c} f : f_1 : f_2 : f_3 : \ldots : f_{n-1} : f_n, \\ -1 : Hf : Hf_1 : Hf_2 : \ldots : Hf_{n-2} : -1. \end{array}$$

I anticipate that it will be found possible to extend the theorem by the addition of a third series for the case of $n = 4$ or 5, a third and fourth for that of $n = 6$ or 7, and, in general, by the use of $\frac{1}{2}(n + 2)$ or $\frac{1}{2}(n + 1)$ series according as n is even or odd. And possibly it may turn out that the maximum number of series available for any given value of n will by the reckoning of the gain of complete permanences of sign (that is, treble, quadruple... permanences for 3, 4... series) as x increases from λ to μ, afford not merely a superior limit to, but the actual number of, real roots passed over in the interval.

As I find that Mr Todhunter uses a single symbol ϖ for the pP employed in my memoir in the second number of the *Proceedings of the London Mathematical Society**, it may be well to advise my readers that I use p, P to signify permanences of sign, and v, V variations of sign in the f and G series respectively; so that double permanences, permanence variations, variation permanences and variation variations would be denoted by the compound symbols pP, pV, vP, vV respectively.

The theorem above given is, I find, only a particular case of the one subjoined.

Let f_i denote $(a_0, a_1, a_2 \dots a_i \between x, y)^i$ and $H_\epsilon(f_{i+\epsilon})$ that covariant of $f_{i+\epsilon}$ whose highest power of x bears the coefficient

$$\begin{vmatrix} a_0, & a_1, & a_2, & \dots a_\epsilon \\ a_1, & a_2, & a_3, & \dots a_{\epsilon+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_\epsilon, & a_{\epsilon+1}, & a_{\epsilon+2}, & \dots a_{2\epsilon} \end{vmatrix};$$

then is

$$\begin{vmatrix} f_{i-\epsilon}, & f_{i-\epsilon+1}, & f_{i-\epsilon+2}, & \dots f_{i+\epsilon} \\ f_{i-\epsilon+1}, & f_{i-\epsilon+2}, & f_{i-\epsilon+3}, & \dots f_{i+\epsilon+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ f_{i+\epsilon}, & f_{i+\epsilon+1}, & f_{i+\epsilon+2}, & \dots f_{i+2\epsilon} \end{vmatrix}$$

equal to $y^{\epsilon^2+\epsilon} H_\epsilon(f_{i+\epsilon})$.

The order in (x, y) of $H_\epsilon f_{i+\epsilon}$, since the weight of its leading coefficient is $\epsilon^2 + \epsilon$ and its degree in the coefficients $\epsilon + 1$, will be $(\epsilon+1)(i+\epsilon) - 2(\epsilon^2+\epsilon)$, that is, $(\epsilon+1)i - \epsilon^2 - \epsilon$, so that multiplied by $y^{\epsilon^2+\epsilon}$ the order becomes $(\epsilon+1)i$, as it ought to be.

The theorem may be proved as follows:

Let ϕ be any homogeneous function of λ dimensions in x, y, and denote $\frac{d}{dx}, \frac{d}{dy}$ by X, Y.

(1) I shall show that in respect of ϕ,

$$y^i \,.\, Y^i = {}^i\lambda - i\,{}^{i-1}(\lambda-1)\,xX + \frac{i\,.\,i-1}{2}\,{}^{i-2}(\lambda-2)\,x^2X^2 \dots + (-)^i x^i X^i,$$

where ${}^i m$ for any positive integer values of m and i denotes the factorial quantity $m(m-1)\dots(m-i+1)$.

Suppose the equation to be true for any assigned value of i, it will be true for $i+1$. For $Y^i\phi$, it will be observed, is of $\lambda - i$ dimensions in x, y; hence

$$y^{i+1} Y^{i+1} = (\lambda - i - xX) * y^i Y^i$$

[* Vol. II. of this Reprint, p. 498.]

for $(\lambda - i)\, Y^i\phi = (xX + yY)\, Y^i\phi$ by Euler's well-known theorem on homogeneous functions.

The $(j+1)$th and $(j+2)$th terms in $y^i Y^i$ are respectively

$$\mp \frac{i(i-1)\ldots(i-j+1)}{1.2\ldots j}(\lambda - j)(\lambda - j - 1)\ldots(\lambda - i + 1)\, x^j X^j,$$

say
$$-Ax^jX^j$$

and $$\pm \frac{i(i-1)\ldots(i-j)}{1.2\ldots(j+1)}(\lambda - j - 1)(\lambda - j - 2)\ldots(\lambda - i + 1)\, x^{j+1}X^{j+1},$$

say
$$Bx^{j+1}X^{j+1}.$$

Now
$$xX * x^{j+1}X^{j+1} = x^{j+2}X^{j+2} + (j+1)\, x^{j+1}X^{j+1}.$$

Hence the $(j+2)$th term in $y^{i+1}Y^{i+1}$ will be

$$A + (\lambda - i - j - 1)\, B,$$

that is, putting $\dfrac{i(i-1)\ldots(i-j+1)}{1.2\ldots j\,(j+1)} = B'$, is $\mu B'$,

where
$$\begin{aligned}\mu &= (j+1)(\lambda - j) + (i - j)(\lambda - i - 1 - j)\\ &= \{-j^2 + (\lambda - 1)j + \lambda\} + \{j^2 - (\lambda - 1)j + \lambda i - i^2 - i\}\\ &= (\lambda - i)(i+1).\end{aligned}$$

Thus the $(j+2)$th term in $y^{i+1}Y^{i+1}$ will be

$$\pm \frac{(i+1)\, i\ldots(i+1-j)}{1.2\ldots(i+1)}(\lambda - j - 1)(\lambda - j - 2)\ldots\{\lambda - (i+1) + 1\};$$

and consequently the equation is true for $i+1$.

Hence, being true for $i = 1$, it is true universally.

(2) Consider a persymmetrical determinant of the order $\epsilon + 1$ formed with the distinct constituents $\phi_0, \phi_1, \phi_2, \ldots \phi_{2\epsilon}$, where ϕ_0 is a constant and in general $\dfrac{d}{dx}\phi_k = \pm k\phi_{k-1}$; as, for example, suppose $\epsilon = 2$, and let the determinant be

$$\begin{vmatrix} a, & ax+b, & P \\ ax+b, & P, & Q \\ P, & Q, & R \end{vmatrix},$$

where P, Q, R stand for

$$ax^2 + 2bx + c,\ ax^3 + 3bx^2 + 3cx + d,\ ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$$

and $\dfrac{d\phi_k}{dx} = k\phi_{k-1}$. If we made $\dfrac{d\phi_k}{dx} = -k\phi_{k-1}$ the effect would be to change the signs of all the odd-degreed functions, but the value of the determinant would not be altered by this change. Calling the columns P_0, P_1, P_2,

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0$$

will represent a determinant equal to the given one, but of the form

$$\begin{vmatrix} a, & b, & c \\ ax+b, & bx+c, & cx+d \\ ax^2+2bx+c, & bx^2+2cx+d, & cx^2+2dx+e \end{vmatrix},$$

and now, calling the lines L_0, L_1, L_2, the equivalent determinant

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0$$

becomes

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

which is the same as if in the original form we made $x=0$.

So in general for the order $\epsilon+1$, calling the ϵ columns $P_0, P_1, P_2 \ldots P_\epsilon$, we may pass to a new determinant by means of the combinations represented by

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0,\ \ldots\ P_\epsilon - \epsilon xP_{\epsilon-1} + \frac{\epsilon(\epsilon-1)}{2}x^2P_{\epsilon-2} \ldots + (-)^\epsilon x^\epsilon P_0,$$

and calling the lines of these new determinants

$$L_0,\ L_1,\ L_2 \ldots L_\epsilon,$$

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0,\ \ldots\ L_\epsilon - \epsilon xL_{\epsilon-1} + \frac{\epsilon(\epsilon-1)}{2}x^2L_{\epsilon-2} \ldots + (-)^\epsilon x^\epsilon L_0,$$

will produce a determinant containing no power of x, and which is what the original one becomes on making $\epsilon=0$.

(3) If we take for our $2\epsilon+1$ distinct elements of the persymmetrical matrix, the quantities

$$X^{2\epsilon}\phi,\ yYX^{2\epsilon-1}\phi,\ y^2Y^2X^{2\epsilon-2}\phi,\ \ldots\ Y^{2\epsilon}\phi,$$

where ϕ is of λ dimensions in x, y, we shall find by virtue of (1) that they will be represented by

$$A_0,\ A_1 - A_0x,\ A_2 - 2A_1x + A_0x^2,\ \ldots$$
$$A_{2\epsilon} - 2\epsilon A_{2\epsilon-1}x + \frac{2\epsilon(2\epsilon-1)}{2}A_{2\epsilon-2}x^2 \ldots + A_0x^{2\epsilon},$$

(where $\phi_k = -k\phi_{k-1}$) on making

$$A_0 = X^{2\epsilon}\phi,\ A_1 = (\lambda - 2\epsilon + 1)X^{2\epsilon-1}\phi,$$
$$A_2 = (\lambda - 2\epsilon + 2)(\lambda - 2\epsilon + 1)X^{2\epsilon-2}\phi,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{2\epsilon} = \lambda(\lambda-1)\ldots(\lambda-2\epsilon+1).$$

Now obviously the persymmetrical determinant in question on striking out each power of y from its several constituents will be diminished in the proportion of 1 to $y^{2+4\ldots+2\epsilon}$, that is, $y^{\epsilon^2+\epsilon}$.

Hence

$$\begin{vmatrix} A_0, & A_1, & A_2, & \dots & A_\epsilon \\ A_1, & A_2, & A_3, & \dots & A_{\epsilon+1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_\epsilon, & A_{\epsilon+1}, & A_{\epsilon+2}, & \dots & A_{2\epsilon} \end{vmatrix}$$

$$= y^{\epsilon^2+\epsilon} \begin{vmatrix} X^{2\epsilon}\phi, & X^{2\epsilon-1}Y\phi, & \dots & X^\epsilon Y^\epsilon \phi \\ X^{2\epsilon-1}Y\phi, & X^{2\epsilon-2}Y^2\phi, & \dots & X^{\epsilon-1}Y^{\epsilon+1}\phi \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ X^\epsilon Y^\epsilon \phi, & X^{\epsilon-1}Y^{\epsilon+1}\phi, & \dots & Y^{2\epsilon}\phi \end{vmatrix}.$$

This is true for any function ϕ homogeneous in x and y*.

If ϕ is a rational integral function of x, y, say

$$(a_0, a_1, a_2, \dots \between x, y)^\lambda,$$

the last written determinant becomes a covariant whose leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, a_2, \dots a_{2\epsilon}$ multiplied by

$$\{\lambda(\lambda-1)\dots(\lambda-2\epsilon+1)\}^{\epsilon+1},$$

and if we write

$$\begin{aligned} \lambda(\lambda-1)\dots(\lambda-2\epsilon+1)B_0 &= X^{2\epsilon}\ \phi, \\ \lambda(\lambda-1)\dots(\lambda-2\epsilon\ \ \ \ )B_1 &= X^{2\epsilon-1}\phi, \\ \lambda(\lambda-1)\dots(\lambda-2\epsilon-1)B_2 &= X^{2\epsilon-2}\phi, \\ \dots\dots\dots\dots\dots\dots \\ B_{2\epsilon} &= \phi, \end{aligned}$$

we shall have the persymmetrical determinant formed with the elements $B_0, B_1, \dots B_{2\epsilon}$, equal to $y^{\epsilon^2+\epsilon}$ multiplied into the covariant of which the leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, \dots a_{2\epsilon}$, as was to be proved.

* It seems to me very likely or almost certain that every covariant of $f(x, y)$, or what becomes such when f is a quantic, may in like manner be converted into a function of f and of its derivatives in respect to one of the variables alone divided by an appropriate power of the other; and, if true, as it can hardly help being, the proof ought not to be far to seek.

It is indeed virtually contained in a formula obvious from inspection of the expression for y^iY^i in (1), namely,

$$\left\{\frac{d}{dx}(y^iY^iX^j)\right\} * = -i\{y^{i-1}Y^{i-1}X^{j+1} *\},$$

whatever homogeneous function is supposed to follow the asterisk. In connexion with this it should be observed that the determinant in (2) is bound to vanish, from the mere fact that on putting $x=0$, it becomes an invariant of $(a_0, a_1, a_2, \dots a_{2\epsilon}\between \xi, \eta)^{2\epsilon}$, and that its several terms are what the a elements become when X becomes $\xi + x\eta$. We are thus led to view the whole subject of invariance under a somewhat broader aspect, as a theory not directly concerned with quantics, but with homogeneous functions in general.

Scholium. The theory of hyperdeterminants teaches us that every in- and co-variant has its source of being in a higher existence, namely, in a pure form typified by

$$F(X^\mu\phi,\ YX^{\mu-1}\phi,\ Y^2X^{\mu-2}\phi,\ \ldots\ Y^\mu\phi),$$

ϕ being a perfectly general operand, or as we may phrase it, an operand absolute. This enables me to express the idea which was struggling into light when I wrote the antecedent footnote. It is this: Let ϕ now be made to do duty for any given homogeneous function of given order λ in x, y.

The value of F will remain unaltered when we write

$$\begin{array}{rcc} \dfrac{\lambda-\mu+1}{y} & \text{in place of} & Y, \\ \dfrac{(\lambda-\mu+2)(\lambda-\mu+1)}{y^2} & " & Y^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \\ \dfrac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-\mu+1)}{y^\mu} & " & Y^\mu. \end{array}$$

This is an immediate consequence of the invariantive property of F combined with the fact that

$$\frac{d}{dx}y^iY^i = -iy^{i-1}Y^{i-1}X,$$

previously shown. The numerators in the above expressions are the first terms in the expression for y^iY^i as a function of xX modified by writing successively $\lambda-\mu+1$, $\lambda-\mu+2$, $\ldots\ \lambda$ in place of λ on account of the powers of X which precede Y, Y^2, $\ldots\ Y^\mu$ in F and lowering the degrees of the operands in respect to these powers by $\mu-1$, $\mu-2$, $\ldots$ 0 units respectively.

Thus, for example, the pure invariant

$$(X^4:)(Y^4:)-4(X^3Y:)(XY^3:)+3(X^2Y^2:)^2$$

where the colon (:) does duty for an operand absolute is equivalent to

$$\begin{array}{lll} \dfrac{1}{y^4}\cdot & (\lambda-3)(\lambda-2)(\lambda-1)\lambda & (X^4:)(X^0:) \\ & -4(\lambda-3)^2(\lambda-2)(\lambda-1) & (X^3:)(X:) \\ & +3(\lambda-3)^2(\lambda-2)^2 & (X^2:)(X^2:), \end{array}$$

the colon now representing a homogeneous function of order λ in x, y.

So in general we may say that a *pure invariant,* or it might be more correct to say the *Schema* of an invariant, *is a function of symbolic inverses* $(X, Y, \ldots)$ *to any number of letters and of any number of unconditional absolutes, possessing the property that when those absolutes become conditioned to stand for homogeneous functions of the letters, of* SPECIFIED *orders, it becomes a function of any one of the letters, of the symbolic inverses to the rest and of the absolutes so conditional.*

This property, which is certainly *necessary*, is in all probability sufficient to define a pure invariant, for I presume (nay I think it is obvious) that when it is satisfied, the only part the arbitrarily selected letter can play is that of contributing a power of itself as a factor to the function in which it figures. This definition of invariance, although it may appear abstruse, is in reality the most complete and simplest, in the sense of exemption from foreign ingredients and unnecessary specifications, that can be given, and may of course be extended without difficulty to systems of sets of letters $(x, y, \ldots)$. Nor should it be overlooked that in our great art, the *ars magna excogitandi*, a gain in expression is a gain in power*.

Returning from this rather wide excursus to our original theme of Newton's theorem, it may be useful to give the values of the G† series as far as required for equations of the 5th order inclusive corresponding to the critical value of the arbitrary parameter, that is, for the case of $C = -n$.

The given form being supposed to be $(a, b, c, \ldots \between x, y)^n$,

when $n = 2$, $-G_1 = ac - b^2$,

when $n = 3$, $-G_1 = (ac - b^2)x^2 + (ad - bc)x + (bd - c^2)$,

$-G_2 = ac - b^2$,

when $n = 4$, $-G_1 = (ac - b^2)x^4 + 2(ad - bc)x^3 + (ae + 2bd - 3c^2)x^2 + 2(be - cd)x + (ce - d^2)$,

$-G_2 = (ac - b^2)x^2 + (ad - bc)x + (bd - c^2)$,

$-G_3 = ac - b^2$,

when $n = 5$, $-G_1 = (ac - b^2)x^6 + 3(ad - bc)x^5 + 3(ae + bd - 2c^2)x^4 + (af + 7be - 8cd)x^3 + 3(bf + ce - 2d^2)x^2 + 3(cf - de)x + (df - e^2)$,

G_2, G_3, G_4 being the G_1, G_2, G_3 of the preceding case‡.

In applying the series of these G's combined with the f series to ascertain the maximum possible number of real roots passed over in going *up* from λ

* The object of pure Physic is the unfolding of the laws of the intelligible world. ["The unseen world" belongs to another province altogether.] The object of pure Mathematic (which is only another name for Algebra) that of unfolding the laws of the human intelligence. With Geometry it fares as it was thought to be probably about to fare with a certain distant land—it is "wiped out" between the two neighbouring powers. Algebra takes for its share Geometry in the abstract. Sensible or empirical Geometry (as, thanks to the Copernican genius of Lobatcheffsky and the sublimated practical sense of Helmholtz, is now beginning to be well understood) falls into the domain of Physic.

So already Logic is divided between Psychology and Algebra; and so eventually with Grammar, whilst Linguistic is handed over to History, Psychology and Physiology; its theoretical part, the laws of syntax, declension or conjugation, regimen and collocation, must be eventually absorbed into Algebra.

† [*In line* 12 *of* p. 415 *above, the first sign should be* $-$, *not* $+$.]

‡ It is thus seen that the G series is formed of the second alliances or "überschiebungen" of the given form (made homogeneous in x, y), and of its successive derivatives each with itself; and I have great reason to believe (as already hinted) that we may append a 3rd, 4th, ... series

to μ it is proper to use simultaneously the three *independent* superior limits (1) the gain of pP's, (2) the loss of vP's, (3) the gain of p's or loss of v's, which two latter numbers are of course identical.

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by substituting the 4th, 6th, ... of such alliances in lieu of the second, filling up the vacant spaces with positive absolutes, and always reckoning the gain of the permanence-permanence-permanence...s in going up from λ to μ as one superior limit, and, as a consequence thereof, the loss of the variation-permanence-permanence...s as another. Thus, for example, for the case of $n=4$, the series would be three in number, namely,

$$\begin{array}{ccccc} f, & f_1, & f_2, & f_3, & f_4, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & 1, \\ 1, & 1, & s, & 1, & 1, \end{array}$$

where $s=ae-4bd+3c^2$ (and it may be noticed that we know from the expression for s in terms of the roots that when they are real, s must be positive).

For $n=5$ the series would be

$$\begin{array}{cccccc} f, & f_1, & f_2, & f_3, & f_4, & f_5, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & -Hf_3, & 1, \\ 1, & 1, & s, & s', & 1, & 1, \end{array}$$

where $$s'=ae-4bd+3c^2,$$

and $$s=(ae-4bd+3c^2)\,x^2+(af-3be+2cd)\,x+(bf-4ce+3d^2).$$

When $n=6$ or $n=7$ a new series would dawn into existence, and so on continually. Thus we set a number of sieves, as it were, successively under each other; it is certain, however, that by this method we can never be assured that no more than the actual number of real roots have fallen through; but there is another method which might be studied, and is, I think, not unworthy of investigation, that is, to take for our third series the covariants of f which have for their common leading coefficient the discriminant of the form $(a, b, c, d \between x, y)^3$, for the fourth series the covariants which have for their common leading coefficient the discriminant of $(a, b, c, d, e \between x, y)^4$, and so on indefinitely, always filling up the vacant spaces with positive absolutes.

In this way I think it not improbable that the gain of compound permanences may be found to give not merely a superior limit to, but the actual number of real roots passed over in any ascent from one value of x to another.

Such a theorem, however, would have no practical value as a method for separating the roots, as its application would entail much greater labour than the ordinary Sturmian process.

43.

ON THE EXACT RELATION WHICH RESULTANTS AND DISCRIMINANTS BEAR TO THE PRODUCT OF DIFFERENCES OF ROOTS OF EQUATIONS.

[*Messenger of Mathematics*, IX. (1880), pp. 164—166.]

First, for Resultants.

Let there be two rational integral functions in x of the degrees r, s respectively; and, for greater simplicity, let the coefficients of x^r, x^s in these functions be each made equal to unity. Call ρ the roots of the one, σ of the other; and denote the product of the differences found by subtracting each σ from each ρ by $D_{\rho,\sigma}$.

Also, by the resultant $R_{r,s}$ understand that irreducible rational integral function of the coefficients, vanishing when the functions have a root in common, in which the highest power of the last coefficient of the "s" equations enters with the positive sign.

We must then have $R_{r,s} = \mu D_{\rho,\sigma}$; and it only remains to determine μ as a function of r, s.

To do this let the r function become x^r, and the s function $x^s + 1$.

For greater distinctness, suppose $r = 4$, $s = 2$.

Then, obviously, $R_{r,s}$ becomes the dialytic resultant of

$$\begin{array}{cccccc}
x^5 & & & & & \\
 & x^4 & & & & \\
x^5 & & +x^3 & & & \\
 & x^4 & & +x^2 & & \\
 & & x^3 & & +x & \\
 & & & x^2 & & +1
\end{array}$$

which is equal to 1.

And in like manner for all values of r, s,

$$R_{r,s} = 1.$$

Again, $$D_{\rho,\sigma} = \{0 - (-1)^{\frac{1}{s}}\}^{rs} = (-)^{rs+r}.$$

Hence μ, which is a function of r, s exclusively, $= (-)^{rs+r}$.

Next, for Discriminants.

By the discriminant of fx of the order n, and where, for greater simplicity, the coefficient of x^n is supposed to be unity, I mean the resultant of fx and $f'x$; or, which is the same thing, of $\dfrac{df(x,1)}{dx}$ and $\dfrac{df(x,1)}{d1}$, when the term in which the highest power of the last coefficient in fx appears is made positive. Let this be called R_n, and the product of the squared differences of the roots Z_n; we have then $R_n = \mu Z_n$, where μ is a function of n to be determined. To find it let us take $fx = x^n - 1$.

R_n is then the resultant of nx^{n-1}, $-ny^{n-1}$, that is, is equal to

$$(-)^{n-1} n^{2n-2}.$$

Again,
$$Z_n = \left.\begin{array}{lll} (1-\rho) & (1-\rho^2) & \ldots (1-\rho^{n-1}) \\ (\rho-\rho^2) & (\rho-\rho^3) & \ldots (\rho-1) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ (\rho^{n-1}-1) & (\rho^{n-2}-\rho) & \ldots (\rho^{n-1}-\rho^{n-2}) \end{array}\right\} \div (-)^{\frac{1}{2}(n\cdot n-1)},$$

ρ representing $\cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$.

Hence
$$Z_n = n^n \,.\, (-)^{\frac{1}{2}n(n-1)} \,.\, \{\rho^{\frac{1}{2}n(n-1)}\}^{n-1}$$
$$= (-)^\theta n^n,$$

where
$$\theta = -\tfrac{1}{2}\{n(n-1)\} + (n-1)^2 = \tfrac{1}{2}\{(n-1)(n-2)\},$$

and
$$\theta + (n-1) = \tfrac{1}{2}\{(n-1)n\}.$$

Hence $R_n = (-)^{\frac{1}{2}(n-1)n} n^{n-2} Z_n$, or $\mu = (-)^{\frac{1}{2}(n-1)n}$, which was to be found.

For ordinary algebraical investigations the determination of μ has little importance, which may account for its value being omitted in the ordinary text books; but for certain investigations concerning the numerical divisors of cyclotomic functions, with which I am occupied, I found it necessary to pay attention to the numerical part at least of this factor, and I have thought that the publication of the result might save others some unnecessary trouble.

44.

SUR LES DIVISEURS DES FONCTIONS CYCLOTOMIQUES.

[*Comptes Rendus*, XC. (1880), pp. 287—289, 345—347.]

SOIT k un nombre quelconque ; formons la série

$$\cos \lambda_1 \frac{2\pi}{k}, \ \cos \lambda_2 \frac{2\pi}{k}, \ \ldots, \ \cos \lambda_i \frac{2\pi}{k},$$

$\lambda_1, \lambda_2, \ldots, \lambda_i$ étant les $\frac{1}{2}\phi(k)$ nombres premiers à k et moindres que $\frac{1}{2}k$. Le produit de tous les facteurs $x - 2\cos\lambda\frac{2\pi}{k}$ est ce que l'on nomme une *fonction cyclotomique*, et k sera nommé son indice. En effet, la fonction cyclotomique en x à l'indice k est ce que devient le facteur primitif de $t^k - 1$ quand on le divise par $t^{\frac{1}{2}\phi(k)}$ et que l'on écrit $t + t^{-1} = x$. A l'indice 1 ou 2 ne correspond aucune fonction cyclotomique, et pour les indices 3, 4, 6, la fonction cyclotomique est linéaire, et conséquemment ne peut posséder aucune propriété arithmétique.

Je distingue les diviseurs de ces fonctions en deux classes. Les nombres qui divisent la fonction sans diviser l'indice se nomment *diviseurs extérieurs* ou *extrinsèques*, ceux qui divisent en même temps une fonction et son indice se nomment *diviseurs intérieurs* ou *intrinsèques.*

Voici les théorèmes que j'ai réussis à établir concernant ces diviseurs.

Quant à la première classe, je démontre :

1°. Que tout nombre dont les facteurs premiers diminués ou augmentés de l'unité sont divisibles par l'indice d'une fonction cyclotomique est diviseur de cette fonction. Je fais dépendre la démonstration de cette proposition du théorème suivant, qui est, pour ainsi dire, la clef de la théorie entière :

En posant $$J(\cos\vartheta) = \cos(p^i\vartheta) - \cos(p^{i-1}\vartheta),$$

$J(\cos\vartheta)$, *regardé comme fonction algébrique de* $\cos\vartheta$, *est divisible par* p^i *pour toute valeur réelle et entière attribuée à* $\cos\vartheta$.

La proposition précédente est une conséquence immédiate de ce théorème, quand on met $2\cos\vartheta = x$ et qu'on substitue, pour la congruence

$$J(\cos\vartheta)\equiv 0 \quad [\text{mod. } p^i],$$

la congruence équivalente

$$(t^{p^i-p^{i-1}}-1)(t^{p^i+p^{i-1}}-1)\equiv 0 \quad [\text{mod. } p^i];$$

de sorte que, a étant un nombre réel quelconque, il faut que l'un ou l'autre des deux facteurs $a^{p^i-p^{i-1}}-1$, $a^{p^i+p^{i-1}}-1$ soit toujours divisible par p^i, car, si les deux facteurs contenaient p, on aurait $a^{2p^i}-1$ divisible par p; c'est-à-dire, puisque $2p^i = 2+\left(2\dfrac{p^i-1}{p-1}\right)(p-1)$, a^2-1 serait divisible par p, et conséquemment $a=\pm 1+\lambda p$, auquel cas $a^{p^{i-1}}\equiv(\pm 1)$ mod. p^i, et les deux facteurs deviennent respectivement congrus à $(\pm 1)^{p^i\pm p^{i-1}}-1$, c'est-à-dire tous les deux congrus à zéro par rapport à ce module, et par conséquent tous les deux divisibles par p^i et congrus à zéro. Avec l'exception de ces valeurs de a, c'est toujours l'un des deux facteurs exclusivement qui s'évanouit pour une valeur donnée de a.

2°. Je démontre, à l'aide du même théorème de forme trigonométrique, mais en faisant $i=1$, que si un diviseur extérieur d'une fonction cyclotomique, disons ψ_k, est de la forme $mk\pm e$, k étant son indice, la congruence

$$\psi_k\equiv 0 \quad [\text{mod. } mk\pm e]$$

aura deux racines congrues l'une à l'autre, à moins que $e=1$. On prouve facilement que cette équivalence est impossible avec l'aide du petit principe additionnel que, si ψ est congru à zéro selon un module quelconque, $\dfrac{d\psi}{dx}$ sera congru à zéro selon le même module.

Quant à la seconde classe des diviseurs, je démontre que, laissant à part les fonctions cyclotomiques linéaires $x+1$, x, $x-1$ appartenant aux indices 3, 4, 6 et la fonction quadratique qui répond à l'indice 12, il n'y a *au plus* qu'*un seul* diviseur intérieur (un nombre premier); bien entendu, la première puissance seulement de ce nombre. J'ai déjà dit que, pour que p^j soit un diviseur extérieur, il faut et il suffit que $p=mk+\epsilon$, k étant l'indice et $\epsilon=\pm 1$. Or, pour que p soit diviseur intérieur de la fonction cyclotomique à l'indice k, je démontre qu'il faut et qu'il suffit que k soit de la forme

$$\frac{p-\epsilon}{m}p^j.$$

En général, il n'y a *au plus* qu'une seule manière de mettre un indice k, donné sous la forme qui met en évidence un diviseur intérieur; mais, quand $k=12$, on peut écrire $m=1$, $j=2$, $p=2$, $\epsilon=-1$ ou bien $m=1$, $j=1$, $p=3$, $\epsilon=-1$; c'est pourquoi ψ_{12} possède les *trois* diviseurs intérieurs 2, 3, 6. En démontrant que la condition donnée plus haut pour que p soit diviseur

intérieur est nécessaire et que la première puissance seulement de p est un diviseur de la fonction, je me sers du même théorème trigonométrique qu'auparavant et en même temps de la seconde proposition sur les facteurs extérieurs. Pour démontrer que cette condition est suffisante, j'ai recours à un théorème purement algébrique, savoir, que si $k = k_1(mk_1 \pm 1)^j$, $mk_1 \pm 1$ étant un nombre premier p, le résultant des deux équations $\psi_k = 0$, $\psi_{k_1} = 0$ est égal à $p^{\frac{1}{2}\phi(k_1)}$, en me servant en même temps d'un second petit principe, qu'afin que deux congruences soient satisfaites simultanément par rapport au même module, le résultant algébrique de ces congruences transformées en équations doit être congru à zéro par rapport au module.

La fonction cyclotomique à l'indice 9, $x^3 - 3x + 1$, m'a amené à faire cette recherche; car j'avais grandement besoin de démontrer apodictiquement (ce que j'avais établi par des épreuves numériques sans fin) que les diviseurs de cette fonction sont 3 et les nombres premiers de la forme $18n \pm 1$ exclusivement. C'est à l'aide de ce théorème que je démontre qu'aucun nombre A de la forme*

$$pq,\ p^2q^2,\ p_1p_2^2,\ q_1q_2^2;\quad 9pq,\ 9p^2q^2,\ 9p_1p_2^2,\ 9q_1q_2^2,$$

où chaque p désigne un nombre premier de la forme $18n - 5$ et chaque q un nombre premier de la forme $18n + 7$, ne peut être décomposé en une somme ou différence de deux cubes rationnels. En effet, je démontre facilement que, si cette décomposition était possible, l'équation

$$x^3 - 3xy^2 + y^3 = 3Az^3$$

serait résoluble en nombres entiers, ce qui est impossible, puisque $x^3 - 3x + 1$ ne contient aucun p ou q. La même équation, en mettant $A = 3$, devrait avoir lieu aussi si 3 était décomposable en deux cubes rationnels; ainsi on voit (comme on sait déjà) que cette décomposition est impossible, puisque $x^3 - 3x + 1$ ne contient pas le diviseur intérieur 9.

Tout ce que j'ai pu trouver sur la question qui a fait le sujet de ma première Communication † est contenu dans le livre classique du professeur Bachmann, *Die Lehre von der Kreistheilung*‡, Leipzig, 1872, pp. 242, 243;

[* See below, p. 437.]

† *Comptes Rendus*, séance du 16 février [p. 428, above.]

‡ *Kreistheilung* = cyclotomie. La fonction à racines réelles qui sert à la division du cercle en parties égales est celle que j'ai nommée *fonction cyclotomique*. Il y a aussi des fonctions cyclotomiques à racines imaginaires; je parle des facteurs primitifs de $x^k - 1$, qu'on pourrait nommer *fonctions cyclotomiques simples* ou *irréduites*, dont les diviseurs sont assujettis à des conditions parallèles, mais non identiques avec celles des fonctions cyclotomiques que j'ai traitées dans le texte. En effet, voici la règle pour les diviseurs des fonctions cyclotomiques non réduites. Afin qu'un nombre quelconque soit diviseur d'une fonction cyclotomique non réduite à l'indice k, il faut et il suffit que chaque facteur premier de ce diviseur soit de la forme $ki + 1$, avec exception d'un seul facteur premier p qui peut figurer aussi comme facteur du diviseur dans le cas, et

mais cela même ne me servait à rien, car cet excellent auteur s'est borné au cas où l'indice est un nombre premier, pour lequel cas il énonce et démontre "qu'en dehors des diviseurs premiers de la forme $2mp \pm 1$" la fonction cyclotomique à l'indice p "contient seulement le diviseur premier p"; mais M. Bachmann n'a nullement démontré ni même affirmé, ce qui cependant est vrai, que tout nombre premier de la forme $2mp \pm 1$, et même un tel nombre élevé à une puissance quelconque *, est diviseur de la fonction cyclotomique à l'indice p.

Reste une remarque à faire. Si l'on prend le produit des facteurs $x - 2\cos\lambda \frac{2\pi}{k} y$, on obtient ce qu'on peut nommer une *forme* cyclotomique. Quand on prend l'indice égal à 5 ou à 10, à 8 ou à 12, de sorte que l'ordre de cette forme, disons $F(x, y)$, devient 2, si D est un diviseur quelconque de la fonction cyclotomique à ces indices, on sait, par la théorie ordinaire des

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seulement dans le cas, que k admet de la représentation (nécessairement et sans exception unique) $\frac{p-1}{m} p^j$. Ainsi, si P, p désignent des nombres premiers, J, j des nombres indéfinis, et k l'indice d'une fonction cyclotomique de l'une ou de l'autre espèce, et si

$$P = mk + \epsilon \quad \text{et} \quad k = \frac{p-\epsilon}{m} p^j,$$

P^J et p seront diviseurs de la fonction dans un cas et dans l'autre, avec la distinction que pour les fonctions cyclotomiques simples $\epsilon = 1$, tandis que pour les fonctions cyclotomiques à racines réelles $\epsilon = \pm 1$. En effet, le cours de la démonstration est précisément le même dans les deux cas, avec la seule exception que pour la première proposition, celle qui affirme que, p étant un nombre premier de la forme $mk + \epsilon$, p^j est diviseur de la fonction à indice k, pour les fonctions cyclotomiques d'une classe on se sert du théorème que la congruence $\cos p^j \vartheta - \cos p \vartheta \equiv 0$ [mod. p^j] a toutes ses racines réelles; pour les fonctions cyclotomiques de l'autre classe on se sert du théorème (mieux connu) que *la congruence*

$$x^{p^j} - x^{p^{j-1}} \equiv 0 \quad [\text{mod. } p^j]$$

a toutes ses racines réelles. Pour tout ce qui suit cette proposition, la méthode de démonstration pour les deux cas est absolument identique. Peut-être serait-il mieux de nommer les fonctions dont je parle spécialement dans le texte *fonctions cyclotomiques de la seconde*, et celles qui sont simplement facteurs primitifs de la forme binôme *fonctions cyclotomiques de la première espèce*. Il y a une raison qui me paraît assez grave pour ce changement de nomenclature, vu qu'il suggère l'idée d'une théorie de diviseurs des fonctions cyclotomiques dont le *rang de l'espèce* sera un nombre q quelconque, où figureront les racines $q^{\text{ièmes}}$ de l'unité, par rapport à l'indice comme module, de laquelle théorie je crois entrevoir assez distinctement et la haute probabilité de son existence et sa nature. J'espère développer cette théorie dans quelque futur Mémoire.

* Il est à peine nécessaire d'observer que la fonction cyclotomique de l'ordre ω [où $\omega = \frac{1}{2}\phi(k)$] étant divisible pour ω valeurs a de la variable incongrues par rapport à p^α, et ω autres valeurs b de la même variable incongrues par rapport à q^β, par p^α, q^β respectivement, on n'a qu'à combiner un a quelconque avec un b quelconque, et, en écrivant $p^\alpha u - a = t = q^\beta v - b$, on obtiendra une valeur réelle de t (et conséquemment ω valeurs réelles de t), qui substituée pour la variable rendra la fonction divisible par $p^\alpha q^\beta$; et de même on déduit que la fonction admettra comme diviseur un nombre quelconque dont les facteurs sont les nombres premiers de la forme $mk \pm 1$ accompagnés ou non (au choix) par le facteur intrinsèque, quand il y en a un, et par l'un ou l'autre ou tous les deux facteurs intrinsèques 2, 3, dans le cas où l'indice est le nombre 12.

formes quadratiques, qu'en écrivant $F(x, y) = Dz^2$ (les valeurs de F étant $x^2 \pm xy - y^2$, $x^2 - 2y^2$ ou $x^2 - 3y^2$), une telle équation est résoluble en nombres entiers.

Or une étude empirique très étendue sur le cas où l'indice est 9, qui mène à l'équation $x^3 - 3xy^2 + y^3 = Dz^3$, m'a donné lieu de croire qu'il y a une probabilité très considérable que cette équation est aussi toujours résoluble en nombres entiers. Si cela était établi, il deviendrait plus que probable que le théorème analogue est vrai pour toutes les formes cyclotomiques, et du cas de l'indice 9, si seulement la résolubilité de l'equation qui y appartient était démontrée, on tirerait la belle conséquence que tout nombre dont les facteurs premiers sont de la forme $18n \pm 1$, accompagné ou non accompagné (au choix) par le facteur 9, est décomposable en une somme de cubes de deux nombres rationnels. Car on démontre facilement qu'en substituant pour X, Y, Z, respectivement, certaines fonctions rationnelles et entières qu'on connaît, du neuvième degré en x, y, z, la fonction $X^3 + Y^3 + AZ^3$ contiendra

$$x^3 - 3xy^2 + y^3 - 3Az^3$$

comme facteur algébrique.

Voici, en quelques mots, le résumé des lois actuellement démontrées:

Tout diviseur de la fonction cyclotomique à l'indice k est de la forme $ik \pm 1$, excepté dans le cas que $k = \dfrac{p \mp 1}{m} p^j$, dans lequel cas p aussi (mais non pas p^2) sera un diviseur. Et réciproquement tout nombre dont les facteurs sont des puissances arbitraires de nombres premiers de la forme $ik \pm 1$ est diviseur de la fonction cyclotomique à l'indice k.

On peut y ajouter que, si l'ordre de la fonction cyclotomique [c'est-à-dire $\frac{1}{2}\phi(k)$] est nommé ω, et N un nombre quelconque qui ne divise pas k, il n'y aura aucune valeur ou ω valeurs de la variable, incongrues par rapport à N, qui rendront la fonction divisible par N. Mais si p, nombre premier, est un diviseur de k, le nombre des valeurs de la variable qui rendent la fonction divisible par p sera ou nul ou le quotient de k par la plus haute puissance qu'il contient de p.

45.

SUR LA LOI DE RÉCIPROCITÉ DANS LA THÉORIE DES NOMBRES.

[*Comptes Rendus*, XC. (1880), pp. 1053—1057, 1104—1106.]

SOIT $\left(\frac{Q}{P}\right)$ le symbole bien connu de Jacobi, généralisation du symbole $\left(\frac{Q}{p}\right)$ de Legendre. Selon que $\left(\frac{Q}{P}\right) = +1$ ou -1, je dirai que l'aspect quadratique ou simplement l'aspect de Q vers P est positif ou négatif. On accorde que Q et P peuvent l'un et l'autre être ou positifs ou négatifs, avec la convention que $\left(\frac{Q}{-P}\right) = \left(\frac{Q}{P}\right)$ et $\left(\frac{Q}{1}\right) = 1$. Alors il est plus ou moins distinctement reconnu que, Q, P étant tous les deux nombres impairs et relativement premiers, si Q et P ne sont pas tous les deux négatifs, $\left(\frac{Q}{P}\right)\left(\frac{P}{Q}\right) = 1$ quand Q et P *ne sont pas*, et $= -1$ quand Q et P *sont*, tous les deux de la forme $4m + 3$.

Mais, si Q et P sont tous les deux négatifs, $\left(\frac{Q}{P}\right)\left(\frac{P}{Q}\right) = -1$ quand Q et P *ne sont pas*, et $= 1$ quand Q et P *sont*, tous les deux de la forme $4m + 3$.

Servons-nous du mot *reste quaternaire* pour exprimer le reste minimum absolu d'un nombre impair par rapport au module 4. Ce reste sera ou $+1$ ou -1. Servons-nous aussi, en général, du symbole $\binom{m}{n}$ ou $\binom{n}{m}$ pour signifier un nombre qui est -1 quand m et n sont tous les deux négatifs et $+1$ dans le cas contraire. Soient a, b deux nombres quelconques positifs ou négatifs, impairs et relativement premiers, a' et b' leurs restes quaternaires; alors, en vertu des théorèmes précédents, on aura

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \binom{a}{b}\binom{a'}{b'},$$

formule qui constitue le véritable théorème de réciprocité et suffit à elle-même comme formule universelle de réduction, sans avoir besoin de supplément (*Ergänzung*) aucun.

Je nomme, en général, *chaîne réductive* une suite de chiffres positifs ou négatifs dont le dernier est l'unité positive ou négative et dont chaque terme intermédiaire est un diviseur de la différence de ses deux termes voisins; une telle suite se nomme *chaîne réductive impaire* quand tous les termes sont impairs. Il est évident qu'on peut toujours former une chaîne réductive impaire dont les deux premiers termes sont des nombres impairs donnés, car dès le second terme on peut trouver des termes continuellement décroissants qui rempliront les conditions imposées.

Or je dis que, pour trouver la valeur de $\left(\frac{b}{a}\right)$, on n'a qu'à former une chaîne réductive impaire commençant avec a, b et une chaîne auxiliaire dont les termes sont les résidus quaternaires des termes de la première; alors, selon que la somme des nombres des permanences des signes *moins* prises dans une suite et dans l'autre est paire ou impaire, l'*aspect* de b vers a sera positif ou négatif.

En voici la preuve. Soient

$$a,\ b,\ c,\ d,\ \ldots,\ h,\ k,\ l,$$
$$a',\ b',\ c',\ d',\ \ldots,\ h',\ k',\ l',$$

la première une suite réductive impaire et la seconde une suite auxiliaire formée avec les restes quaternaires de l'autre. Alors on aura

$$\left(\frac{b}{a}\right)=\binom{b}{a}\binom{b'}{a'}\left(\frac{a}{b}\right)=\binom{b}{a}\binom{b'}{a'}\left(\frac{c}{b}\right),$$
$$\left(\frac{c}{b}\right)=\binom{c}{b}\binom{c'}{b'}\left(\frac{b}{c}\right)=\binom{c}{b}\binom{c'}{b'}\left(\frac{d}{c}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left(\frac{k}{h}\right)=\binom{k}{h}\binom{k'}{h'}\left(\frac{h}{k}\right)=\binom{k}{h}\binom{k'}{h'}\left(\frac{l}{k}\right),$$
$$\left(\frac{l}{k}\right)=\binom{l}{k}\binom{l'}{k'}\left(\frac{k}{l}\right)=\binom{l}{k}\binom{l'}{k'}.$$

Donc

$$\begin{aligned}\left(\frac{b}{a}\right)&=\binom{b}{a}\binom{c}{b}\cdots\binom{k}{h}\binom{l}{k}\\&\quad\times\binom{b'}{a'}\binom{c'}{b'}\cdots\binom{k'}{h'}\binom{l'}{k'}\\&=(-1)^{n+n'},\end{aligned}$$

n étant le nombre de fois que les successions a, b; b, c; ...; h, k; k, l contiennent deux signes $-$ et n' le nombre correspondant pour a', b'; b', c'; ...; h', k'; k', l'; c'est-à-dire l'aspect de b vers a sera positif ou négatif, selon que $n + n'$ (que je nommerai ν) est pair ou impair, ce qui était à démontrer.

Je ferai l'application de cette méthode de calculer le symbole $\left(\frac{b}{a}\right)$ à des exemples tirés du Traité (*Zahlentheorie*) de Lejeune-Dirichlet. Pour trouver $(\frac{195}{1901})$, on forme la chaîne réductive

$$\begin{matrix} + & + & - & - \\ 1901 & 195 & 49 & 1, \end{matrix}$$

qui donne la chaîne auxiliaire

$$\begin{matrix} + & - & - & - \\ 1 & 1 & 1 & 1. \end{matrix}$$

On a donc $n=1$, $n'=2$, $\nu = n+n' = 3$; conséquemment $(\frac{195}{1901}) = -1$, et, puisque 1901 est nombre premier, 195 est non-résidu quadratique de ce nombre. Pour trouver $\left(\frac{74}{101}\right) = \left(\frac{-27}{101}\right)$, on obtient les deux chaînes (omettant dans la seconde le chiffre constant 1)

$$\begin{matrix} + & - & - & + \\ 101, & 27, & 7, & 1; \\ + & + & + & + \end{matrix}$$

$\nu = 1+0 = 1$, et, comme auparavant, 74 est non-résidu au nombre premier 101.

Si $b > a$, les suites prendront la forme

$$\begin{matrix} a, & b, & a, & d, \ldots, l, \\ a', & b', & a', & d', \ldots, l', \end{matrix}$$

et, puisque la somme des permanences négatives dans aba et $a'b'a'$ est évidemment 0, 2 ou 4, on peut faire abstraction de ces parties de la chaîne double dans le calcul. Ainsi, par exemple, on aura pour $(\frac{27}{103})$

$$\begin{matrix} + & + & - & - & + \\ 103, & 27, & 5, & 3, & 1, \\ - & - & - & + & + \end{matrix}$$

et pour $(\frac{103}{27})$

$$\begin{matrix} + & - & - & + \\ 27, & 5, & 3, & 1. \\ - & - & + & + \end{matrix}$$

Comme dernier exemple, je trouverai la valeur générale de $\left(\frac{2}{k}\right)$, c'est-à-dire de $\left(\frac{2-k}{k}\right)$. Si l'on donne à n les valeurs 1, 3, 5, 7, on obtient les chaînes doubles

$$\begin{matrix} + \\ 1; \\ + \end{matrix} \quad \begin{matrix} + & - \\ 3, & 1; \\ - & - \end{matrix} \quad \begin{matrix} + & - & - \\ 5, & 3, & 1; \\ + & + & - \end{matrix} \quad \begin{matrix} + & - & - & + \\ 7, & 5, & 3, & 1; \\ - & - & + & + \end{matrix}$$

et, en général, pour $n = 2i+1$, 3, 5, 7, on trouvera très facilement que les valeurs des quatre chaînes doubles de signes qui y correspondent seront

$$\begin{pmatrix} + & - & - & + \\ + & + & - & - \end{pmatrix}^i \begin{matrix} + \\ + \end{matrix}; \quad \begin{pmatrix} + & - & - & + \\ - & - & + & + \end{pmatrix}^i \begin{matrix} + & - \\ - & - \end{matrix};$$

$$\begin{pmatrix} + & - & - & + \\ + & + & - & - \end{pmatrix}^i \begin{matrix} + & - & - \\ + & + & - \end{matrix}; \quad \begin{pmatrix} + & - & - & + \\ - & - & + & + \end{pmatrix}^i \begin{matrix} + & - & - & + \\ - & - & + & + \end{matrix},$$

où l'indice supérieur i signifie que les signes contenus dans les parenthèses doivent être i fois répétés. Il est à remarquer que dans ces suites répétées de quatre signes il n'arrive jamais que le premier et le dernier signe sont tous les deux négatifs; de sorte qu'on n'obtiendra aucune permanence négative à la jonction de deux de ces suites.

On aura donc la somme des permanences négatives pour ces quatre cas égale à

$$2i,\ 2i+1,\ 2i+1,\ 2i+2,$$

respectivement: de sorte que l'aspect de 2 vers $8i+1, 7$ est positif et vers $8i+3, 5$ négatif: résultat qu'on a ainsi déduit avec l'aide de la seule formule de réduction pour les nombres impairs.

Il est digne de remarque que, puisque $\left(\frac{b}{a}\right)=\left(\frac{b}{-a}\right)$, il s'ensuit que, si, dans une série réductive impaire quelconque et la série de ses restes quaternaires, on change simultanément le signe des termes alternés en commençant avec le premier terme en chacune, la somme des permanences des signes négatifs sera augmentée ou diminuée par un nombre pair.

Il y a tant d'analogie entre la méthode exposée dans un précédent article et celles qu'on emploie dans les théorèmes de Newton et Fourier sur les racines réelles des équations algébriques, qu'on se sent très porté à soupçonner que le nombre que j'ai nommé ν est la limite supérieure à quelque affection de a, b à laquelle elle reste toujours congrue par rapport au module 2; mais de la nature de cette affection, si toutefois elle existe, je n'ai nulle connaissance.

De même qu'on a trouvé une expression générale pour l'aspect de $2-k$ vers k, on peut, avec l'aide du théorème de la chaîne, construire, d'une infinité de manières, des fonctions algébriques de k, dont on saura d'avance les aspects des unes vers les autres. Ainsi, pour prendre un exemple très simple, formons la série

$$1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378,\ \ldots,$$

où
$$u_k=2u_{k-1}+u_{k-2},\quad u_1=2,\quad u_0=1,$$

et conséquemment

$$u_k=2^k+(k-1)\,2^{k-2}+\frac{(k-1)(k-3)}{1.2}\,2^{k-4}+\ldots.$$

On peut se demander l'expression générale pour l'*aspect quadratique* de u_{2i-1} vers u_{2i} pour une valeur quelconque de i.

On trouvera sans peine que les suites de signes qui donnent les valeurs de $(\frac{2}{5})$, $(\frac{12}{29})$, $(\frac{70}{169})$, $(\frac{408}{985})$ sont

$$\begin{matrix}+--\\++-\end{matrix};\quad \begin{matrix}+--++\\+---+\end{matrix};\quad \begin{matrix}+--++--\\++-+++-\end{matrix},$$

$$\begin{matrix}+--++--++\\+---+---+\end{matrix};$$

et, en général, que $\left(\frac{u_{4i+1}}{u_{4i+2}}\right)$ donne naissance à la chaîne double

$$\begin{pmatrix}+ & - & -\\ + & + & -\end{pmatrix}\begin{pmatrix}+ & + & - & -\\ + & + & + & -\end{pmatrix}^i$$

et $\left(\frac{u_{4i-1}}{u_{4i}}\right)$ à

$$\begin{pmatrix}+\\ +\end{pmatrix}\begin{pmatrix}- & - & + & +\\ - & - & - & +\end{pmatrix}^i.$$

Dans le premier cas, ν est égal à $i+1$, et dans le second à $3i$; ainsi les valeurs successives de ν étant 1, 3, 2, 6, 3, 9, 4, 12, 5, ..., l'aspect de u_{8j+1} à u_{8j+2} et de u_{8j+3} à u_{8j+4} est positif, mais de u_{8j+5} à u_{8j+6} et de u_{8j+7} à u_{8j+8} négatif.

Dans le *Zahlentheorie* de Lejeune-Dirichlet, rédigé par M. Dedekind (3e édition, p. 110; Braunschweig, 1879), on rencontre cette phrase: "Es zeigt sich nun, dass die damals nothwendige Zerlegung in Primzahlfactoren (abgesehen von dem Factor 2) ganz überflüssig geworden." Ce qui précède ici rend évident (il me semble) que cette exclusion du nombre 2 (due probablement à quelque mésintelligence de la part des auditeurs de l'illustre Dirichlet) est elle-même (*überflüssig*) superflue.

Je profite de cette occasion pour corriger la liste que j'ai donnée dans une Note précédente des nombres qu'on démontre, par le moyen des diviseurs de x^3-3x+1, être indécomposables dans une somme de cubes rationnels. Dans cette liste*, $9pq$, $9p_1p_2^2$, $9q_1q_2^2$, $9p^2q^2$ étaient insérés par erreur; la démonstration, en un seul coup, de l'irrésolubilité des seize formes générales qui restent a paru† dans le dernier fascicule de l'*American Journal of Mathematics.*

Post-scriptum.—Dans les exemples très nombreux que j'ai calculés de l'application de mon algorithme pour déterminer l'aspect de Q vers P, j'ai toujours trouvé que la *différence* δ de n et n' (les nombres de permanences négatives dans les deux suites), prise positivement, est une *limite inférieure* au nombre de cas où q est non-résidu de p (q étant un facteur premier quelconque de Q et p de P).

Si cette remarque est démontrée de validité universelle, elle fournira un moyen de mettre à l'épreuve, d'une infinité de manières, si un nombre donné P est un nombre premier. Car, en combinant P avec un nombre premier arbitraire Q, si δ est plus grand que 1, P, devant contenir au moins δ facteurs auxquels Q est non-résidu, sera nécessairement un nombre composé. Au contraire, quand P est nombre premier, δ sera toujours ou 0 ou 1, selon la valeur de Q, ce qui constituerait un théorème nouveau sur le symbole $\left(\frac{q}{p}\right)$ de Legendre.

[* Above, p. 430.] [† Above, p. 347.]

46.

SUR LES ÉQUATIONS À 3 ET À 4 PÉRIODES DES RACINES DE L'UNITÉ.

[*Compte Rendu de la Association Française* (1880), *Reims*, pp. 96—98.]

DÉSIGNONS par p l'ordre des racines de l'unité,

„ „ e le nombre des périodes,

„ „ $\eta_0, \eta_1, \eta_2, \ldots, \eta_{e-1}$, les périodes elles-mêmes;

„ „ $\omega_0, \omega_1, \omega_2, \ldots, \omega_{e-1}$ les périodes que j'appelle affectées et définies par l'équation $\omega = e\eta + 1$.

§ 1.

I. On prouve facilement que l'on a

$$\Sigma\eta^i \equiv \left(\frac{p-1}{e}\right)^{e-1} \pmod{p},$$

i étant un nombre entier quelconque.

II. On en conclut $$\Sigma\omega = 0,$$
et aussi $$\Sigma\omega^i \equiv 0 \pmod{p}.$$

III. On démontre facilement que

$$\Sigma\omega_0\omega_1 = \tfrac{1}{2}\Sigma\omega^2 = -\frac{e}{2}\,p, \text{ lorsque } \frac{p-1}{2} \text{ est impair};$$
$$= \frac{e^2-e}{2}\,p, \text{ lorsque } \frac{p-1}{2} \text{ est pair.}$$

IV. Enfin, on démontre encore les relations

$$3\Sigma\omega_0\omega_1\omega_2 = \Sigma\omega^3 \equiv 3 \pmod{9}, \quad \text{pour } e = 3;$$
$$3\Sigma\omega_0\omega_1\omega_2 \equiv 8 \pmod{24}, \quad \text{pour } e = 4.$$

§ 2.

Si une fonction rationnelle et entière des périodes ne change pas de valeur par une substitution circulaire, on sait que cette fonction est nécessairement un nombre réel. Il est également vrai et on démontre facilement que si une telle fonction change de signe, mais conserve sa valeur absolue par une substitution circulaire, elle est un multiple entier de $\sqrt{p}$ ou de $\sqrt{(-p)}$, selon que $\dfrac{p-1}{e}$ est pair ou impair.

Ainsi par exemple, le produit des différences des périodes est un nombre entier lorsque e est impair, et un multiple entier de $\sqrt{(\pm p)}$ lorsque e est pair. De même, dans le cas de 4 périodes, le produit $(\eta_0 - \eta_2)(\eta_1 - \eta_3)$ est un multiple de $\sqrt{p}$.

§ 3.

La forme de l'équation à 3 périodes affectées en vertu de (II) et de (III) sera

$$\omega^3 - 3p - Ap = 0$$

dont le discriminant est $4A^2p^2 - p^3$; donc, en vertu de § 2,

$$\frac{p^3 - 4A^2p^2}{27} = M^2 = B^2p^2;$$

donc $4p = A^2 + 27B^2$; ainsi A^2 est déterminé. De plus, $3Ap$ en vertu de (III) sera congru à 3 (mod. 9), c'est-à-dire Ap, et conséquemment A sera congru à $+1$ (mod. 3): donc A est parfaitement déterminé.

V. Pour $e = 4$ on voit facilement que les équations dont ω_0, ω_2, ω_1, ω_3 sont les racines seront de la forme

$$\omega^2 - 2\sqrt{(p)}\omega + Ap - B\sqrt{p} = 0,$$
$$\omega^2 + 2\sqrt{(p)}\omega + Ap + B\sqrt{p} = 0,$$

où A et B seront des nombres entiers, et en vertu du § 2 on aura

$$\{(A-1)p + B\sqrt{p}\}\{(A-1)p - B\sqrt{p}\},$$

c'est-à-dire

$$(A-1)^2p^2 - pB^2 = m^2p.$$

Selon que $\frac{p-1}{2}$ est pair ou impair

$$2A = -\frac{e}{2} = -2 \text{ ou } \frac{e^2 - e}{2} = 6;$$

ainsi dans l'un et l'autre cas $(A-1)^2 = 4$, c'est-à-dire $4p = B^2 + m^2$, de sorte que si $p = f^2 + g^2$, $B = 2f$ et $m = 2g$. Donc, pour les deux cas respectivement, l'équation aux périodes affectées sera

$$(\omega^2 - p)^2 - 4p(\omega + 2f)^2 = 0, \quad (\omega^2 + 3p)^2 - 4p(\omega + 2f)^2 = 0,$$

et avec l'aide de (IV) on trouve facilement que $f \equiv -1$ (mod. 4), de sorte qu'en mettant $p = f^2 + g^2$ on sait que f doit être impair et congru à 1 (mod. 4). Donc, les équations sont parfaitement déterminées.

Ces formules s'accordent, comme il est nécessaire, avec les résultats connus depuis longtemps, mais qu'on n'obtient par les méthodes de Gauss et de Jacobi qu'avec des calculs pénibles, pour ainsi dire fortuits, ou des raisonnements un peu détournés.

47.

ON A POINT IN THE THEORY OF VULGAR FRACTIONS.

[*American Journal of Mathematics*, III. (1880), pp. 332—335, 388—389.]

THE reciprocal of an integer I call a simple fraction; any other fraction, whether rational or irrational, may be termed complex; but it is to be understood that only proper fractional quantities of either sort, that is, fractions greater than zero and less than unity, will be considered in what follows.

Suppose Q to represent any fractional quantity; if Q lies between $\frac{1}{u_0 - 1}$ and $\frac{1}{u_0}$, we may make $Q = \frac{1}{u_0 + \delta} + Q'$, where δ is zero or a positive integer, and Q' will continue a proper fraction, which in like manner may be resolved into $\frac{1}{u_1 + \delta_1} + Q''$, and so on continually.

But if we make δ_0, δ_1, ... each zero, the process of expansion becomes determinate. Any such determinate representation of a fractional quantity I shall term a *sorites*. It is obvious that in expanding a given fraction under the form of a sorites, the successive denominators, which I shall call the *elements*, may be obtained by a process of division; if the fraction to be expanded is rational, the real divisor will be an integer which continually decreases*, and consequently every complex rational fraction can be expanded (and only in one way) under the form of a finite sorites.

The elements of a sorites are analogous to the partial quotients of a regular continued fraction; but there is this difference between the two cases, that whilst the latter quantities are perfectly arbitrary, the elements in question are subject to a certain law which I shall proceed to examine.

* See examples of development of sorites, page [443].

Let $n, p, q, \ldots r, s, \ldots t, u$ be the elements of a sorites. It is clear that the last remainder being the reciprocal of $\frac{1}{t}+\frac{1}{u}$, we must have $\frac{1}{t}+\frac{1}{u}<\frac{1}{t-1}$, that is to say, u greater than t^2-t, that is, u is equal to or greater than t^2-t+1. Again, if we look to the residue which gives birth to the element r, that must be of the form $\frac{1}{s-\epsilon}$, where ϵ is some fraction, and we must now have $\frac{1}{r}+\frac{1}{s-\epsilon}<\frac{1}{r-1}$, or $s-\epsilon$ equal to or greater than r^2-r. Hence s is equal to or greater than r^2-r+1, so that the relation between any two contiguous elements is the same, whether they are or are not the final two; and if u_x, u_{x+1} be any two consecutive integers in a series, the one necessary and sufficient condition for the possibility of the existence of the sorites, of which those terms shall be elements, is that we must have for all values of x, u_{x+1} equal to or greater than $u_x^2-u_x+1$.

If u_{x+1} is throughout equal to $u_x^2-u_x+1$, we obtain a series which may be termed a limiting sorites.

It is obvious that any simple fraction $\frac{1}{u_0-1}$ may be expanded under the form of an infinite sorites, of which the elements are $u_0, u_1, u_2 \ldots$ subject to the above relation. An infinite sorites read in the limiting case is therefore expressible under the form of a finite fraction, and the same will be true for a sorites in which the right-hand branch beginning from any term u_i, namely, $\frac{1}{u_i}+\frac{1}{u_{i+1}}+\frac{1}{u_{i+2}}\ldots$, forms a limiting sorites.

But in every other case of a sorites the sum cannot be a finite fraction; for such fraction can be expanded in only one way under the form of a sorites, and such sorites is necessarily finite in the number of its terms.

Hence it is impossible that the sum of the reciprocals of an ascending series of positive integers, such that the square root of the difference between any one of them and its immediate antecedent is greater than the difference between that antecedent and unity, can represent a rational quantity; for if so, we have $u_{x+1}-u_x$ greater than $(u_{x-1}-1)^2$, that is, $u_{x+1}>u_x^2-u_x+1$, and the series will form a sorites not belonging to the limiting class.

I proceed to examine some of the properties of the series of terms defined by the condition $u_{x+1}=u_x^2-u_x+1$.

In the first place, I observe that any term u_{x+i} may be expressed under the form Pu_x+1: for suppose this to be true for one value of i; then, since $u_{x+i+1}-1=u_{x+i}(u_{x+i}-1)$, it is obviously true for the next above; here the proposition, being true when i is unity, is true universally.

It follows from this that each element of a limiting sorites is prime to all that follow it, and consequently any two terms of the sorites are prime to one another.

Again, for greater simplicity, let $v_0, v_1, v_2, \ldots$ be used to represent $(u_0 - 1)$, $(u_1 - 1)$, $(u_2 - 1)$, ...; we have, then,

$$v_1 - v_0 = v_0^2, \quad v_2 - v_1 = v_1^2, \quad v_3 - v_2 = v_2^2, \ldots.$$

Hence $v_2 - v_0$, $v_3 - v_0$, ... $v_x - v_0$ (as is obvious from successive addition of the above equations) will each of them be of the form Pv_0^2, where P is a rational integral function of v_0, and v_x will be of the form $Pv_0^2 + v_0$. This conclusion leads to a representation of the sum of any given number of terms of a limiting sorites by a fraction in its lowest terms. For

$$\frac{1}{v_x} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_x}{v_x v_{x+1}} = \frac{v_x^2}{v_x v_{x+1}} = \frac{v_x}{v_x + v_x^2} = \frac{1}{v_x + 1} = \frac{1}{u_x}.$$

Hence

$$\frac{1}{u_0} + \frac{1}{u_1} + \ldots + \frac{1}{u_x} = \frac{1}{v_0} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_0}{v_0 v_{x+1}} = \frac{(v_{x+1} - v_0) \div v_0^2}{v_{x+1} \div v_0},$$

which is of the form $\dfrac{P}{Pv_0 + 1}$ and is consequently a fraction in its lowest terms.

Again, if we denote the product of the elements $u_0, u_1, u_2, \ldots, u_x$ by Πu_x and the sum of their $(x-1)$-ary combinations by $\Pi' u_x$, $\dfrac{\Pi' u_x}{\Pi u_x}$ will also be the same fraction in its lowest terms, because (as has been shown) all the elements of the sorites are prime to one another.

Hence we may deduce the equations

$$u_{x+1} = u_0 + (u_0 - 1)^2\, \Pi' u_x,$$

$$u_{x+1} = 1 + (u_0 - 1)\, \Pi u_x.$$

The second of these serves to give an inferior limit to the rate of convergence of any sorites. For in the limiting case we have

$$u_1 > (u_0^2 - u_0),$$

$$u_2 > (u_0 - 1)\; u_0 u_1 \; > (u_0^2 - u_0)^2,$$

$$u_3 > (u_0 - 1)\, u_0 u_1 u_2 > (u_0^2 - u_0)^4,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and so in general $u_x > (u_0^2 - u_0)^{2^{x-1}}$, because the solution of the equation

$$\theta_i = \theta_{i-1} + \theta_{i-2} + \ldots + \theta_0 \text{ is } \theta_i = 2^{i-1}\theta_0.$$

In any other sorites in which the initial element remains u_0, the value of the element at x-places distant must be *à fortiori* greater than the value $(u_0^2 - u_0)^{2^{x-1}}$ last obtained for the limiting case.

The preceding matter was suggested to me by the chapter in Cantor's *Geschichte der Mathematik* which gives an account of the singular method in use among the ancient Egyptians for working with fractions. It was their curious custom to resolve every fraction into a sum of simple fractions according to a certain traditional method, not leading, I need hardly say, except in a few of the simplest cases, to the expansion under the special form to which I have, in what precedes, given the name of a fractional *sorites.*

I subjoin examples of development of a rational fraction under the form of a sorites.

Let $\frac{4699}{7320}$ be the fraction to be expanded. The work may be arranged as follows:—

(2)	(8)	(60)	(3660)
4699	2078	1984	1920
7320	14640	117120	7027200
9398	16624	119040	7027200

(2) is the number one unit greater than $E\frac{7320}{4699}$; 9398 is 2×4699; 2078 is $9398 - 7320$; 14640 is 2×7320.

One element (2) is now determined, and the fraction $\frac{2078}{14640}$ remains to be expanded.

(8) is the number one unit greater than $E\frac{14640}{2078}$; 16624 is 8×2078; 1984 is $16624 - 14640$; 117120 is 8×14640.

A second element (8) is now found, and $\frac{1984}{117120}$ remains over to be expanded. Proceeding in this manner, and with numerators 4699, 2078, 1984, 1920, necessarily diminishing at each step, we come at last to the element 3660 with a remainder zero. The required sorites is therefore

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{60} + \frac{1}{3660}.$$

As a second example take the fraction $\frac{335}{336}$.

The work may be arranged in a similar manner to that of the foregoing example, and will be as follows:—

(2)	(3)	(7)	(48)
335	334	330	294
336	672	2016	14112
670	1002	2310	14112

and accordingly it will be found that

$$\frac{335}{336} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{48}.$$

Postscript.

Let ϕx represent $x^2 - x + 1$, $\phi^n c$ will then be the general term of the "limiting sorites" whose first term is c, for which, if we please, $1-c$ may be substituted. The properties of the numbers $\phi^n c$ seem to be worthy of some attention. I confine my observations in what follows to the lowest of such series, namely, where $c = 2$ or -1.

The first five terms in such series then become $\bar{1}$ or 2, 3, 7, 43, 1807, 3263443, of which all but 1807, which $= 13 . 139$, are prime numbers. Every term in the series must contain only factors of the form $6i+1$, and this, joined to the fact that a prime factor which has once appeared in any term can never reappear in any other, favours a tendency, so to say, of the numbers to remain primes, or at all events, to be of very limited frangibility into a product of primes.

It is easy to determine whether any proposed prime can occur as a factor of any term whatever in the series; for taking that number, say p, as a modulus, if r is a remainder of any term to that modulus, the remainder of the next term will be $r^2 - r + 1$, and as soon as any remainder reappears the series of remainders becomes periodic; so that necessarily in less than the number p of remainders, if p does divide any term of the sorites, we must arrive at a remainder zero, subsequent to which all the remainders are unity. I give the remainders and periods in the annexed table for all values of p of the form $6i+1$ up to 139, from which it will be seen that, under that limit, 13 and 73 are the only prime numbers which are contained as factors in the terms of the series.

p	Remainders of $\phi^n(2)$ to modulus p
2	0.
3	2, 0.
7	2, 3, 0.
13	2, 3, 7, 4, 0.
19	2, 3, 7, 5; 2, 3, 7, 5; ...
31	2, 3, 7, 12, 9, 11, 18, 28, 13; 2, 3, 7, ..., 13; ...
37	2, 3, 7; 6, 31; 6, 31; ...
43	2, 3, 7, 0.
61	2, 3, 7, 43, 38, 4, 13, 35, 32; 17, 29, 20, 15, 28, 25, 52, 30; ...
67	2, 3; 7, 43, 65; 7, 43, 65; ...
73	2, 3, 7, 43, 55, 51, 69, 21, 56, 15, 65, 0.
79	2, 3, 7; 43, 69, 32, 45, 6, 31, 61, 27, 71, 73; ...
97	2, 3, 7, 43, 61; 72, 69, 37; 72, 69, 37; ...
103	2, 3; 7, 43, 56, 94, 91, 54, 82, 51, 79, 86, 101; 7, 43, ...; ....
109	2, 3, 7, 43, 63, 92, 89, 94, 23, 71, 66, 40, 35, 101, 73, 25, 56, 29, 50, 53; 32, 12, 24, 8; 32, 12, 24, 8; ...
127	2, 3, 7, 43, 29, 51, 11; 111, 19, 89, 86, 72, 33, 41, 117; ...
139	2, 3, 7, 43, 0.
151	2, 3, 7, 43, 146; 31, 25, 148, 13, 6; ...
157	2, 3; 7, 43, 80, 41, 71, 104, 37, 77, 44, 9, 73, 76, 49, 155; ...
163	2, 3; 7, 43, 14, 20, 55, 37, 29, 161; ...
181	2, 3, 7, 43, 178, 13, 157, 58, 49, 0.
193	2; 3, 7, 43, 70, 6, 31, 159, 33, 92, 74, 192; ...
199	2, 3; 7, 43, 16, 42, 131, 116, 8, 57, 9, 73, 83, 41, 49, 164, 67, 45, 190, 91, 32, 197; ...

48.

INSTANTANEOUS PROOF OF A THEOREM OF LAGRANGE ON THE DIVISORS OF THE FORM $Ax^2+By^2+Cz^2$, WITH A POSTSCRIPT ON THE DIVISORS OF THE FUNCTIONS WHICH MULTISECT THE PRIMITIVE ROOTS OF UNITY.

[*American Journal of Mathematics,* III. (1880), pp. 390—392.]

IF possible, let p be not a divisor of x^2+y^2+1, and consequently not of the form $4i+1$, since, if it were of that form, x^2+1 would contain it.

Let ρ be any primitive pth root of unity.

Call $R=\Sigma\rho^{x^2}$, where x^2 means any one of the quadratic residues of and inferiors to p, and let the period conjugate to R be called R'.

Let R^2 be expanded as a sum of powers of ρ. Then, because p is not of the form $4i+1$, we cannot have $x^2+y^2=p$, so that no pth power of ρ can occur in that expansion; again, because by hypothesis neither $2x^2$ nor x^2+y^2 can be congruous to -1 [mod. p], no such power as ρ^{p-1} which belongs to R', nor consequently any other term of R', can appear in R^2; and as each power of ρ in R^2 belonging to the same period must appear a like number of times, we must have

$$R^2=\frac{p-1}{2}R, \text{ that is, } R=0, \text{ or } R=\frac{p-1}{2},$$

each of which suppositions is in the highest degree absurd. Hence p is a divisor of x^2+y^2+1. Q.E.D.

Compare Legendre's *Théorie des Nombres,* Ed. 1830, Tom. I. pp. 211—213, and again Serret's *Cours d'Algèbre Supérieure,* Tom. II. pp. 94—99, for proofs of the more general similar theorem due to Lagrange, concerning u^2+Bt^2+C. These proofs are highly ingenious, but long and laboured in no slight degree; and as the sole apparent object of either author in proving the general theorem is to make use of the particular case of it to which this note refers as a foundation to the proof of Fermat's law of the four squares, I have

thought that an intuitive proof of so important a lemma might not be without interest to some of the junior readers of the *Journal**.

But in fact the general theorem may be proved with scarcely any greater trouble than the particular case disposed of.

For, supposing A, B, C to be all quadratic residues to p, we may write

$$A \equiv \alpha^2, \quad B \equiv \beta^2, \quad C \equiv \gamma^2 \text{ [mod. } p],$$

$$\alpha x = u, \quad \beta y = v, \quad \gamma z = w;$$

and the congruence $u^2 + v^2 + w^2 \equiv 0$, as previously shown, being soluble, evidently

$$Ax^2 + By^2 + Cz^2 \equiv 0$$

will be so too, since

$$\alpha x \equiv u, \quad \beta y \equiv v, \quad \gamma z \equiv w \text{ [mod. } p],$$

give integer values for x, y, z; and as obviously the case of A, B, C being all non-residues falls into the previous case by multiplying the congruence by any non-residue, we have only to consider the case of two of the three coefficients being residues and the third a non-residue, or the converse case, which, however, by multiplication as above, may be reduced to the former one.

Suppose, then, $A = \alpha^2$, $B = \beta^2$, C a non-residue, and that

$$Ax^2 + By^2 + Cz^2 \equiv 0 \text{ [mod. } p]$$

is insoluble. For simplicity, let $z = 1$. Then $u^2 + v^2 + C = 0$ must be insoluble; if p is of the form $4i + 3$, we shall obtain, precisely as before,

$$R^2 = \frac{p-1}{2} R,$$

and if p is of the form $4i + 1$,

$$R^2 = 2\frac{p-1}{4} + \left\{\left(\frac{p-1}{2}\right)^2 - \left(\frac{p-1}{2}\right)\right\} \div \frac{p-1}{2} \,.\, R,$$

or $$R^2 - \frac{p-3}{2} R + \frac{p-1}{2} = 0, \text{ that is, } R = \frac{p-1}{2}, \text{ or } R = -1,$$

any of which conclusions are eminently absurd.

* From this lemma there is scarcely more than a step to the theorem in question. If P is contained as a factor in the sum of four squares, it is easy to see that we may write

$$PQ = f^2 + g^2 + h^2 + k^2,$$

where $Q < P$, and

$$QQ' = (f - \alpha Q')^2 + (g - \beta Q')^2 + (h - \gamma Q')^2 + (k - \delta Q')^2,$$

where $Q' < Q$, and consequently, applying the Quaternion law of multiplication,

$$PQ' = f'^2 + g'^2 + h'^2 + k'^2,$$

and so we may form a continually decreasing series of quantities Q, Q', Q'', ... any one of which multiplied by P is a sum of four squares. Hence any divisor of such sum is itself such a sum, but by the lemma any prime number is a divisor of the sum of three, which plays the same part for present purposes as a sum of four squares, and is therefore a sum of four squares; consequently any number whatever, by the rule of multiplication already alluded to in this note, will be a sum of four squares.

Hence $Ax^2 + By^2 + Cz^2 \equiv 0$ [mod. p] *cannot* be insoluble; that is, the left-hand side of the congruence must contain p as a divisor.

P.S. In a future communication I will prove very simply that if a prime number $p = ef + 1$, and e is itself a prime number such that $(e - 1)$ contains no odd square number, then every divisor, without exception (other than p), of the function whose roots are the e periods of the primitive pth roots of unity, must be an eth power residue of p. If $(e - 1)$ contains any square number, the proof still holds good, except as regards the factors of such square, and there is no reason at present for supposing that the theorem may not be extended to the case of these excepted factors*. The same kind of reasoning may be applied also to the theory of period-functions for which e (the number of the periods) is not a prime number, and I find for the case of $e = 4$, that, leaving out of account the number 2 (which is always a divisor of the four-period function to p when p is of the form $8i + 1$, but never when it is of the form $8i + 5$, and may be or not a biquadratic residue of p, according to a well-known law), the divisors of the four-period function (excepting p) which do not divide g (the even term in the equation $[f^2 + g^2 = p]$), are necessarily biquadratic residues of p; as is also true of the prime-number divisors of g which are of the form $4i + 1$; but the prime-number divisors of g (all of which are necessarily divisors of the four-period function), of the form $4i - 1$, are quadratic only, and not biquadratic residues of p when p is of the form $8i + 5$; whereas for the case of $p = 8i + 1$ *all* the odd divisors of the four-period function (not counting p) are biquadratic residues of p†. The same investigation leads to the remarkable conclusion that if $p = f^2 + 4\gamma^2$, where f and γ are both of them odd and p a prime number, every divisor of $\frac{f^2 + 3\gamma^2}{4}$ is a biquadratic residue of p,—a theorem which I imagine would be difficult to prove by any other method.

* Thus, for example, if e is a prime number of the form $2^{2^x} + 1$, I am able to prove that *every* divisor of the e-period function (not excepting 2, if 2 should happen to be such a divisor) is an eth-power residue of p. Thus for $e = 2, 3, 5, 7, 11, 17$ we may be certain that there are none but eth-power-residue divisors of the period-function.

† Of course in a certain sense p or zero is an any-power residue. But there is good reason for separating p from the residues proper, inasmuch as only the *first* power of p, but an *unlimited* power of any true eth-power residue is a divisor of the e-period function,—a most important fact, which I presume must have been known to Bachmann, but has not been stated by him (in his *Kreistheilung*, 1872). An exceedingly simple proof of this and of the corresponding theorem for any cyclotomic function was given by Mr Hathaway at a recent meeting of the Mathematical Seminarium, at the Johns Hopkins University.

49.

SUR L'ENTRELACEMENT D'UNE FONCTION PAR RAPPORT À UNE AUTRE.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 1—3.]

SOIENT $f(\lambda)$ et $\phi(\lambda)$ deux fonctions rationnelles et entières de λ. Désignons par $\lambda = A$ les racines réelles de l'équation $f(\lambda) = 0$ et par $\lambda = B$ les racines réelles de l'équation $\phi(\lambda) = 0$, et concevons que ces racines soient représentées par des points situés sur l'axe réel du plan. Les racines A et les racines B se suivront sur cet axe d'une manière quelconque. Supposons que toutes les fois qu'un nombre pair de points A forment une suite non interrompue par des points B on supprime ces points A, et que toutes les fois qu'un nombre pair de points B forme une suite non interrompue par des points A on supprime ces points B, de sorte qu'à la fin de ces suppressions on arrive à une suite alternative de points A et de points B, c. à d. à une suite finale dans laquelle on ne trouve ni un A suivi d'un A, ni un B suivi d'un B. Cela posé, le nombre des points A qui restent dans la suite finale sera ce que l'on peut nommer l'indice de l'entrelacement effectif des racines de l'équation $f = 0$ par rapport aux racines de l'équation $\phi = 0$, mais que pour plus de brièveté je nommerai plutôt *l'entrelacement de f par ϕ**.

Construisons la courbe $y = f(x)$ et la courbe $y = \phi(x)$ et supposons que chacune de ces courbes soit représentée par un fil flexible infiniment mince et fixé en deux points assez éloignés de l'axe des x. Désignons les deux courbes par f et par ϕ, et supposons que dans les points de rencontre des deux courbes situés au nord de l'axe des x la courbe f passe au-dessus de la courbe ϕ, qu'au contraire dans les points de rencontre situés au midi de l'axe des x la courbe ϕ passe au-dessus de la courbe f. Cela posé, si deux points de rencontre consécutifs se trouvent tous les deux du même côté de l'axe des x, ils ne contribuent point au nombre que je viens de nommer l'entrelacement de f par ϕ, et l'on peut ôter ces deux points par une flexion convenable de l'une des deux courbes. Cette construction donne donc une signification géométrique et intuitive au nombre que j'ai défini comme l'entrelacement de f par ϕ. En effet, si f est d'ordre pair ou si f et ϕ sont tous les deux d'ordre

* De même que je viens de définir l'entrelacement total de f par ϕ, de même on pourra définir l'entrelacement de f par ϕ entre les deux limites p et q en se bornant aux racines réelles A et B situées entre les deux limites $\lambda = p$ et $\lambda = q$.

impair, ce nombre est en même temps le nombre des intersections permanentes des deux courbes f et ϕ. Si f est d'ordre impair et ϕ d'ordre pair, il faut selon les circonstances augmenter ou diminuer d'une unité le nombre des intersections permanentes des deux courbes pour obtenir le nombre analytique défini comme l'entrelacement de f par ϕ, ce que j'exposerai plus amplement dans la suite.

Pour donner plus de précision à la construction expliquée ci-dessus on peut faire l'hypothèse que dans toute l'étendue de l'axe des x les deux parties du plan des xy soient séparées par une fente (désignée dorénavant sous le nom de fente de X), que des fils flexibles ou rubans représentant les courbes f et ϕ soient assujettis à passer par cette fente toutes les fois qu'ils traversent l'axe des x et que le fil ou ruban ϕ soit collé aux deux faces du plan des xy. De cette hypothèse on tire comme conséquence que f se trouve au-dessus de ϕ pour des y positifs et au-dessous de ϕ pour des y négatifs comme on a supposé antérieurement.

Dans le cas où la fonction f est d'ordre impair et où la fonction ϕ est d'ordre pair on a déjà avancé que le nombre des intersections permanentes diffère d'une unité positive ou négative de l'entrelacement de f par ϕ. Pour décider de cette ambiguïté, changeons d'abord, s'il est nécessaire, les signes des fonctions f et ϕ, ce qui est permis dans cette recherche, de sorte qu'après le changement de signe dans l'une comme dans l'autre des deux fonctions la plus haute puissance de x se trouve multipliée par un coefficient positif. Poursuivons le cours des deux courbes f et ϕ en marchant du côté positif de l'axe des x vers le côté négatif et désignons sous le nom de nœud (knot en anglais) les intersections permanentes des deux courbes en nous rappelant que le nombre total de toutes leurs intersections est pair. Cela posé, si le premier nœud précède le premier passage des deux courbes par la fente de X ou, ce qui est la même chose, si le dernier nœud suit le dernier passage, le nombre des nœuds *diminué* d'une unité est égal à l'entrelacement de f par ϕ, dans le cas contraire le nombre des nœuds *augmenté* d'une unité est égal à l'entrelacement de f par ϕ. Il y a un criterium (physique) auquel on peut réduire la distinction des deux cas dont il s'agit. Imaginons que l'on réunit le bout positif de f (ruban libre) au même bout de ϕ (ruban collé au plan). De cette manière on formera une aire (loop) comprise entre les parties extrêmes positives des deux courbes. Dans le premier cas discuté ci-dessus ce loop est transitoire et peut être supprimé par une déformation convenable de f. Dans le cas contraire il est impossible de supprimer le loop sans rupture des rubans réunis. Cela posé, l'entrelacement analytique de f par ϕ est égal au nombre des nœuds qui se trouvent dans les deux rubans réunis, si l'on fait la convention de ne compter du tout le loop quand il est transitoire, mais de le compter comme équivalent à deux nœuds quand il est permanent.

50.

PREUVE INSTANTANÉE D'APRÈS LA MÉTHODE DE FOURIER, DE LA RÉALITÉ DES RACINES DE L'ÉQUATION SÉCULAIRE.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 4, 5.]

SOIT M un carré de termes dont le déterminant est Δ, m un carré mineur quelconque de M composé des éléments

$$\begin{array}{llll} \lambda, & \lambda_{1,2} & \ldots & \lambda_{1,\epsilon}, \\ \mu_{2,1}, & \mu & \ldots & \mu_{2,\epsilon}, \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ \rho_{\epsilon,1} & \rho_{\epsilon,2} & \ldots & \rho, \end{array}$$

et considérons les coefficients différentiels de Δ pris par rapport à chacun des ϵ^2 éléments qui se trouvent dans le carré écrit ci-dessus; d'après un théorème connu on sait que le déterminant de l'ordre ϵ, formé de tous ces coefficients différentiels, est égal à

$$\Delta^{\epsilon-1}\frac{d^\epsilon\Delta}{d\lambda\, d\mu \ldots d\rho}.$$

Soient λ, μ, ... ρ des éléments qui se trouvent dans la diagonale de M, et supposons que M soit symétrique par rapport à cette diagonale, faisons de plus $\epsilon = 2$, le théorème énoncé ci-dessus se change en la formule élémentaire

$$\Delta\frac{d^2\Delta}{d\lambda\, d\mu} = \frac{d\Delta}{d\lambda}\frac{d\Delta}{d\mu} - \left(\frac{d\Delta}{d\lambda_{1,2}}\right)^2,$$

où le carré qui forme le dernier terme de la seconde partie de l'equation est écrit au lieu du produit des deux expressions $\frac{d\Delta}{d\lambda_{1,2}}$, $\frac{d\Delta}{d\mu_{2,1}}$ devenues égales en vertu de la symétrie des éléments compris dans M.

De cette équation on tire les deux conclusions suivantes:

(θ) Quand $\Delta = 0$, $\frac{d\Delta}{d\lambda}$ et $\frac{d\Delta}{d\mu}$ ont le même signe.

(ϕ) Quand $\frac{d\Delta}{d\lambda} = 0$, Δ et $\frac{d^2\Delta}{d\lambda\, d\mu}$ ont des signes contraires.

Supposons à présent que α, β, ... ρ soient tous les éléments compris dans la diagonale de M, que α, β, ... ρ soient remplacés par $\alpha + x$, $\beta + x$, ... $\rho + x$

et que D soit la valeur correspondante du déterminant Δ. Cela posé, si r est une valeur de x pour laquelle $D=0$, toutes les expressions $\frac{dD}{d\alpha}, \frac{dD}{d\beta}, \dots \frac{dD}{d\rho}$ et par conséquent $\frac{dD}{dx} = \frac{dD}{d\alpha} + \frac{dD}{d\beta} + \dots + \frac{dD}{d\rho}$ auront le même signe en vertu de (θ).

Soit de plus h une quantité positive infiniment petite, la valeur de D pour $x = r + h$ aura le même signe que $\frac{dD}{d\alpha}$ et la valeur de D pour $x = r - h$ aura le signe contraire. Formons la suite

$$D, \frac{dD}{d\alpha}, \frac{d^2D}{d\beta d\alpha}, \dots \frac{d^nD}{d\rho \dots d\beta d\alpha},$$

n désignant l'ordre du déterminant D, et faisons croître la variable x depuis une limite inférieure quelconque en la faisant passer par toutes les valeurs successives jusqu'à une limite supérieure quelconque. Toutes les fois que x passe par la valeur d'une racine r de $D=0$, il y aura après le passage une permanence de signe de plus dans la suite écrite ci-dessus qu'il n'y en avait avant; si au contraire x passe par une valeur pour laquelle D ne s'évanouit point, il y aura avant et après le passage le même nombre de permanences. En effet prenons dans la suite dont il s'agit trois termes consécutifs quelconques, p. e.

$$\frac{d^2D}{d\beta d\alpha}, \frac{d^3D}{d\gamma d\beta d\alpha}, \frac{d^4D}{d\delta d\gamma d\beta d\alpha},$$

et soit

$$D' = \frac{d^2D}{d\beta d\alpha};$$

en vertu de (ϕ) les deux expressions D' et $\frac{d^2D'}{d\delta d\gamma}$ auront des signes contraires quand $\frac{\partial D'}{\partial\gamma}$ s'évanouit. Mais les deux premiers termes présenteront, comme l'on a vu ci-dessus, une variation de signe avant le passage d'une racine de l'equation $D=0$ et une permanence après ce passage. Conséquemment la série dont il s'agit gagnera depuis la limite inférieure jusqu'à la limite supérieure autant de permanences de signes qu'il y a entre ces limites de racines de l'équation $D=0$.

Soit $-\infty$ la limite inférieure, $+\infty$ la limite supérieure de la variable x, n sera le nombre des permanences que la série dont il s'agit aura gagné, donc les racines de l'équation $D=0$ sont toutes réelles, ce qu'il fallait démontrer.

Post-scriptum. Je dois remarquer que la preuve donnée par M. Salmon (*Lessons on Higher Algebra*, 3^{me} édition p. 43), que je n'avais pas remarquée précédemment, est encore plus simple que celle donnée en haut, cependant elle est tant soit peu moins directe: dans cette autre preuve on ne fait usage que de la conclusion (ϕ).

51.

SUR UN DÉTERMINANT SYMÉTRIQUE QUI COMPREND COMME CAS PARTICULIER LA PREMIÈRE PARTIE DE L'ÉQUATION SÉCULAIRE.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 6—9.]

LA théorie de l'équation séculaire s'étend aisément à un déterminant symétrique beaucoup plus général. Dans le théorème de Sturm ou dans le théorème plus complet sur les intercalations que j'ai donné dans mon mémoire "Sur les rapports syzygétiques, etc.", inséré aux *Philosophical Transactions**, on considère une suite de fonctions telles que trois fonctions consécutives quelconques P, Q, R soient liées par l'équation

$$P = Q'Q - R.$$

Dans le cas présent je m'occuperai de même d'une suite de fonctions telles qu'entre trois fonctions consécutives quelconques P, Q, R on ait l'équation

$$PR = Q'Q - Q''^2,$$

équation qui présente avec la première cette circonstance commune que pour $Q = 0$ le produit PR est négatif.

Soit D un déterminant symétrique quelconque dont les éléments sont des fonctions rationnelles et entières de λ, et désignons par $a, b, c, \ldots l$ les termes constants, c. à d. indépendants de λ, des éléments situés dans la diagonale de symétrie et rangés dans un ordre quelconque. Formons, comme je l'ai fait dans la preuve instantanée [p. 451 above], la suite

$$D,\ \delta_a D,\ \delta_b \delta_a D,\ \ldots\ \delta_l \delta_k \ldots \delta_a D$$

et soient p, q, r trois quantités consécutives prises dans la série $a, b, \ldots k, l$; cela posé, on aura

$$(\delta_r \delta_q \delta_p \ldots \delta_a)\, D \,.\, (\delta_p \ldots \delta_a)\, D = (\delta_r \delta_p \ldots \delta_a)\, D \,.\, (\delta_q \delta_p \ldots \delta_a)\, D - M^2,$$

M étant une fonction entière de λ. La loi du signe contraire de deux termes, voisins d'un terme qui s'évanouit, de la suite considérée ci-dessus ne subit donc aucun changement.

[* Vol. I. of this Reprint, p. 545.]

La méthode dont on s'est servi dans la preuve instantanée conduit par conséquent à ce résultat que l'entrelacement de D par $\delta_a D$ entre des limites quelconques p et q est égal à la valeur numérique absolue de la différence entre le nombre des permanences de signe que présente la suite considérée ci-dessus pour les deux valeurs $\lambda = p$ et $\lambda = q$.

Je dis de plus que l'entrelacement de D par $\delta_a D$ est égal à l'entrelacement de D par $\delta_b D$, a et b étant deux quelconques des quantités $a, b, \ldots l$.

En effet le premier de ces deux nombres dépend uniquement du rapport des signes de D et de $\delta_a D$ dans le voisinage des valeurs de λ pour lesquelles D s'évanouit, et pour le second de ces deux nombres on n'a qu'à substituer $\delta_b D$ au lieu de $\delta_a D$. Mais on sait que le produit $\delta_a D \,.\, \delta_b D$ excède le produit $D \,.\, (\delta_a \delta_b)\, D$ d'un carré positif. Les deux quantités $\delta_a D$ et $\delta_b D$ sont donc du même signe dans le voisinage des valeurs de λ pour lesquelles $D = 0$, ce qui prouve l'assertion qu'il s'agissait de démontrer.

En se bornant pour plus de simplicité au cas des entrelacements absolus, c. à d. au cas où $-\infty$ et $+\infty$ sont les limites de λ, on en tire le théorème suivant:

Soient $$\theta,\ \theta_1,\ \theta_2,\ \ldots\ \theta_n$$
les degrés et $$\mu,\ \mu_1,\ \mu_2,\ \ldots\ \mu_n$$
les coefficients des plus hautes puissances de λ dans
$$D,\ \delta_a D,\ (\delta_b \delta_a)\, D,\ \ldots\ (\delta_l \ldots \delta_b \delta_a)\, D;$$
cela posé, l'entrelacement de D par une quelconque des quantités $\delta_a D$, $\delta_b D, \ldots\ \delta_l D$ est la valeur numérique de la différence entre le nombre des permanences de signes que présentent les deux suites
$$\begin{array}{ccccc} \mu, & \mu_1, & \mu_2, & \ldots & \mu_n \\ (-1)^{\theta}\mu, & (-1)^{\theta_1}\mu_1, & (-1)^{\theta_2}\mu_2, & \ldots & (-1)^{\theta_n}\mu_n. \end{array}$$

Considérons le cas particulier dans lequel les éléments du déterminant D sont des fonctions linéaires de λ. Supposons de plus que tous les coefficients $\mu, \mu_1, \mu_2, \ldots \mu_n$ ont le signe positif. Dans ce cas l'entrelacement de D par l'une quelconque des déterminants dérivés $\delta_a D, \delta_b D, \ldots \delta_l D$ est égal à n, et par conséquent toutes les racines de l'équation $D=0$ sont réelles, résultat que l'on verifie aisément.

En effet, soit D_1 le déterminant formé des parties constantes des éléments de D ou, ce qui est la même chose, la valeur de D pour $\lambda = 0$, soit de plus Δ le déterminant formé des coefficients de λ des éléments de D ou, ce qui est la même chose, le coefficient de λ^n dans le développement de D; cela posé, D peut être regardé comme l'invariant de la forme quadratique à n variables $\Phi + \lambda\Omega$, Φ et Ω étant des formes quadratiques dont les invariants sont D_1 et Δ. Représentons d'une manière symbolique par
$$(\alpha x + \beta y + \gamma z + \ldots + \lambda u)^2$$

la forme quadratique Ω, et par

$$\alpha,\ \beta,\ \gamma,\ \ldots\ \lambda \ \text{⁎}\ p,\ q,\ r,\ \ldots\ t$$

le déterminant

$$\begin{vmatrix} \alpha p, & \alpha q, & \alpha r, \ldots & \alpha t \\ \beta p, & \beta q, & \beta r, \ldots & \beta t \\ \gamma p, & \gamma q, & \gamma r, \ldots & \gamma t \\ \vdots & \vdots & \vdots & \vdots \\ \lambda p, & \lambda q, & \lambda r, \ldots & \lambda t \end{vmatrix};$$

posons de plus

$$\begin{aligned} \xi &= \alpha \ \text{⁎}\ (\alpha x + \beta y + \gamma z + \ldots + \lambda u) \\ \eta &= \alpha\beta \ \text{⁎}\ (\alpha\beta y + \alpha\gamma z + \ldots + \alpha\lambda u) \\ \zeta &= \alpha\beta\gamma \ \text{⁎}\ (\alpha\beta\gamma z + \ldots + \alpha\beta\lambda u) \\ &\vdots \\ v &= \alpha\beta\gamma \ldots \lambda \ \text{⁎}\ (\alpha\beta\gamma \ldots \lambda u) \end{aligned}$$

et enfin

$$\frac{1}{A} = \alpha^2, \quad \frac{1}{B} = \alpha^2 (\alpha,\ \beta)^2,$$

$$\frac{1}{C} = (\alpha,\ \beta)^2 (\alpha,\ \beta,\ \gamma)^2, \ldots \frac{1}{L} = (\alpha,\ \beta,\ \ldots\ \kappa)^2 . (\alpha,\ \beta,\ \ldots\ \kappa,\ \lambda)^2,$$

où la notation $(\alpha,\ \beta,\ \ldots\ \lambda)^2$ désigne le déterminant

$$\alpha,\ \beta,\ \ldots\ \lambda \ \text{⁎}\ \alpha,\ \beta,\ \ldots\ \lambda\,;$$

cela posé, on aura $\qquad \Omega = A\xi^2 + B\eta^2 + \ldots + Lv^2.$

Les expressions désignées d'une manière symbolique par

$$\alpha^2,\ (\alpha,\ \beta)^2,\ (\alpha,\ \beta,\ \gamma)^2,\ \ldots\ (\alpha,\ \beta,\ \gamma,\ \ldots\ \lambda)^2$$

étant identiques aux quantités désignées antérieurement par

$$\mu_n,\ \mu_{n-1},\ \mu_{n-2},\ \ldots\ \mu,$$

il est aisé de voir que l'invariant de $\Phi + \lambda\Omega$ est, à un facteur numérique près, égal à l'invariant de $\Phi_1 + \lambda\Omega_1$, Φ_1 représentant une fonction quadratique de fonctions linéaires à coefficients réels de $x,\ y,\ z,\ \ldots\ u$ et Ω_1 une somme de carrés de ces mêmes fonctions linéaires.

En admettant les hypothèses faites ci-dessus on retombe donc sur le cas de l'équation séculaire.

Je me suis servi dans les calculs précédents de la notation ⁎ pour représenter une multiplication symbolique entre $\alpha,\ \beta,\ \gamma,\ \ldots\ \lambda$; $\alpha',\ \beta',\ \gamma',\ \ldots\ \lambda'$, tandis que je préfère garder le signe simple $\times$ pour représenter l'espèce d'opération disjonctive dont on se sert dans la théorie ordinaire de la multiplication des déterminants et qui donne naissance aux produits

$$(\alpha\alpha',\ \beta\beta',\ \gamma\gamma',\ \ldots\ \lambda\lambda').$$

52.

SUR LES DÉTERMINANTS COMPOSÉS.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 49—67.]

DANS l'article qui va suivre je m'occuperai de la théorie des déterminants composés, regardée sous un point de vue très général. Comme on sait, les déterminants composés sont des déterminants dont les éléments sont eux-mêmes des déterminants puisés dans la même matrice ou, ce qui est la même chose*, des sous-déterminants ou déterminants-mineurs d'un même déterminant primitif.

Servons-nous en premier lieu de la notation ombrale ordinaire pour représenter un déterminant simple—dans cette notation *une ligne double* c. à d. une paire de lignes de n ombres

$$\begin{matrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{matrix}$$

servira à représenter un déterminant du n^{me} ordre. On peut aussi se servir avantageusement de la notation

$$a_1 a_2 \ldots a_n \divideontimes \alpha_1 \alpha_2 \ldots \alpha_n$$

pour représenter la même chose.

Un système de n quantités a étant donné, on se sert ordinairement de la notation

$$\Sigma a_1; \ \Sigma a_1 a_2; \ \ldots$$

pour signifier $\quad a_1 + a_2 + \ldots + a_n; \ a_1 a_2 + a_1 a_3 + \ldots + a_{n-1} a_n; \ldots$

En changeant un peu cette notation, les expressions

$$\Sigma a_1, \ \Sigma a_1 a_2, \ \ldots$$

seront employées pour représenter l'ensemble des termes

$$(a_1, a_2, \ldots a_n), \ (a_1 a_2, a_1 a_3, \ldots a_{n-1} a_n), \ \ldots$$

au lieu de leur somme.

* Autant que la matrice est quadratique et non rectangulaire.

Un déterminant composé qui se rapporte à une seule ligne double

$$\begin{matrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{matrix}$$

sera représenté par la notation:

$$\Sigma a_1 a_2 \ldots a_i, \ \divideontimes \ \Sigma \alpha_1 \alpha_2 \ldots \alpha_i.$$

Soit p. e. $n = 4$, la notation

$$\Sigma a_1 a_2 a_3, \ \divideontimes \ \Sigma \alpha_1 \alpha_2 \alpha_3,$$

signifiera le déterminant composé

$$\begin{matrix}
a_1 a_2 a_3 & a_1 a_2 a_4 & a_1 a_3 a_4 & a_2 a_3 a_4 \\
\alpha_1 \alpha_2 \alpha_3 & \alpha_1 \alpha_2 \alpha_3 & \alpha_1 \alpha_2 \alpha_3 & \alpha_1 \alpha_2 \alpha_3 \\
a_1 a_2 a_3 & a_1 a_2 a_4 & a_1 a_3 a_4 & a_2 a_3 a_4 \\
\alpha_1 \alpha_2 \alpha_4 & \alpha_1 \alpha_2 \alpha_4 & \alpha_1 \alpha_2 \alpha_4 & \alpha_1 \alpha_2 \alpha_4 \\
a_1 a_2 a_3 & a_1 a_2 a_4 & a_1 a_3 a_4 & a_2 a_3 a_4 \\
\alpha_1 \alpha_3 \alpha_4 & \alpha_1 \alpha_3 \alpha_4 & \alpha_1 \alpha_3 \alpha_4 & \alpha_1 \alpha_3 \alpha_4 \\
a_1 a_2 a_3 & a_1 a_2 a_4 & a_1 a_3 a_4 & a_2 a_3 a_4 \\
\alpha_2 \alpha_3 \alpha_4 & \alpha_2 \alpha_3 \alpha_4 & \alpha_2 \alpha_3 \alpha_4 & \alpha_2 \alpha_3 \alpha_4 .
\end{matrix}$$

Ce déterminant est du 4$^{\text{me}}$ ordre par rapport aux lignes doubles qui forment ses éléments et qui sont elles-mêmes des déterminants simples du 3$^{\text{me}}$ ordre. La notation

$$\Sigma a_1 a_2, \ \divideontimes \ \Sigma \alpha_1 \alpha_2,$$

signifiera de même un déterminant du 4$^{\text{me}}$ ordre dont les éléments sont des déterminants simples du 2$^{\text{me}}$ ordre. Enfin la notation

$$\Sigma a_1, \ \divideontimes \ \Sigma \alpha_1,$$

signifiera le déterminant

$$\begin{matrix}
a_1 & a_2 & a_3 & a_4 \\
\alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\
a_1 & a_2 & a_3 & a_4 \\
\alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
a_1 & a_2 & a_3 & a_4 \\
\alpha_3 & \alpha_3 & \alpha_3 & \alpha_3 \\
a_1 & a_2 & a_3 & a_4 \\
\alpha_4 & \alpha_4 & \alpha_4 & \alpha_4 .
\end{matrix}$$

La dernière notation étant équivalente à

$$a_1, a_2, a_3, a_4 \ \divideontimes \ \alpha_1, \alpha_2, \alpha_3, \alpha_4$$

représente le même déterminant que l'on écrit plus simplement sous la forme

$$a_1 a_2 a_3 a_4 \divideontimes \alpha_1 \alpha_2 \alpha_3 \alpha_4.$$

L'identité des valeurs de

$$\Sigma a_1, \divideontimes \Sigma \alpha_1, \text{ et de } a_1 a_2 \ldots a_n \divideontimes \alpha_1 \alpha_2 \ldots \alpha_n$$

est un cas particulier (le cas extrême) d'un théorème général qui se rapporte aux déterminants composés à une seule ligne double (et que je nommerai déterminants composés à un seul argument). Dans le cas de n ombres a et d'autant d'ombres α le théorème relatif à cette classe (la plus simple qu'on puisse former) de déterminants composés s'énonce comme il suit:

$$\Sigma a_1 a_2 \ldots a_i, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_i, = \begin{pmatrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{pmatrix}^{\frac{n-1 \,.\, n-2 \ldots n-i+1}{1 \,.\, 2 \ldots i-1}}.$$

Ainsi si $i + j = n + 1$ on aura

$$\Sigma a_1 a_2 \ldots a_i, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_i, = \Sigma a_1 a_2 \ldots a_j, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_j;$$

car l'indice de la puissance de

$$\begin{pmatrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{pmatrix}$$

a la même valeur $\dfrac{\Pi(n-1)}{\Pi(i-1)\,\Pi(j-1)}$ dans les deux cas.

Un cas bien connu de ce théorème est celui où $i = n - 1$. Dans ce cas l'indice de la puissance

$$\begin{pmatrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{pmatrix}$$

devient $n - 1$; c'est le théorème qui affirme qu'en désignant par D un déterminant du n^{me} ordre et par Δ le déterminant dont les éléments sont les dérivées de D par rapport à ses éléments, on aura $\Delta = D^{n-1}$ *.

Avant de passer au théorème plus général qui se rapporte aux déterminants composés à un nombre quelconque de lignes doubles (ou disons plutôt à un nombre quelconque d'arguments) considérons d'abord un cas spécial qui est d'un grand intérêt par les applications qu'il admet, à savoir le cas dans lequel on attache à chaque ligne double dans le développement de

$$\Sigma a_1 a_2 \ldots a_i, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_i,$$

une ligne double constante

$$\begin{matrix} b_1 b_2 \ldots b_p \\ \beta_1 \beta_2 \ldots \beta_p. \end{matrix}$$

La matrice ainsi modifiée peut être désignée par la notation

$$[b_1 b_2 \ldots b_p \Sigma a_1 a_2 \ldots a_i,] \divideontimes [\beta_1 \beta_2 \ldots \beta_p \Sigma \alpha_1 \alpha_2 \ldots \alpha_i,].$$

* Le théorème pour les valeurs générales de i a été retrouvé récemment et inséré dans les *Comptes Rendus de l'Académie des Sciences de Paris* par un auteur distingué; depuis on a porté à sa connaissance que j'avais déjà publié le même théorème dans le *Philosophical Magazine* de 1851. [Vol. I. of this Reprint, p. 252.]

Pour $n = 2$ et $p = 2$ la notation

$$[b_1 b_2 \Sigma a_1,] \divideontimes [\beta_1 \beta_2 \Sigma \alpha_1,]$$

représentera p. e. le déterminant:

$$\begin{array}{cc} \begin{array}{c} b_1\, b_2\, a_1 \\ \beta_1 \beta_2 \alpha_1 \end{array} & \begin{array}{c} b_1\, b_2\, a_2 \\ \beta_1 \beta_2 \alpha_1 \end{array} \\ \\ \begin{array}{c} b_1\, b_2\, a_1 \\ \beta_1 \beta_2 \alpha_2 \end{array} & \begin{array}{c} b_1\, b_2\, a_2 \\ \beta_1 \beta_2 \alpha_2. \end{array} \end{array}$$

Cette classe de déterminants composés peut être désignée sous le nom de déterminants composés *à deux arguments dont l'un est non-distribué.*

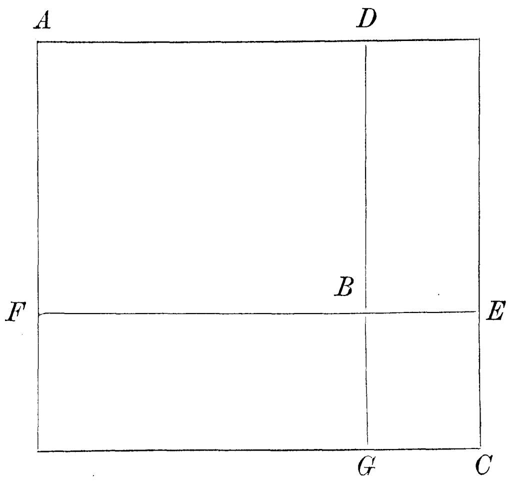
Je nommerai p et n les indices de l'étendue de B et de A respectivement et i l'indice de distribution de A; je me servirai pour le déterminant de la notation $B_p \,.\, {}^iA_n$; cela posé, je dis que l'on a*

$$B_p \,.\, {}^iA_n = B^\sigma (AB)^\tau,$$

où $\sigma = \dfrac{(n-1)(n-2)\ldots(n-i)}{1\,.\,2\ldots i}$, $\tau = \dfrac{(n-1)(n-2)\ldots(n-i+1)}{1\,.\,2\ldots(i-1)}$.

Il y a deux cas particuliers qui présentent un intérêt spécial, ce sont les cas de $i = n - 1$ et de $i = 1$.

Dans le premier cas on a $\sigma = 1$, $\tau = n - 1$, dans le second $\sigma = n - 1$, $\tau = 1$. Le premier cas de $i = n - 1$ donne le théorème auquel on est parvenu antérieurement. Le cas de $i = 1$ fournit une preuve immédiate de la règle connue pour trouver la valeur du déterminant de la matrice qui résulte de la multiplication de deux matrices carrées ou rectangulaires, comme on verra facilement à l'aide du diagramme suivant.



Imaginons que AB soit rempli de n lignes et de n colonnes, DE de n lignes et de p colonnes, FG de n colonnes et de p lignes d'éléments quelconques, enfin BC de p lignes et de p colonnes de zéros. Cela posé, et en se servant de AB, AC pour représenter les déterminants des éléments qui se trouvent dans ces carrés, le produit $(AB)^{n-1} AC$ sera égal au déterminant composé dont chaque élément est le déterminant du carré AB bordé par l'une des n colonnes de DE et par l'une des n lignes de FG.

Supposons que tous les éléments de AB s'évanouissent à l'exception des n éléments qui se trouvent dans la diagonale AB, et que ces derniers soient égaux à l'unité. Dans ce cas, si n est plus petit que p, le déterminant AC

[* As proved, pp. 249, 650 of Vol. I. of this Reprint.]

s'évanouira; de plus le carré AB bordé de la colonne $c_1c_2 \dots c_n$ et de la ligne $l_1l_2 \dots l_n$ produira le déterminant

$$(c_1, c_2, \dots c_n)(l_1, l_2, \dots l_n) \text{ c. à d. } c_1l_1 + c_2l_2 + \dots + c_nl_n:$$

de là on tire la conséquence que si l'on multiplie deux matrices rectangulaires selon la direction de leurs axes mineurs, le déterminant qui en résulte est égal à zéro. Dans le cas contraire où n est égal ou plus grand que p, AC devient la somme des produits de chaque déterminant de l'ordre p puisé dans DE multiplié par le déterminant correspondant puisé dans FG: et dans le cas particulier de $n = p$, c. à d. lorsque les matrices DE, FG deviennent des carrés, cette somme de produits se réduit au produit des déterminants DE, FG: ce qui est la règle élémentaire de multiplication des déterminants.

Voilà le point extrême jusqu'auquel j'avais précédemment avancé la théorie des déterminants composés—c. à d. jusqu'au cas dans lequel une ligne double non-distribuée s'attache comme une espèce de caput-mortuum aux lignes doubles puisées dans les combinaisons des ombres supérieures avec des ombres inférieures d'une seule ligne double: mais en y réfléchissant je me suis senti dans la nécessité morale d'étendre la théorie d'abord au cas où toutes les deux lignes doubles sont distribuées et puis au cas le plus général où l'on puise dans un nombre quelconque de lignes doubles les combinaisons des ombres supérieures et inférieures (chaque combinaison d'un ordre donné), et je vais donner l'expression générale des déterminants ainsi composés en termes des déterminants simples qui correspondent aux lignes doubles prises séparément ou combinées d'une manière quelconque entr'elles. Soient $A, B, C, \dots L, M, \dots Z$ un nombre quelconque i de lignes doubles; soient $a, b, c, \dots l, m, \dots z$ leurs indices d'étendue: ainsi par exemple A représentera une ligne double de la forme

$$\begin{matrix} \theta_1 \theta_2 \dots \theta_a \\ \phi_1 \phi_2 \dots \phi_a. \end{matrix}$$

Construisons le déterminant composé

$$^{\alpha}A^{\beta}B^{\gamma}C \dots {}^{\zeta}Z,$$

$\alpha, \beta, \gamma, \dots \zeta$ étant les indices de *distribution* de $A, B, C, \dots Z$, c. à d. que l'on a un déterminant dont les éléments sont des déterminants représentés par des lignes doubles dans chacune desquelles la ligne supérieure est formée par la juxtaposition de α des ombres supérieures de A, β des ombres supérieures de B, ... ζ des ombres supérieures de Z, et de même la ligne inférieure par la juxtaposition des lignes inférieures correspondantes: par exemple $^{\alpha}A^{\beta}B^{\gamma}C$ signifiera un déterminant composé de la forme

$$\Sigma p_1 p_2 \dots p_\alpha, \times \Sigma q_1 q_2 \dots q_\beta, \times \Sigma r_1 r_2 \dots r_\gamma$$
$$⁂ \; \Sigma p_1' p_2' \dots p_\alpha', \times \Sigma q_1' q_2' \dots q_\beta', \times \Sigma r_1' r_2' \dots r_\gamma'.$$

Or je dis que le déterminant $^{\alpha}A^{\beta}B^{\gamma}C \dots {}^{\zeta}Z$ sera* un produit des puissances de

[* See however, Borchardt: *Remarque relative au mémoire de M. Sylvester sur les déterminants composés*, *Crelle*, Bd. LXXXIX. (1880).]

A, de B, ... de Z, de AB, AC, BC, ... AZ, BZ, CZ et en général des lignes doubles en nombre $2^i - 1$ qu'on peut former par la juxtaposition de toutes les manières possibles d'un nombre quelconque des A, B, ... Z.

Pour obtenir les exposants de ces puissances voici la règle. Servons-nous en général de la notation (l, λ) pour réprésenter le nombre du binôme

$$\frac{l(l-1)\dots(l-\lambda+1)}{1.2\dots\lambda},$$

et posons
$$(a-1,\, \alpha)(b-1,\, \beta)\dots(z-1,\, \zeta) = \varpi,$$
l'exposant de la puissance de $AB \dots L$ sera

$$\frac{(a-1,\, \alpha-1)(b-1,\, \beta-1)\dots(l-1,\, \lambda-1)}{(a-1,\, \alpha)(b-1,\, \beta)\dots(l-1,\, \lambda)}\varpi.$$

Comme vérification numérique je remarquerai que l'on a

$$\frac{(a-1,\, \alpha)}{(a-1,\, \alpha-1)} = \frac{a-\alpha}{\alpha}.$$

Or en regardant une puissance P^n comme répétition n-tuple de P, on verra facilement que dans le produit de puissances donné ci-dessus le nombre de fois que les combinaisons α^{mes} des ombres de A et les combinaisons β^{mes} des ombres de B etc. se trouvent répétées, sera le produit de toutes les quantités de la forme

$$(a-1,\, \alpha-1)\left(1 + \frac{(a-1,\, \alpha)}{(a-1,\, \alpha-1)}\right) = \frac{a}{\alpha}(a-1,\, \alpha-1) = (a,\, \alpha),$$

résultat qui s'accorde bien avec la remarque que l'ordre du déterminant composé dont il est question est évidemment égal au produit $(a, \alpha)\,(b, \beta) \dots (z, \zeta)$.

Je conclurai en appliquant à un exemple le théorème énoncé ci-dessus. Considérons le déterminant composé ${}^mA_m\,{}^2B_3\,{}^1C_3$ où m, 2, 1 sont les indices de distribution et m, 3, 3 les indices d'étendue de A, B, C. On forme les trois couples de nombres binômes consécutifs *

$$\begin{array}{l} 1,\ 0 \\ 2,\ 1 \\ 1,\ 2; \end{array}$$

alors en remarquant que

$$\begin{array}{lll} 1.1.2 = 2, & 1.2.2 = 4, & \\ 2.0.2 = 0, & 1.1.1 = 1, & 1.2.1 = 2, \\ 1.0.1 = 0, & 1.2.0 = 0, & \end{array}$$

on en déduit la conséquence que

$${}^mA_m\,{}^2B_3\,{}^1C_3 = A^2\,(AB)^4\,(AC)\,(ABC)^2.$$

Soient
$$A = \frac{a_1 a_2 \dots a_m}{\alpha_1 \alpha_2 \dots \alpha_m}, \quad B = \frac{b_1\, b_2\, b_3}{\beta_1 \beta_2 \beta_3}, \quad C = \frac{c_1\, c_2\, c_3}{\gamma_1 \gamma_2 \gamma_3},$$

* On remarquera que 1, 2 sont les coefficients de t^0, t^1 dans $(1+t)^{3-1}$, 2, 1 de t^1, t^2 dans $(1+t)^{3-1}$ et 1, 0 de t^{m-1}, t^m dans $(1+t)^{m-1}$.

on aura le déterminant composé du 9^me^ ordre dont la première ligne est

$$a_1 \dots a_m b_1 b_2 c_1,\ a_1 \dots a_m b_1 b_2 c_2,\ a_1 \dots a_m b_1 b_2 c_3,\ a_1 \dots a_m b_1 b_3 c_1,\ a_1 \dots a_m b_1 b_3 c_2,$$
$$\alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,$$
$$a_1 \dots a_m b_1 b_3 c_3,\ a_1 \dots a_m b_2 b_3 c_1,\ a_1 \dots a_m b_2 b_3 c_2,\ a_1 \dots a_m b_2 b_3 c_3,$$
$$\alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,\ \alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1,$$

et dont on forme les autres lignes en remplaçant $\alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_1$ successivement par

$$\alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_2,\quad \alpha_1 \dots \alpha_m \beta_1 \beta_3 \gamma_1,\quad \alpha_1 \dots \alpha_m \beta_2 \beta_3 \gamma_1,$$
$$\alpha_1 \dots \alpha_m \beta_1 \beta_2 \gamma_3,\quad \alpha_1 \dots \alpha_m \beta_1 \beta_3 \gamma_2,\quad \alpha_1 \dots \alpha_m \beta_2 \beta_3 \gamma_2,$$
$$\alpha_1 \dots \alpha_m \beta_1 \beta_3 \gamma_3,\quad \alpha_1 \dots \alpha_m \beta_2 \beta_3 \gamma_3.$$

La valeur de ce déterminant est donnée par le produit:

$$\begin{pmatrix} a_1 a_2 \dots a_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^2 \begin{pmatrix} a_1 a_2 \dots a_m b_1\ b_2\ b_3 \\ \alpha_1 \alpha_2 \dots \alpha_m \beta_1 \beta_2 \beta_3 \end{pmatrix}^4 \begin{pmatrix} a_1 a_2 \dots a_m\ c_1\ c_2\ c_3 \\ \alpha_1 \alpha_2 \dots \alpha_m \gamma_1 \gamma_2 \gamma_3 \end{pmatrix} \begin{pmatrix} a_1 a_2 \dots a_m b_1\ b_2\ b_3\ c_1\ c_2\ c_3 \\ \alpha_1\ \alpha_2 \dots \alpha_m \beta_1 \beta_2 \beta_3 \gamma_1 \gamma_2 \gamma_3 \end{pmatrix}^2.$$

Le théorème général énoncé ci-dessus contient le résultat le plus général sur l'évaluation des déterminants composés du deuxième rang, c. à d. des déterminants dont les éléments sont des déterminants simples. On pourrait étendre ces recherches aux déterminants du 3^me^ rang, c. à d. dont les éléments sont des déterminants du 2^me^ rang, et même aux déterminants composés d'un rang quelconque.

J'introduirai une légère modification dans l'énoncé du théorème général. On peut toujours supposer qu'aux i arguments distribués désignés par A, B, ... Z soit associé un $i+1^{\text{me}}$ argument non distribué Ω, bien entendu que dans un cas donné ce dernier peut disparaître. On n'a pas besoin de supposer l'existence de plus d'un seul argument non distribué, parce que, s'il y en avait plusieurs, on pourrait toujours les réunir en un seul.

Soient a, b, ... z les indices d'étendue et α, β, ... ζ les indices de distribution des arguments A, B, ... Z, et désignons comme ci-dessus par (a, α) le nombre binôme

$$(a, \alpha) = \frac{a \,.\, a-1 \dots a-\alpha+1}{1 \,.\, 2 \dots \alpha},$$

par ϖ le produit

$$\varpi = (a-1, \alpha)(b-1, \beta) \dots (z-1, \zeta),$$

et par a', b', ... z' les quotients

$$a' = \frac{\alpha}{a-\alpha},\ b' = \frac{\beta}{b-\beta},\ \dots z' = \frac{\zeta}{z-\zeta}.$$

Cela posé, le théorème général pour l'évaluation des déterminants composés prendra la forme:

$$(\Omega^{a} A_{a}{}^{\beta} B_{b} \dots {}^{\zeta} Z_{z}) = P^{\varpi},$$

P désignant le produit des 2^i facteurs

$$
\begin{array}{l}
(\Omega) \\
(\Omega A)^{a'}(\Omega B)^{b'}\ldots(\Omega Z)^{z'} \\
(\Omega AB)^{a'b'}(\Omega AC)^{a'c'}\ldots(\Omega BC)^{b'c'}\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(\Omega AB\ldots Z)^{a'b'\ldots z'}.
\end{array}
$$

Quand Ω disparaît il faut remplacer (Ω) par l'unité.

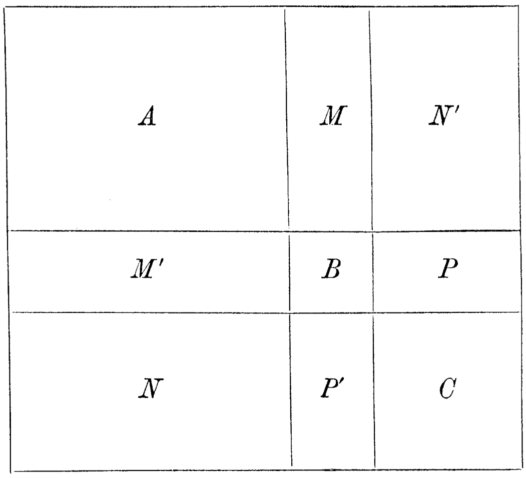
Cette formule s'écrit plus aisément sous la forme logarithmique suivante:

$$
\frac{1}{\varpi}\cdot\log(\Omega^{\alpha}A_a{}^{\beta}B_b\ldots{}^{\zeta}Z_z) = \log(\Omega) + \Sigma\frac{\alpha}{a-\alpha}\log(\Omega A)
$$
$$
+\Sigma\frac{\alpha}{a-\alpha}\cdot\frac{\beta}{b-\beta}\log(\Omega AB) + \ldots + \frac{\alpha}{a-\alpha}\cdot\frac{\beta}{b-\beta}\cdots\frac{\zeta}{z-\zeta}\log(\Omega AB\ldots Z),
$$

les sommations s'étendant à tous les expressions semblables.

Il est bon de remarquer que la valeur du déterminant simple représenté par la juxtaposition d'un nombre quelconque d'arguments A, B, … ne dépend, ni pour sa valeur, ni pour son signe algébrique, de l'ordre de la juxtaposition, car on a $(AB) = (BA)$ et de même pour un nombre quelconque d'arguments.

Pour donner une idée nette du théorème général, faisons disparaître Ω et considérons le cas de trois arguments A, B, C aux indices d'étendue a, b, c et aux indices de distribution α, β, γ. Dans la figure les carrés A, B, C correspondent aux déterminants (A), (B), (C); $AMBM'$, $AN'CN$, $BPCP'$ aux déterminants (AB), (AC) (BC); enfin le carré complet au déterminant (ABC).



Les éléments du déterminant composé ${}^{\alpha}A_a{}^{\beta}B_b{}^{\gamma}C_c$ sont des déterminants simples de l'ordre $\alpha+\beta+\gamma$, formés avec

$$\alpha^2,\ \alpha\beta,\ \alpha\gamma,\ \beta\alpha,\ \beta^2,\ \beta\gamma,\ \gamma\alpha,\ \gamma\beta,\ \gamma^2,$$

éléments puisés respectivement dans

$$A,\ M,\ N',\ M',\ B,\ P,\ N,\ P',\ C.$$

Les éléments des déterminants simples se trouvent aux intersections de

α lignes qui passent par A, M, N',
β » » » » M', B, P,
γ » » » » N, P', C,

avec $\qquad\alpha$ colonnes qui passent par $A,\ M',\ N,$

$\qquad\beta\quad$ „ „ „ „ $M,\ B,\ P',$

$\qquad\gamma\quad$ „ „ „ „ $N',\ P,\ C.$

Posons $\qquad \alpha' = a - \alpha,\quad \beta' = b - \beta,\ \ldots\ \zeta' = z - \zeta$

et nommons déterminants complémentaires du déterminant $(\Omega^{\alpha}A_a{}^{\beta}B_b\ldots{}^{\zeta}Z_z)$ tous ceux dans lesquels un nombre quelconque d'indices de distribution $\alpha, \beta, \ldots \zeta$ sont remplacées par leurs indices complémentaires $\alpha', \beta', \ldots \zeta'$. Cela posé, il existe une relation très-simple qui lie ensemble les 2^i déterminants complémentaires. En effet, en ajoutant les expressions que fournit le théorème général pour le logarithme de chacun de ces 2^i déterminants composés on trouve par un calcul facile que la somme de ces logarithmes est égale à

$$\frac{\Pi a\,.\,\Pi b\ldots\Pi z}{\Pi\alpha\,.\,\Pi\alpha'\,.\,\Pi\beta\,.\,\Pi\beta'\ldots\Pi\zeta\,.\,\Pi\zeta'}\{\log(\Omega)+\Sigma\log(\Omega A)+\Sigma\log(\Omega AB)+\ldots$$
$$+\log(\Omega AB\ldots Z)\}.$$

La quantité, qui se trouve en parenthèse, étant indépendante de $\alpha, \beta, \ldots \zeta$, on est arrivé à ce résultat remarquable : de quelque manière que l'on fasse la partition en deux nombres

$$\alpha,\ \alpha',\ \beta,\ \beta',\ \ldots\ \zeta,\ \zeta'$$

des nombres donnés $a, b, \ldots z$, la somme des logarithmes des 2^i déterminants mutuellement complémentaires sera toujours, à un facteur numérique près, la même.

En faisant disparaître dans le théorème général l'argument non-distribué Ω, on revient à la forme première dans laquelle le résultat a été énoncé, c. à d. à l'équation

$$\frac{\log({}^{\alpha}A_a{}^{\beta}B_b\ldots{}^{\zeta}Z_z)}{(a-1,\,\alpha)(b-1,\,\beta)\ldots(z-1,\,\zeta)}=\Sigma\frac{\alpha}{a-\alpha}\log(A)+\Sigma\frac{\alpha}{a-\alpha}\,\frac{\beta}{b-\beta}\log(AB)+\ldots$$
$$+\Sigma\frac{\alpha}{a-\alpha}\,\frac{\beta}{b-\beta}\cdots\frac{\zeta}{z-\zeta}\log(AB\ldots Z).$$

L'indice de distribution α pouvant prendre toutes les valeurs depuis 0 jusqu'à a, considérons le cas particulier de $\alpha = a$. En multipliant par $(a-1,\,\alpha)$, nombre qui s'évanouit pour $\alpha = a$, et en posant $\alpha = a$, de tous les termes de la seconde partie de l'équation tous ceux qui ne contiennent pas la lettre A s'évanouissent et il vient

$$\frac{\log({}^{a}A_a{}^{\beta}B_b\ldots{}^{\zeta}Z_z)}{(b-1,\,\beta)\ldots(z-1,\,\zeta)}=\log(A)+\Sigma'\frac{\beta}{b-\beta}\log(AB)+\ldots$$
$$+\Sigma'\frac{\beta}{b-\beta}\cdots\frac{\zeta}{z-\zeta}\log(AB\ldots Z)$$

où les sommations Σ' se rapportent seulement aux arguments $B\ldots Z$ à l'exclusion de A.

Mais dans le cas général dans lequel la valeur de α est quelconque la seconde partie de l'équation qui donne la valeur de $\log({}^{\alpha}A_a{}^{\beta}B_b \ldots {}^{\zeta}Z_z)$ peut être présentée sous la forme

$$\frac{\alpha}{a-\alpha}\left\{\log(A) + \Sigma' \frac{\beta}{b-\beta}\log(AB) + \ldots + \Sigma' \frac{\beta}{b-\beta} \cdots \frac{\zeta}{z-\zeta}\log(AB\ldots Z)\right\}$$

$$+ \Sigma' \frac{\beta}{b-\beta}\log(B) + \Sigma' \frac{\beta}{b-\beta}\frac{\gamma}{c-\gamma}\log(BC) + \ldots$$

$$+ \Sigma' \frac{\beta}{b-\beta}\frac{\gamma}{c-\gamma}\cdots\frac{\zeta}{z-\zeta}\log(BC\ldots Z)$$

équivalente à

$$\frac{\alpha}{a-\alpha}\cdot\frac{\log({}^{\alpha}A_a{}^{\beta}B_b \ldots {}^{\zeta}Z_z)}{(b-1,\,\beta)\ldots(z-1,\,\zeta)} + \frac{\log({}^{\beta}B_b{}^{\gamma}C_c \ldots {}^{\zeta}Z_z)}{(b-1,\,\beta)\ldots(z-1,\,\zeta)}.$$

Donc puisque $\dfrac{\alpha}{a-\alpha}(a-1,\,\alpha) = (a-1,\,\alpha-1)$, on est conduit à la relation

$$(\vartheta.)\quad \log({}^{\alpha}A_a{}^{\beta}B_b \ldots {}^{\zeta}Z_z) = (a-1,\,\alpha)\log({}^{\beta}B_b \ldots {}^{\zeta}Z_z)$$
$$+ (a-1,\,\alpha-1)\log({}^{\alpha}A_a{}^{\beta}B_b \ldots {}^{\zeta}Z_z).$$

Je me bornerai au cas particulier de cette formule dans lequel le nombre des arguments ne s'élève qu'à deux, et je passerai à une formule plus générale qui forme une extension de l'équation (ϑ.) relative au cas de deux arguments. Concevons un carré composé contenant b^2 carrés simples chacun de κ^2 éléments. Rangeant ces b^2 carrés simples en b lignes et b colonnes et choisissant β de ces b lignes et β de ces b colonnes on est conduit à un déterminant composé de l'ordre $(b,\,\beta)$ et que je désignerai par ${}^{\kappa^2.\beta}B_{\kappa^2.b}$. Les éléments de ce déterminant sont eux-mêmes des déterminants composés de l'ordre β dont chaque élément est un déterminant simple comprenant κ^2 éléments.

Ayant défini le sens de ${}^{\kappa^2.\beta}B_{\kappa^2.b}$, passons à l'interprétation de la notation ${}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}$. Concevons un carré de $a+\kappa b$ lignes et d'autant de colonnes, divisé en deux carrés A et B_κ, de a^2 et de $\kappa^2 b^2$ termes respectivement, et en deux rectangles de $a\,.\,\kappa b$ termes. Choisissons α lignes et autant de colonnes qui passent par A, choisissons β groupes de κ lignes et autant de groupes de κ colonnes qui passent par B_κ. Les déterminants formés des $(\alpha+\kappa b)^2$ termes choisis formeront les éléments d'un déterminant composé, de l'ordre $(a,\alpha)(b,\beta)$ par rapport à ses éléments composés, et que je désignerai par ${}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}$. Pour les déterminants ainsi définis on peut énoncer le théorème suivant

$$(\eta.)\quad \log({}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}) = (a-1,\,\alpha)\log({}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}) + (a-1,\,\alpha-1)\log({}^{\kappa^2.\beta}B_{\kappa^2.b}).$$

Considérons l'exemple de

$$a=2,\quad b=2,\quad \kappa=2,\quad \alpha=1,\quad \beta=1,$$

qui se rapporte aux déterminants mineurs de la matrice

$$\left|\begin{array}{cc|cccc} p & q & r & s & t & u \\ p' & q' & r' & s' & t' & u' \\ \hline p'' & q'' & l & l' & m & m' \\ p''' & q''' & l'' & l''' & m'' & m''' \\ p^{\mathrm{IV}} & q^{\mathrm{IV}} & \lambda & \lambda' & \mu & \mu' \\ p^{\mathrm{V}} & q^{\mathrm{V}} & \lambda'' & \lambda''' & \mu'' & \mu''' \end{array}\right|.$$

Dans ce cas les déterminants

$$d=({}^{\kappa^2.\beta}B_{\kappa^2.b}),\quad D=({}^{a}A_{a}{}^{\kappa^2.\beta}B_{\kappa^2.b}),\quad \Delta=({}^{\alpha}A_{a}{}^{\kappa^2.\beta}B_{\kappa^2.b})$$

ont les valeurs

$$d=\left|\begin{array}{cc|cc} l & l' & m & m' \\ l'' & l''' & m'' & m''' \\ \hline \lambda & \lambda' & \mu & \mu' \\ \lambda'' & \lambda''' & \mu'' & \mu''' \end{array}\right|,$$

$$D=\left|\begin{array}{cccc|cccc} p & q & r & s & p & q & t & u \\ p' & q' & r' & s' & p' & q' & t' & u' \\ p'' & q'' & l & l' & p'' & q'' & m & m' \\ p''' & q''' & l'' & l''' & p''' & q''' & m'' & m''' \\ \hline p & q & r & s & p & q & t & u \\ p' & q' & r' & s' & p' & q' & t' & u' \\ p^{\mathrm{IV}} & q^{\mathrm{IV}} & \lambda & \lambda' & p^{\mathrm{IV}} & q^{\mathrm{IV}} & \mu & \mu' \\ p^{\mathrm{V}} & q^{\mathrm{V}} & \lambda'' & \lambda''' & p^{\mathrm{V}} & q^{\mathrm{V}} & \mu'' & \mu''' \end{array}\right|,$$

$$\Delta = \left|\begin{array}{c|c|c|c}
\begin{matrix} p & r & s \\ p'' & l & l' \\ p''' & l'' & l''' \end{matrix} &
\begin{matrix} q & r & s \\ q'' & l & l' \\ q''' & l'' & l''' \end{matrix} &
\begin{matrix} p & t & u \\ p'' & m & m' \\ p''' & m'' & m''' \end{matrix} &
\begin{matrix} q & t & u \\ q'' & m & m' \\ q''' & m'' & m''' \end{matrix} \\ \hline
\begin{matrix} p' & r' & s' \\ p'' & l & l' \\ p''' & l'' & l''' \end{matrix} &
\begin{matrix} q' & r' & s' \\ q'' & l & l' \\ q''' & l'' & l''' \end{matrix} &
\begin{matrix} p' & t' & u' \\ p'' & m & m' \\ p''' & m'' & m''' \end{matrix} &
\begin{matrix} q' & t' & u' \\ q'' & m & m' \\ q''' & m'' & m''' \end{matrix} \\ \hline
\begin{matrix} p & r & s \\ p^{\text{IV}} & \lambda & \lambda' \\ p^{\text{V}} & \lambda'' & \lambda''' \end{matrix} &
\begin{matrix} q & r & s \\ q^{\text{IV}} & \lambda & \lambda' \\ q^{\text{V}} & \lambda'' & \lambda''' \end{matrix} &
\begin{matrix} p & t & u \\ p^{\text{IV}} & \mu & \mu' \\ p^{\text{V}} & \mu'' & \mu''' \end{matrix} &
\begin{matrix} q & t & u \\ q^{\text{IV}} & \mu & \mu' \\ q^{\text{V}} & \mu'' & \mu''' \end{matrix} \\ \hline
\begin{matrix} p' & r' & s' \\ p^{\text{IV}} & \lambda & \lambda' \\ p^{\text{V}} & \lambda'' & \lambda''' \end{matrix} &
\begin{matrix} q' & r' & s' \\ q^{\text{IV}} & \lambda & \lambda' \\ q^{\text{V}} & \lambda'' & \lambda''' \end{matrix} &
\begin{matrix} p' & t' & u' \\ p^{\text{IV}} & \mu & \mu' \\ p^{\text{V}} & \mu'' & \mu''' \end{matrix} &
\begin{matrix} q' & t' & u' \\ q^{\text{IV}} & \mu & \mu' \\ q^{\text{V}} & \mu'' & \mu''' \end{matrix}
\end{array}\right|;$$

et la relation qui existe entre eux se réduit à

$$\Delta = d\,.\,D.$$

Du théorème (η.) relatif à deux arguments dont l'un est complexe* on pourrait faire l'extension à un nombre quelconque d'arguments complexes, mais sans m'y arrêter je vais au contraire passer à un cas particulier du théorème (η.) et que je nommerai le théorème du gnomon†. Faisons coincider respectivement dans l'exemple donné ci-dessus les huit quantités

$$r,\quad s,\quad r',\quad s',\quad p'',\quad q'',\quad p''',\quad q'''$$

avec

$$t,\quad u,\quad t',\quad u',\quad p^{\text{IV}},\quad q^{\text{IV}},\quad p^{\text{V}},\quad q^{\text{V}}.$$

Dans ce cas chaque carré dont le déterminant forme un élément de d doit être enveloppé par le gnomon non distribué

$$(G.)\qquad \left\{\begin{matrix} p & q & r & s \\ p' & q' & r' & s' \\ p'' & q'' & & \\ p''' & q''' & & \end{matrix}\right.$$

pour passer des éléments de d aux éléments de D.

* Je me sers ici du mot complexe dans un autre sens que dans celui qui est généralement en usage dans l'analyse.

† γνώμων (équerre), voyez le premier livre des éléments d'Euclide.

De même pour obtenir les éléments de Δ déduisons du gnomon ($G.$) les gnomons partiels

$$\begin{array}{lll} p & r & s \\ p'' & & \\ p''' & & \\ p' & r' & s' \\ p'' & & \\ p''' & & \end{array} \qquad \begin{array}{lll} q & r & s \\ q'' & & \\ q''' & & \\ q' & r' & s' \\ q'' & & \\ q''' & & \end{array}$$

et enveloppons successivement chaque élément de d par ces quatre gnomons partiels, ce procédé que l'on peut nommer enveloppement distributif nous conduit aux éléments de Δ.

Les mêmes lois de formation existent pour un gnomon d'ordre quelconque que je désignerai par ${}^{\alpha,\kappa}G_{a,\kappa}$ et que l'on combinera avec un carré de l'ordre b de carrés de l'ordre κ.

En dénotant pour plus de brièveté par B_κ le carré de b^2 carrés de κ^2 éléments on a donc l'équation

$$\log({}^{\alpha,\kappa}G_{a,\kappa}B_\kappa) = (a-1,\,\alpha)\log(B_\kappa) + (a-1,\,\alpha-1)\log({}^{\alpha,\kappa}G_{a,\kappa}B_\kappa),$$

résultat que l'on peut énoncer de la manière suivante. Étant donné un carré de carrés simples et un gnomon ayant ses deux branches rectangulaires de longueur égale au côté des carrés simples: le déterminant composé du second rang dont les éléments sont les déterminants des carrés simples donnés enveloppés par un carré de gnomons partiels dérivés du gnomon donné sera égal au produit de puissances entières de deux déterminants composés. Les éléments de ces déterminants sont les déterminants des carrés simples donnés et les déterminants de ces carrés simples enveloppés par le gnomon donné.

Les exposants des puissances qui entrent dans la formule en question ne dépendent que de l'indice a d'étendue et de l'indice α de distribution de la partie carrée du gnomon.

Le théorème du gnomon contient comme cas particulier l'équation ($\vartheta.$). Pour s'en convaincre on n'a qu'à former le carré composé qui correspond à ${}^{\beta}B_b \ldots {}^{\zeta}Z_z$, carré qui porte la notation $(b,\,\beta)\,(c,\,\gamma)\ldots(z,\,\zeta)$ et qui comprend des carrés simples de l'ordre $\beta+\gamma+\ldots+\zeta$. Ces deux nombres devront remplacer les nombres b et κ que l'on a considérés dans l'étude du gnomon. En appliquant à ce carré composé un gnomon dont l'étendue des branches est égale à $\beta+\gamma+\ldots+\zeta$ et l'étendue de la partie carrée égale à $(b,\,\beta)\,(c,\,\gamma)\ldots(z,\,\zeta)$ on retrouve l'équation ($\vartheta.$).

En désignant par Φ le symbole

$${}^{\beta}B_b{}^{\gamma}C_c \ldots {}^{\zeta}Z_z$$

l'équation (ϑ.) prend la forme

$$\log({}^{\alpha}A_a\Phi) = (a-1,\ \alpha)\log(\Phi) + (a-1,\ \alpha-1)\log({}^{a}A_a\Phi).$$

On peut énoncer le résultat plus général qu'entre les trois expressions

$$\log{}^{\alpha_1}A_a\Phi,\quad \log{}^{\alpha_2}A_a\Phi,\quad \log{}^{\alpha_3}A_a\Phi$$

il y a toujours une relation linéaire à coefficients indépendants de b, β, c, γ, ... z, ζ. En effet soit

$$\alpha_1' = a - \alpha_1,\quad \alpha_2' = a - \alpha_2,\quad \alpha_3' = a - \alpha_3,$$

la relation dont il s'agit s'exprime par l'équation

$$\sum_1^3 \Pi\alpha_3 \,.\, \Pi\alpha_3' (\alpha_1\alpha_2' - \alpha_2\alpha_1') \log{}^{\alpha_3}A_a\Phi = 0,$$

le signe $\sum\limits_1^3$ exprimant la somme que l'on obtient en ajoutant au terme écrit les deux autres qui en dérivent par une permutation cyclique des trois indices 1, 2, 3. Cette équation ne change pas quand on échange α_1, α_2, α_3 avec α_1', α_2', α_3', de plus elle peut être présentée dans cette forme plus simple

$$\Sigma\Pi\alpha_3\Pi\alpha_3'(\alpha_1 - \alpha_2)\log{}^{\alpha_3}A_a\Phi = 0.$$

Désignant par Ψ le symbole

$${}^{\gamma}C_c \ldots {}^{\zeta}Z_z$$

on trouvera aisément la relation

$$\log{}^{\alpha}A_a{}^{\beta}B_b\Psi = (a-1,\ \alpha)(b-1,\ \beta)\log\Psi + (a-1,\ \alpha-1)(b-1,\ \beta)\log A\Psi$$
$$+ (a-1,\ \alpha)(b-1,\ \beta-1)\log B\Psi + (a-1,\ \alpha-1)(b-1,\ \beta-1)\log AB\Psi$$

dans laquelle A, B sont écrits au lieu de ${}^{a}A_a{}^{b}B_b$. On peut énoncer le résultat plus général qu'entre cinque expressions de la forme

$$\log{}^{\alpha_r}A_a{}^{\beta_r}B_b\Psi \qquad (r = 1, 2, 3, 4, 5)$$

il existe une relation linéaire. En effet, soit

$$\alpha_r + \alpha_r' = a,\quad \beta_r + \beta_r' = b,$$

la relation dont il s'agit prend la forme

$$\Sigma\Pi\alpha_5\Pi\alpha_5'\Pi\beta_5\Pi\beta_5' D_{1,2,3,4}\log{}^{\alpha_5}A_a{}^{\beta_5}B_b\Psi = 0.$$

Dans cette équation $D_{1,2,3,4}$ désigne le déterminant

$$\begin{matrix} \alpha_1\beta_1 & \alpha_1\beta_1' & \alpha_1'\beta_1 & \alpha_1'\beta_1' \\ \alpha_2\beta_2 & \alpha_2\beta_2' & \alpha_2'\beta_2 & \alpha_2'\beta_2' \\ \alpha_3\beta_3 & \alpha_3\beta_3' & \alpha_3'\beta_3 & \alpha_3'\beta_3' \\ \alpha_4\beta_4 & \alpha_4\beta_4' & \alpha_4'\beta_4 & \alpha_4'\beta_4'. \end{matrix}$$

En divisant par a^2b^2 il prend la forme plus simple

$$\begin{matrix} 1 & \alpha_1 & \beta_1 & \alpha_1\beta_1 \\ 1 & \alpha_2 & \beta_2 & \alpha_2\beta_2 \\ 1 & \alpha_3 & \beta_3 & \alpha_3\beta_3 \\ 1 & \alpha_4 & \beta_4 & \alpha_4\beta_4. \end{matrix}$$

Ce résultat peut être aisément étendu à un nombre quelconque d'arguments. En effet désignons par Θ le symbole

$$^{\alpha_{i+1}}A^{(i+1)}_{a_{i+1}}\,{}^{\alpha_{i+2}}A^{(i+2)}_{a_{i+2}}\cdots{}^{\alpha_m}A^{(m)}_{a_m}$$

et soit

$$j = 2^i;$$

cela posé, il y aura toujours entre les $j+1$ expressions

$$\log{}^{\alpha_{1,q}}A^{(1)}_{a_1}\,{}^{\alpha_{2,q}}A^{(2)}_{a_2}\cdots{}^{\alpha_{i,q}}A^{(i)}_{a_i}\,\Theta \qquad (q = 1,\ 2,\ \ldots j+1)$$

une relation linéaire exprimée par l'équation

$$\Sigma\left\{\begin{matrix}\Pi\alpha_{1,j+1}\Pi\alpha'_{1,j+1}\Pi\alpha_{2,j+1}\Pi\alpha'_{2,j+1}\cdots\Pi\alpha_{i,j+1}\Pi\alpha'_{i,j+1}\\ \times D_{1,2,\ldots j}\log{}^{\alpha_{1,j+1}}A^{(1)}_{a_1}\,{}^{\alpha_{2,j+1}}A^{(2)}_{a_2}\cdots{}^{\alpha_{i,j+1}}A^{(i)}_{a_i}\end{matrix}\right\} = 0$$

dans laquelle

$$\alpha'_{p,q} + \alpha_{p,q} = a_p$$

et où $D_{1,2,\ldots j}$ désigne le déterminant dialytique formé par les développements des expressions:

$$\begin{matrix}(1+\alpha_{1,1}x_1)(1+\alpha_{2,1}x_2)\ldots(1+\alpha_{i,1}x_i)\\ (1+\alpha_{1,2}x_1)(1+\alpha_{2,2}x_2)\ldots(1+\alpha_{i,2}x_i)\\ \vdots\\ (1+\alpha_{1,j}x_1)(1+\alpha_{2,j}x_2)\ldots(1+\alpha_{i,j}x_i)\end{matrix}$$

dans lesquelles les j produits différents des variables sont considérés comme j variables indépendantes.

Remarquons encore que lorsque $i = 2$ ou > 2 on aura même une relation entre $j = 2^i$ logarithmes, pourvu qu'entre les nombres $\alpha_{p,q}$ une condition se trouve remplie.

Pour $i = 2$, p. e. si α_1, β_1, α_2, β_2, α_3, β_3, α_4, β_4 sont les coordonnées de quatre points d'une hyperbole dans un système de coordonnées parallèles aux asymptotes, on aura l'équation

$$\begin{vmatrix} U_1 & \alpha_1 & \beta_1 & 1 \\ U_2 & \alpha_2 & \beta_2 & 1 \\ U_3 & \alpha_3 & \beta_3 & 1 \\ U_4 & \alpha_4 & \beta_4 & 1 \end{vmatrix} = 0$$

U_r désignant l'expression

$$U_r = \Pi\alpha_r \Pi(a-\alpha_r)\,\Pi\beta_r \Pi(b-\beta_r)\log {}^{\alpha_r}A_a{}^{\beta_r}B_b\Psi.$$

Des valeurs particulières des α_r, β_r qui satisfont à la condition indiquée sont p. e. les suivantes :

$$\alpha_1 = 1+\lambda, \quad \alpha_2 = p+\lambda, \quad \alpha_3 = q+\lambda, \quad \alpha_4 = pq+\lambda,$$
$$\beta_1 = pq+\mu, \quad \beta_2 = q+\mu, \quad \beta_3 = p+\mu, \quad \beta_4 = 1+\mu;$$

pour ces valeurs la relation linéaire entre U_1, U_2, U_3, U_4 se réduit à la forme

$$\begin{vmatrix} U_1 & p & q & 1 \\ U_2 & q & p & 1 \\ U_3 & 1 & pq & 1 \\ U_4 & qp & 1 & 1 \end{vmatrix} = 0$$

que l'on obtient par la remarque que les coefficients des U ne changent point lorsqu'on fait coincider avec le centre de l'hyperbole l'origine des co-ordonnées. De plus en divisant par le facteur commun $(1-p)(1-q)$ l'équation devient

$$(pq-1)(U_2-U_1) = (q-p)(U_4-U_3),$$

ce qui fait voir que deux déterminants $({}^{\alpha_r}A_a{}^{\beta_r}B_b\Psi)$ élevés chacun à une puissance entière et multipliés ensemble donnent un produit égal à une expression semblable relative aux deux autres déterminants de la même forme.

Le cas de $i=1$ fait exception. Dans ce cas une relation linéaire entre moins de trois expressions différentes est impossible.

Sans sortir de la sphère des déterminants composés du 2^{me} rang il reste à faire une grande généralisation de la théorie précédente.

Jusqu'ici on n'a considéré que des matrices à forme carrée ou, ce qui est la même chose, on ne s'est servi que d'arguments à un seul indice d'étendue et à un seul indice de distribution.—Mais il y a une théorie à construire relative aux arguments ayant chacun deux indices distincts et d'étendue et de distribution. Pour le moment je me borne au cas comparativement simple dans lequel il n'y a qu'un seul indice de distribution tandis que chaque argument a deux indices distincts d'étendue et se rapporte par conséquent à une matrice de forme rectangulaire. Comme il n'y a qu'un seul indice de distribution, les matrices que l'on forme au moyen des matrices rectangulaires données seront des matrices carrées représentant des déterminants comme dans le cas traité jusqu'à présent.

Il sera utile de se servir de l'expression 'déterminant virtuel' ou 'valeur virtuelle d'une matrice rectangulaire.' Cette dénomination ne définit point une quantité que l'on peut directement mettre en évidence, mais plutôt une

valeur de nature ombrale ou idéale : cependant, comme je vais faire voir, on pourra établir des rapports actuels entre ces valeurs idéales.

Soient $\mathfrak{D}$ et D deux matrices rectangulaires qui contiennent en dernier lieu les mêmes éléments simples (réels ou ombrals). Accentuons les éléments primitifs et désignons par $\mathfrak{D}'$, D' ce que deviennent alors $\mathfrak{D}$, D.

Multiplions ensemble $\mathfrak{D}$ et $\mathfrak{D}'$, D et D', suivant la règle ordinaire pour la multiplication des matrices, appliquée dans la direction de leur plus grande étendue, et comparons les valeurs des déterminants $(\mathfrak{D}\,.\,\mathfrak{D}')$, $(D\,.\,D')$.

Supposons que ces déterminants remplissent identiquement l'équation

$$(\mathfrak{D}\,.\,\mathfrak{D}')^i = (D\,.\,D')^j,$$

comprenant comme cas particulier l'équation

$$(\mathfrak{D}^2)^i = (D^2)^j,$$

dans ce cas j'écrirai l'équation idéale

$$\mathfrak{D}^i = D^j.$$

Avec cette notion des valeurs virtuelles on peut donner avec un trait de plume une grande extension au théorème général établi dans le mémoire précédent.

En effet, considérons Ω, $A, B, C, \ldots Z$ comme des matrices non plus carrées mais rectangulaires avec cette convention que A_a représente une matrice aux indices d'étendue a et a' et *que a' ne soit pas moindre que a.* Alors ${}^{\alpha}A_a$ représentera une matrice dont les deux indices d'étendue sont (a, α), (a', α) respectivement.

En définissant de cette manière le sens des notations Ω, ${}^{\alpha}A_a$, ${}^{\beta}B_b$, ... *je dis que la valeur fournie par le théorème général pour*

$$\frac{\log \Omega\, {}^{\alpha}A_a {}^{\beta}B_b \ldots {}^{\zeta}Z_z}{\varpi}$$

ne subit point de changement, quand on remplace les déterminants réels par les déterminants virtuels dans le cas où les matrices carrées se changent en matrices rectangulaires et que l'on n'a pas besoin de tenir compte de l'excès de a' sur a, de b' sur b, etc.

Pour donner un exemple bien simple des déterminants virtuels j'énoncerai l'extension que l'on peut donner à l'équation élémentaire qui dit que le déterminant actuel

$$\mathfrak{C} = \begin{vmatrix} ab' - a'b & ac' - a'c & bc' - b'c \\ ab'' - a''b & ac'' - a''c & bc'' - b''c \\ a'b'' - a''b' & a'c'' - a''c' & b'c'' - b''c' \end{vmatrix}$$

est égal au carré du déterminant actuel

$$C = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

Cette extension consiste à dire que le déterminant virtuel de

$$\mathfrak{D} = \begin{vmatrix} ab'-a'b & ac'-a'c & ad'-a'd & bc'-b'c & bd'-b'd & cd'-c'd \\ ab''-a''b & ac''-a''c & ad''-a''d & bc''-b''c & bd''-b''d & cd''-c''d \\ a'b''-a''b' & a'c''-a''c' & a'd''-a''d' & b'c''-b''c' & b'd''-b''d' & c'd''-c''d' \end{vmatrix}$$

est égal au carré du déterminant virtuel de

$$D = \begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{vmatrix}.$$

En effet désignons par $\mathfrak{D}_1$, D_1 ce que deviennent les matrices $\mathfrak{D}$, D lorsqu'on y remplace les a, b, c, d par des $\alpha, \beta, \gamma, \delta$; alors des déterminants actuels $(\mathfrak{D} . \mathfrak{D}_1)$ et $(D . D_1)$ sont liés par l'équation

$$(\mathfrak{D} . \mathfrak{D}_1) = (D . D_1)^2,$$

d'où l'on est en droit de tirer la conséquence énoncée ci-dessus relativement aux déterminants virtuels.

Comme second exemple je considère la ligne-couple

$$\begin{bmatrix} a_1 a_2 \dots a_m b_1 b_2 \dots b_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{bmatrix}$$

qui représente une matrice de longueur double de sa largeur. Formons le déterminant composé

$$[(\Sigma a_1 a_2 \dots a_i,) \times (\Sigma b_1 b_2 \dots b_j,)] \divideontimes (\alpha_1 \alpha_2 \dots \alpha_m)$$

où $$i + j = m.$$

D'après un théorème bien connu ce déterminant est égal à

$$\begin{pmatrix} a_1 a_2 \dots a_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1, i)} . \begin{pmatrix} b_1 b_2 \dots b_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1, j)}.$$

Or considérons la ligne-couple

$$\begin{bmatrix} a_1 a_2 \dots a_\nu b_1 b_2 \dots b_\pi \\ \alpha_1 \alpha_2 \dots \alpha_m \end{bmatrix}$$

et supposons que m ne surpasse ni ν ni π, soit de plus comme auparavant

$$i + j = m,$$

cela posé, je dis qu'on aura toujours l'équation

$$[(\Sigma a_1 a_2 \dots a_i,) \times (\Sigma b_1 b_2 \dots b_j,)] \divideontimes (\alpha_1 \alpha_2 \dots \alpha_m)$$
$$= \begin{pmatrix} a_1 a_2 \dots a_\nu \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1, i)} . \begin{pmatrix} b_1 b_2 \dots b_\pi \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1, j)}.$$

Dans cette équation et relativement à chacune des trois matrices qu'elle contient il faut substituer lorsqu'il est nécessaire la valeur virtuelle du déterminant au déterminant même, lorsque la matrice est rectangulaire au lieu d'être carrée.

53.

ON THE TRIANGLES IN- AND EX-SCRIBABLE TO A GENERAL CUBIC CURVE.

[*Johns Hopkins University Circulars,* I. (1880), p. 49.]

THE general cubic being thrown into the form $xy^2 + yz^2 + zx^2 + mxyz = 0$, the lines $x = 0$, $y = 0$, $z = 0$ will constitute an in- and ex-scribable triangle to the curve. The number of such was stated to be 24; consisting of 12 pairs of conjugates, each pair being in triple-perspective position with respect to each other, and the centres of the perspective projection being three collinear points of inflexion. Accordingly, the twenty-four triangles will consist of four groups of three pairs of conjugate in- and ex-scripts. The law for the number of polygonal in- and ex-scripts, with any assigned number of sides, will be found stated in No. 4, Vol. II. of the *American Journal of Mathematics* [above, p. 341].

54.

ON THE RESULTANT OF TWO CONGRUENCES.

[*Johns Hopkins University Circulars*, I. (1881), p. 131.]

LET an integer function of a variable be understood to mean an integral rational function thereof whose coefficients are all of them positive or negative integers.

Suppose p to be a fixed prime number; any integer function which is contained in $Fx + p\psi x$, where ψ is an arbitrary integer form, may be termed a modular factor of Fx and all modular factors which are equivalent (quâ the fixed modulus) may be regarded as identical.

An integer function containing no modular factor (except itself) may be regarded as modularly irreducible, and as a very advantageous *façon de parler* may be affirmed to have as many modular roots as there are units in its degree. If linear, there is one modular root which is *actual*, in other cases the modular roots may be termed *hypothetic*, (words which seem preferable to *real* and *imaginary* for the purpose in view). The theorem of Galois, that the number of modular roots of any integer function is the same as the number of units in its degree, is then tantamount to the affirmation that just as an integer number is capable of being resolved in only one way into a product of prime integer factors, so an integer function can be resolved in only one way into a product of modularly irreducible factors.

If one integer root of an irreducible integer function is also a root of a second function, it is well known that all the roots of the first are roots of the second: from that it follows that, *If the resultant of two integer functions vanishes, they must have an irreducible factor in common.* This is analogous to, or, rather is, so to say, an exaltation of, the fact that if the resultant of two real functions of a variable vanishes, they must have a real factor, linear or quadratic, in common*; indissoluble association of pairs of imaginary roots in the world of real quantity being the analogue of indissoluble association of groups of hypothetic roots in the world of integer numbers. In what immediately precedes, the factors spoken of are ordinary algebraical factors. If now we pass from ordinary to modular factors or roots, the theorem above stated, on the introduction of the word 'modular', becomes the theorem referred to by Professor Smith, in the *British Association Report*, 1860, p. 162, and by Mr Hathaway at the last meeting, which may be thus expressed: "*If the resultant of two integer functions is modularly zero* (that is, contains the modulus), *they must have a modular factor in common.*"

* So as a particular exemplification, if one of two integral rational functions with only real coefficients has no real root and their resultant vanishes, they must have two roots in common.

55.

ON THE PREROGATIVE OF A TERNARY DENOMINATIONAL SYSTEM OF COINAGE.

[*Johns Hopkins University Circulars*, I. (1881), p. 132.]

PROFESSOR SYLVESTER drew attention to the fact that a system of coinage in which each coin is three times the value of the one below it would possess a superiority *above every other* in so far as it would admit of all payments up to any assigned limit, being effected with the smallest possible number of pieces, this advantage increasing with the size of the limit. Thus suppose the limit of 10 dollars to be selected, two persons each possessed of 7 coins of the respective value of 1, 3, 9, 27, 81, 243, 729 cents could pay each other by interchange of their coins any sum from 1 cent up to this limit. The full amount so capable of being paid being of course, $\frac{3^7-1}{2}$ cents, that is, \$10.93. Whereas with 7 coins doubling at each step, the extreme limit would be \$1.27.

Again if each coin were quadruple the value of its antecedent, the extreme limit attainable with 8 coins would be only $2(1+4+16+64)$, or \$1.70. Or, if each coin were five times the value of its antecedent, the sum of the geometrical progression $1+5+25+125+625$ being 781, 10 coins at least would be required to be possessed by each of two persons to enable one of them to pay the other any amount from 1 cent up to \$7.81, whereas as previously shown, 7 would be more than sufficient to allow of this being done on the ternary scale.

Thus the absolutely perfect system of coinage, so far as this depends on the smallness of the number of coins necessary to be used, is that which proceeds in a geometrical progression according to the ternary scale.

The following problem in arithmetic is suggested by the preceding considerations.

What is the condition that the sums and differences of the integers a_1, a_2, a_3, ... a_n, not subject to any defined law of progression, may comprise between them all the numbers from 1 up to $a_1 + a_2 + a_3 \ldots + a_n$?

56.

ON THE MULTISECTION OF THE ROOTS OF UNITY.

[*Johns Hopkins University Circulars*, I. (1881), pp. 150, 151.]

IF p be a prime number, e a divisor of $p-1$, and the e periods into which the primitive pth roots of unity may be distributed are the roots of $\eta^e + B\eta^{e-1} + \ldots$, I call this last written function (say E), the e-period function to p. Every divisor of such function, it is well known, if not p itself or an eth power residue of p, must be a divisor of the discriminant of E.

Every divisor q of the discriminant is necessarily a divisor of E but may or may not be, according to circumstances, an eth power residue to p; if it is not, then q may be called an exceptional divisor of the period-function.

When $e=2$ the discriminant is p itself so that (as is well known) there are no exceptional factors to the two-period function. When $e=3$, it may be shown that every factor of the discriminant is necessarily a cubic residue of p.

This may be proved by the Law of Reciprocity for cubic residues, although obtained in quite a different manner. It follows that the three-period function has no exceptional divisor.

When $e=4$ it is better to distinguish between the two cases of $p=8i+1$ and $p=8i+5$.

In the former case 2 is not necessarily a biquadratic but may be only a quadratic residue of p, although a divisor of the 4-period function, and consequently 2 may be an exceptional divisor. When $p=8i+5$, if $p=f^2+4\gamma^2$, every divisor of γ is necessarily a divisor of the function inasmuch as γ is contained in the discriminant, but whilst divisors of γ of the form $4i+1$ are biquadratic, those of the form $4i-1$ will be only quadratic and not biquadratic residues of p. The results for $e=4$ so far as yet stated may be proved by the law of reciprocity for biquadratic residues.

But when $p = 8i + 5$, or in other words, when $p = f^2 + 4\gamma^2$ where γ is odd, it may be shown that $\frac{3p^2 + f^2}{16}$ is also that factor of the discriminant which is represented by $(\eta_0 - \eta_1)(\eta_1 - \eta_2)(\eta_2 - \eta_3)(\eta_3 - \eta_0)$, ($\eta_0$, η_1, η_2, η_3 being the four periods taken in natural order), and it is capable of proof that every divisor of this chain of products cannot but be a biquadratic residue to p*, or in other words, every divisor of $\frac{f^2 + 4\gamma^2}{4}$ is a biquadratic residue of $f^2 + 4\gamma^2$ when this last quantity is a prime number. This theorem, deduced from the method applied to the divisors of period-functions, does not appear to be referable to any known theorem concerning biquadratic residues. Professor Sylvester finally stated that he had under consideration the question of the existence or otherwise of exceptional factors to the e-period function in the general case of e being a prime number.

* For suppose q, a prime-number divisor of the "chain-product," to be not a biquadratic residue of p; then if q is a quadratic residue of p, it may be shown that q must be also a divisor of $(\eta_0 - \eta_2)^2 (\eta_1 - \eta_3)^2$ and therefore of $p\gamma^2$, which is impossible because γ is prime to f and p, and if q is a non-quadratic residue of p, it may be shown that all four roots of the congruence, which expresses that the 4-period function contains q, must be equal to one another, which admits of easy disproof. Hence q cannot but be a biquadratic residue of p.

57.

SUR LES DIVISEURS DES FONCTIONS DES PÉRIODES DES RACINES PRIMITIVES DE L'UNITÉ.

[*Comptes Rendus,* XCII. (1881), pp. 1084—1086.]

SOIT p un nombre premier égal à $ef+1$; la fonction du $e^{\text{ième}}$ degré, dont les racines sont les e périodes entre lesquelles on peut distribuer les ef $p^{\text{ièmes}}$ racines primitives de l'unité, est ce que je désigne comme la fonction à e périodes par rapport à p.

On connaît bien que p et un $e^{\text{ième}}$ résidu quelconque par rapport à p sont toujours diviseurs de cette fonction. Tout autre diviseur se nomme *diviseur exceptionnel* de la fonction. On sait que tout diviseur exceptionnel d'une fonction de périodes doit être contenu comme facteur dans le discriminant de cette fonction et, de plus, que pour les cas où $f=1$, ou $f=2$, ou $e=2$, il n'y a pas de facteurs exceptionnels. Si l'on en connaît davantage au sujet de ces facteurs exceptionnels, je n'en suis pas instruit. On ne trouve rien de plus dans le Livre classique de Bachmann (*Kreistheilung,* 1872)*.

Or je trouve facilement, pour le cas de $e=3$, qu'il n'y a pas de facteurs exceptionnels, de sorte que tout diviseur premier de la fonction bien connue

$$\eta^3+\eta^2-\frac{p-1}{3}\eta+\ldots$$

est nécessairement ou p ou un résidu cubique de p. Pour $e=4$, la même chose n'a pas lieu.

* Dans cet excellent ouvrage, M. Bachmann démontre que, si Ω est la fonction à e périodes par rapport à p et q nombre premier qui est une $e^{\text{ième}}$ puissance résidu de p, la congruence $\Omega\equiv 0$ (mod. q) aura e racines réelles, mais, chose extraordinaire, omet de démontrer ou même de dire que la même chose a lieu pour la congruence $\Omega\equiv 0$ (mod. q^i), i étant un nombre entier positif quelconque. En effet, cette propriété de q (que *toutes* ses puissances sont diviseurs) est le *caractère distinctif* de la classe *principale* de diviseurs, non pas seulement pour les fonctions des périodes de racines d'unité par rapport à un nombre premier, mais aussi pour les fonctions cyclotomiques en général. Dans le cas que nous considérons, ni p ni aucun *diviseur exceptionnel* ne possède cette propriété.

Quand $p = f^2 + g^2$, où f est impair, si g est divisible par 4, mais non pas par 8, le nombre 2 divisera la fonction des quatre périodes, mais ne sera pas (comme on sait bien) un résidu biquadratique, mais seulement un résidu quadratique de p; de plus, si g n'est pas divisible par 4, tout nombre premier contenu dans $\frac{g}{2}$ sera un diviseur de la fonction des périodes, et, si ce nombre premier est de la forme $4i + 3$, il sera seulement un résidu quadratique et non biquadratique de p. Pour $e = 4$, il n'y a pas d'autres diviseurs exceptionnels au delà de ceux que j'ai donnés ci-dessus. En établissant ce fait, j'ai été amené à cette proposition curieuse, qu'il serait difficile (il me semble) d'établir par un autre genre de considérations, mais qui est indubitablement vraie, c'est-à-dire :

Si $p = f^2 + (2g)^2$ (p étant un nombre premier et g impair), *tout nombre contenu dans le nombre impair* $\frac{f^2 + 3g^2}{4}$ *est un résidu biquadratique de* p.

Mais je passe à un théorème général, qui me paraît très intéressant et que voici :

1°. *Si* e (le nombre des périodes) *est un nombre premier de la forme* $2^{2^x} + 1$, *le nombre* 2 *ne peut pas être un diviseur exceptionnel de la fonction des* e *périodes.*

2°. *Si* e *est un nombre premier, un facteur exceptionnel* K (si un tel cas peut exister) *doit entrer à la seconde puissance au moins comme facteur dans* $e - 1$, *de sorte qu'on sait que, pour* $e = 2, 3, 5, 7, 11, 17$, *il n'existe pas de diviseur de la fonction des* e *périodes en dehors de* p *et des résidus* $e^{\text{ièmes}}$ *de* p.

Quand $e = 19$, puisque $19 - 1$ contient 3^2, le théorème n'exclut pas la possibilité que 3 soit un diviseur de la fonction à dix-neuf périodes sans être une dix-neuvième puissance résidu de p. De même, quand $e = 13$, le théorème ne dit rien sur le caractère du diviseur 2, dont le carré 4 est contenu dans 13. Cependant, je n'ai pas la moindre raison pour conclure que les diviseurs exceptés sont vraiment des facteurs exceptionnels.

On doit regarder le cas où, e étant un nombre premier, $e - 1$ contient K^2, non pas comme un cas exceptionnel, mais comme un cas réservé pour un examen ultérieur.

58.

SUR LES COVARIANTS IRRÉDUCTIBLES DU QUANTIC BINAIRE DU HUITIÈME ORDRE.

[*Comptes Rendus*, XCIII. (1881), pp. 192—196, 365—369.]

M. VON GALL a récemment calculé les dérivées invariantives irréductibles qui appartiennent à la forme binaire du huitième ordre (voir *Mathem. Annalen*, 1880, 1881), et s'est mis en parfait accord avec les résultats que j'avais déjà obtenus, sinon qu'il a trouvé un covariant du *degré-ordre* 10 . 4 qu'il affirme ne pas avoir réussi à décomposer. Je vais donc démontrer que nul covariant irréductible de ce degré-ordre ne peut exister.

Selon M. von Gall et moi-même, on a un seul invariant irréductible de chacun des degrés 2, 3, 4, 5, 6, 7, 8; il y a aussi un seul invariant des degrés 9, 10 respectivement, dont je n'aurai pas besoin de parler. On a aussi un seul covariant irréductible du degré-ordre 2 . 4 et du degré-ordre 3 . 4 et deux des degrés-ordres 4 . 4, 5 . 4, 6 . 4, 7 . 4, 8 . 4 respectivement. Il y a aussi un covariant du degré-ordre 5 . 2, mais nul covariant quadratique d'un degré inférieur à 5 et, comme on le sait bien *à priori*, nul covariant de l'ordre impair 1 ou 3.

En combinant ensemble ces covariants et invariants, on peut obtenir* trente-deux covariants composés, chacun du degré-ordre 10 . 4, car

2 peut être formé avec	......	2,
3	"	3,
4	"	4 et 2, 2,
5	"	5 et 3, 2,
6	"	6 ; 4, 2 ; 3, 3 ; 2, 2, 2,
7	"	7 ; 5, 2 ; 4, 3 ; 3, 2, 2,
8	"	8 ; 6, 2 ; 5, 3 ; 4, 4 ; 4, 2, 2 ; 3, 3, 2 ; 2, 2, 2, 2.

[* See p. 485, below. Also p. 509.]

Donc les covariants irréductibles des ordres 8, 7, 6, 5, 4 donnent naissance à $2(1+1+2+2+4)$, c'est-à-dire vingt covariants du degré-ordre 10.4, et les covariants irréductibles des ordres 2 et 3 à $4+7$, c'est-à-dire dix covariants de ce même degré-ordre, et il y en a aussi un de plus qui s'obtient en prenant le carré du covariant irréductible du degré-ordre 5.2. Le nombre total est donc $20+11+1=32$.

Je me propose d'établir catégoriquement que ces formes sont linéairement indépendantes entre elles. En suivant la notation de M. von Gall, on aura

		Degré-ordre
f		1.8
i	$=(f, f)_4$	2.8
k	$=(f, f)_6$	2.4
Δ	$=(k, k)_2$	4.4
A	$=(f, f)_8$	2.0
B	$=(f, i)_8$	3.0
C	$=(k, k)_4$	4.0
f_4	$=(f, k)_4$	3.4
i_4	$=(i, k)_4$	4.4
$f_{k,2}$	$=(f_4, k)_2$	5.4
$f_{k,3}$	$=(f_4, k)_3$	5.2
$f_{k,k}$	$=(f, k^2)_8$	5.0
f_Δ	$=(f, \Delta)_4$	5.4
$i_{k,2}$	$=(i_4, k)_2$	6.4
$i_{k,k}$	$=(i_4, k)_4$	6.0
i_Δ	$=(i, \Delta)_4$	6.4
$f_{k,\Delta}$	$=(f_4, \Delta)_4$	7.0
$i_{k,\Delta}$	$=(i_4, \Delta)_4$	8.0

Dans cette table, f représente la forme primitive $*(x, y)^8$ et, en général, $(\phi, \psi)_\mu$ signifie l'invariant linéo-linéaire par rapport à u, v des deux formes

$$\left(u\frac{d}{dx}+v\frac{d}{dy}\right)^\mu \phi, \quad \left(u\frac{d}{dx}+v\frac{d}{dy}\right)^\mu \psi.$$

Je ne donne pas la genèse du covariant du degré-ordre 7.4 ni de celui du degré-ordre 8.4, car, par la méthode dont je vais me servir, on n'aura pas occasion d'introduire explicitement ces deux formes dans le calcul.

Je commence en attribuant à f la forme spéciale

$$(0,\ r,\ s,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\between x,\ y)^8,$$

c'est-à-dire

$$8rx^7y + 28sx^6y^2 + y^8.$$

Cette supposition réduira à zéro, comme on va le voir, les trois invariants des degrés 2, 3, 4 respectivement.

En suivant les indications de la table donnée et en négligeant des multiplicateurs numériques, on trouvera facilement

$$\begin{aligned}
A &= (f, f)_8 = 0,\\
i &= (f, f)_4 = 3r^2x^8 + 4qx^3y^5 + br^2x^2y^6,\\
B &= (i, f)_8 = 0,\\
k &= (f, f)_6 = 2qxy^3 + ry^4,\\
i_4 &= (i, k)_4 = 84r^3x^4 - q^2y^4,\\
C &= (k, k)_4 = 0,\\
\Delta &= (k, k)_2 = q^2y^4,\\
i_{k,k} &= (i_4, k)_4 = r^4,\\
f_{k,k} &= (f, k^2)_8 = q^2r,\\
i_{k,\Delta} &= (i_4, \Delta)_4 = q^2r^3,\\
f_4 &= (f, k)_4 = q^2x^4 - 4qrx^3y,\\
f_{k,2} &= (f_4, k)_2 = 2q^3x^3y - q^2rx^2y^2 - 4qr^2xy^3,\\
f_{k,\Delta} &= (f_4, \Delta)_4 = q^4,\\
f_\Delta &= (f, \Delta)_4 = 2q^3x^3y + 3q^2rx^2y^2,\\
f_{k,3} &= (f_4, k)_3 = q^4xy + q^3ry^2.
\end{aligned}$$

Or, soit Ω la fonction (si une telle existe) du degré-ordre $10 . 4$, composée avec les produits des invariants et covariants irréductibles de $*(x, y)^8$, qui est identiquement égale à zéro.

Les seuls termes dans Ω qui continueront à subsister pour la forme spéciale attribuée à f seront (en addition au carré du covariant du degré-ordre $5 . 2$) des multiples numériques des produits des invariants irréductibles des degrés 5, 6, 7, 8 multipliés respectivement par chacun des deux covariants du degré 5, par chacun des deux covariants du degré 4, par le covariant du degré 3 et par le covariant du degré 2.

On obtiendra ainsi les sept expressions suivantes:

$$q^8x^2y^2 + 2q^7rxy^3 + q^6r^2y^4, \tag{1}$$

$$2q^5rx^3y - q^4r^2x^2y^2 - 4q^3r^3xy^3, \tag{2}$$

$$2q^5rx^3y + 3q^4r^2x^2y^2, \tag{3}$$

$$84r^7x^4 - q^2r^4y^4, \tag{4}$$

$$q^2r^4y^4, \tag{5}$$

$$q^6x^4 - 4q^5rx^3y, \tag{6}$$

$$2q^3r^3xy^3 + q^2r^4y^4. \tag{7}$$

Supposons que les multiplicateurs numériques, dans Ω, de ces fonctions soient $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7$ respectivement. Alors, en se souvenant que Ω est identiquement zéro, on voit immédiatement que $\mu_1 = 0$, et à cause du

terme seul r^7x^4 dans (4), que $\mu_4 = 0$, et, à cause du terme seul q^6x^4 dans (6), que $\mu_6 = 0$, et conséquemment, à cause des termes q^5rx^3y et $q^4r^2x^2y^2$ dans (2) et (3), que $\mu_2 = 0$, $\mu_3 = 0$. Il ne reste donc que μ_5 et μ_7 à considérer, lesquels évidemment, à cause du terme $q^3r^3xy^3$ dans (7), seront tous les deux zéro.

Ainsi on voit que l'expression Ω ne peut contenir que des multiples des invariants irréductibles A, B et C.

Pour démontrer que Ω ne contient pas de termes qui ne contiennent ni A ni B, considérons la forme spéciale nouvelle

$$f = x^8 + 8\lambda x^7 y + 8\mu x y^7 + y^8,$$

où je suppose que $\lambda\mu$ est égal à $\frac{1}{8}$.

Alors

$$\begin{aligned}
A &= (f, f)_8 = 1 - 8\lambda\mu = 0,\\
i &= (f, f)_4 = 4\mu x^5 y^3 + x^4 y^4 + 4\lambda x^3 y^5,\\
B &= (i, f)_8 = 0,\\
K &= (f, f)_6 = 16\mu x^3 y + 6x^2y^2 + 16\lambda x y^3,\\
C &= (k, k)_4 = -64\mu\lambda + 3 = -5,\\
\Delta &= (k, k)_2 = 96\mu^2 x^4 + 48\mu x^3 y - 6x^2y^2 + 48\lambda x y^3 + 96\lambda^2 y^4,\\
i_4 &= (i, k)_4 = 20\mu^2 x^4 - 4\mu x^3 y + 24x^2y^2 - 4\lambda x y^3 + 20\lambda^2 y^4,\\
i_\Delta &= (i, \Delta)_4 = (48\mu^2 - 30\mu)\, x^4 + \ldots + (48\lambda^2 - 30\lambda)\, y^4,\\
i_{k,2} &= (i_4, k)^2 = 28\mu^2 x^4 + \ldots + 28\lambda^2 y^4,
\end{aligned}$$

et les seuls termes qui peuvent subsister dans Ω (vu qu'on a démontré que $\mu_1, \mu_2, \ldots, \mu_7$ sont égaux chacun à zéro) seront des multiples numériques de C^2k, $Ci_{k,2}$, Ci_Δ.

Mais le terme μx^4 paraît seulement dans i_Δ, et $\mu^2 x^4$ ne paraît pas dans k; conséquemment les trois termes doivent disparaître spontanément.

Il s'ensuit que Ω peut être mis sous la forme $AV + BU$, où U et V sont des covariants du degré 7 et 8 respectivement, et, puisque $AV + BU$ est identiquement zéro, il faut que $\dfrac{V}{B} + \dfrac{U}{A} = 0$, où $\dfrac{V}{B}$ et $\dfrac{U}{A}$ sont tous les deux covariants entiers, c'est-à-dire qu'on aura une équation de l'ordre 5 entre les covariants irréductibles, ce qui impliquerait un rapport numériquement linéaire entre les valeurs générales de Af_4 et Bk, ce qui est évidemment absurde. Donc l'expression supposée Ω ne peut pas exister, et les trente-deux covariants composés du degré-ordre 10.4, dont j'ai parlé, seront linéairement indépendants entre eux, pourvu, du moins, que nul rapport linéaire ne lie ensemble les invariants dont nul n'est d'un degré aussi élevé que 8; évidemment, le seul rapport possible de cette nature serait de la forme $C^2 =$ une fonction de A et B, mais on a vu que A et B peuvent disparaître sans que C disparaisse. Donc un tel rapport ne peut pas avoir

lieu, et les trente-deux covariants dont il est question sont linéairement indépendants.

Or, par le théorème célèbre de M. Cayley (dont l'exactitude a été établie catégoriquement* dans le *Journal de Borchardt* et dans le *Philosophical Magazine*), on sait que le nombre total des covariants du degré-ordre 10.4 linéairement indépendants les uns des autres, appartenant au quantic de l'ordre 8, est représenté par $\left(\frac{ij-\epsilon}{2}\,;\ i, j\right)-\left(\frac{ij-\epsilon}{2}-1\,;\ i, j\right)$, où $i=8$, $j=10$, $\epsilon=4$, et où, en général, $(w : i, j)$ signifie le nombre de représentation de w comme somme de j ou moins de j, chiffres dont nul n'excède 8 en grandeur, c'est-à-dire, selon un théorème bien connu d'Euler, sera le coefficient de $t^{\frac{8.10-4}{2}}$, c'est-à-dire de t^{38}, dans le développement en série de puissances ascendantes de t de la fraction

$$\frac{(1-t^{11})(1-t^{12})(1-t^{13})(1-t^{14})(1-t^{15})(1-t^{16})(1-t^{17})(1-t^{18})}{(1-t^{2})(1-t^{3})(1-t^{4})(1-t^{5})(1-t^{6})(1-t^{7})(1-t^{8})},$$

ce qu'on trouvera égal à 32. Conséquemment, un covariant quelconque du degré-ordre 10.4 sera une fonction linéaire des trente-deux composés dont j'ai parlé, et ne peut pas être irréductible, ce qu'il fallait démontrer.

Il existe, dans la Note [p. 481, above] sur ce sujet insérée dans les *Comptes rendus* du 25 juillet dernier, des erreurs de calcul qui rendent la conclusion que je voulais établir tout à fait illusoire; cependant j'ai réussi, par le travail plus pénible qui suit, à parvenir au même résultat.

Je prends $(0, b, 2c, d, 0, 0, 0, 0, 1 \between x, y)^8$ avec la condition $bd=3c^2$ pour la forme spéciale de f. Alors les invariants du deuxième et du troisième degré, comme on va voir, deviendront nuls; les invariants des degrés 3, 4, 5, 6, 7, 8 ne seront pas nuls, et, en les combinant avec les deux covariants des *degrés-ordres* 7.4, 6.4, 5.4 et avec les seuls covariants des degrés-ordres 4.4, 3.4 dans toutes les manières possibles pour former 1 covariant du degré-ordre 10.4, on aura 9 covariants de ce type, de sorte que, en y ajoutant le carré du covariant du degré-ordre 5.2, il y aura en tout 10 covariants composés du degré-ordre 10.4, dans lesquels les invariants des degrés 2 et 3 ne figurent pas.

Je vais démontrer qu'aucun de ces 10 covariants ne peut paraître dans la fonction Ω (voir *Comptes rendus*, p. 194) [p. 483, above], et conséquemment cette fonction, si elle existe, contiendra dans chaque terme ou l'invariant du deuxième degré ou l'invariant du troisième, et conduira à une équation syzygétique du degré-ordre 5.4, comme je l'ai déjà remarqué, à moins qu'elle ne soit un multiple exact de l'un ou l'autre de ces 2 invariants, dans lequel cas il conduira à une telle équation du degré-ordre 8.4 ou 7.4.

[* pp. 232, 117 above.]

Mais, en tout cas, il y aura un rapport syzygétique d'un degré-ordre inférieur à 10 . 4 entre les covariants composés, ce qui, selon la loi de Cayley dont j'ai parlé, aurait pour effet d'augmenter le nombre de covariants irréductibles trouvés également par M. Von Gall et moi-même, et dont l'exactitude n'a pas été discutée. Donc tout se ramène à prouver que les 10 covariants composés du degré-ordre 10 . 4, qui correspondent à la forme spéciale que j'ai adoptée, sont linéairement indépendants l'un de l'autre, ce que je vais établir.

On trouvera facilement, pour la forme spéciale supposée,

$$I_2 = 0, \quad I_3 = bd - 3c^2 = 0, \quad I_4 = cd^2, \quad I_5 = d^4,$$
$$I_6 = c^4, \quad I_7 = b^4 + 200c^3d^2, \quad I_8 = b^2c^3,$$

où I_j signifie un invariant du degré j, et en suivant la notation et les procédés de M. Von Gall, négligeant, en outre, des multiplicateurs numériques,

$$\begin{aligned}
k &= (-20d^2,\ 0,\ 0,\ b,\ 4c \between x, y)^4,\\
\Delta &= (0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2 \between x, y)^4,\\
f_4 &= (b^2,\ bc,\ c^2,\ -c\delta,\ 5\delta^2 \between x, y)^4,\\
f_{k,2} &= (-120c^2d^2,\ 3b^3 + 60cd^3,\ 6b^2c - 100d^4,\ 9bc^2,\ 60c^3 \between x, y)^4,\\
f_{k,3} &= (b^3 - 20cd^3,\ b^2c + 50d^4,\ bc^2 \between x, y)^2,\\
f_\Delta &= (40c^2d^2,\ b^3 + 80cd^3,\ 2b^2c,\ 3bc^2,\ 0 \between x, y)^4,\\
i_\Delta &= (240cd^4,\ 21bc^3,\ -6x^4,\ -120c^3d,\ -120c^2d^2 \between x, y)^4,\\
i_4 &= (33bc^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2 \between x, y)^4,\\
i_{k,2} &= (-360cd^4,\ 42bc^3 - 350d^5,\ 49c^4,\ 19c^3d,\ -138c^2d^2 \between x, y)^4.
\end{aligned}$$

Désignons $b^6x^4 + 4b^5cx^3y + 6b^4c^2x^2y^2$, c^7x^4, c^2d^6, $4c^6dx^3y$, $4cd^7x^3y_1$, $6c^5d^2x^2y^2$; $6d^8x^2y^2$, $4d^8xy^3$, $4b^3c^3xy^3$; $b^2c^4y^4$, $c^3d^4y^4$ par les lettres α, β, γ, δ, ϵ, η, ζ, θ, κ, λ, μ et, au lieu des valeurs actuelles des covariants composés du degré-ordre 10 . 4, prenons leurs valeurs par rapport au module 11 ; alors on trouvera que les valeurs peuvent être représentées par le Tableau suivant :

α.	η.	θ.	β.	λ.	γ.	δ.	ϵ.	ζ.	κ.	μ.
3	6	3	5	3	1	1	3	2	10	.
1	.	8	7	1	.	6	.	2	9	10
.	10	.	.	.	1	4	5	10	5	5
.	.	1	7	4	.	.	.	.	.	.
.	.	.	6	4	.	4	.	6	8	.
.	.	.	.	3	.	2	.	7	.	.
.	.	.	.	.	2	.	.	.	3	4
.	.	.	.	.	7	5	3	7	9	.
.	.	.	.	.	9	8	.	8*	1	9
.	.	.	.	.	3	5	2	5	8	5

[* See below, p. 517.]

où la première ligne des chiffres représente la fonction

$$3\alpha + 6\eta + 3\theta + \ldots + 10\kappa,$$

la seconde,

$$\alpha + 8\theta + \ldots + 9\kappa + 10\mu,$$

et ainsi pour toutes les autres.

Il ne reste donc qu'à examiner si les déterminants mineurs du *matrix* écrit au-dessus de l'ordre 10 s'évanouissent tous par rapport au module 11; sinon les 10 fonctions seront nécessairement indépendantes par rapport au module 11 et à plus forte raison absolument aussi: or ce petit problème numérique se ramène facilement à la question de déterminer si les déterminants mineurs de l'ordre 5 du *matrix*

$$\begin{vmatrix} 5 & 2 & . & . & 5 & . \\ 2 & . & . & . & 3 & 4 \\ 7 & 5 & 3 & 7 & 9 & . \\ 9 & 8 & . & 8 & 1 & 9 \\ 3 & 5 & 2 & 5 & 8 & 5 \end{vmatrix}$$

s'évanouissent tous par rapport au module 4, ce qui ne peut avoir lieu si le déterminant

$$\begin{vmatrix} 2 & . & . & . & . \\ . & . & . & 1 & 4 \\ 5 & 3 & 7 & 2 & . \\ 8 & . & 8 & 3 & 9 \\ 5 & 2 & 5 & 5 & 5 \end{vmatrix},$$

ou bien si le déterminant

$$\begin{vmatrix} . & . & 1 & 4 \\ 3 & 7 & 2 & . \\ . & 8 & 3 & 9 \\ 2 & 5 & 5 & 5 \end{vmatrix},$$

ou finalement si le déterminant

$$\begin{vmatrix} 3 & 7 & 8 \\ . & 8 & 3 \\ 2 & 5 & 4 \end{vmatrix},$$

c'est-à-dire si le nombre $51 - 86$ ou bien -35 ne contient pas 11. Donc les 10 fonctions dont je parle sont linéairement indépendantes entre elles. Mais il serait très périlleux d'admettre cette preuve sans confirmation de l'exactitude des chiffres qui résultent de l'immense calcul dont je n'ai qu'indiqué la marche. En effet, j'ai consacré de longues heures à la confirmation de chaque pas de ce calcul, et j'ai appelé à mon aide un calculateur

habile; mais ce qui est le plus important, j'ai pu le vérifier de la manière suivante.

J'ai calculé pour ma forme spéciale la valeur du covariant i_4'', donné par M. Von Gall [p. 518, below] et jusqu'à présent trouvé par lui irréductible; cette valeur est

$$\begin{pmatrix} -128520c^7 - 25600c^2d^6, & 37590c^6d^2, & -10920c^5d^2, \\ 63b^3c^3 - 25000c^4d^3, & 1638b^2c^4 + 9600c^3d^4 & \end{pmatrix}(x, y)^4.$$

En combinant cette fonction avec les dix autres du même type, on obtiendra un déterminant de l'ordre 11 qui doit s'évanouir si mes chiffres sont exacts.

J'ai calculé très consciencieusement la valeur de ce déterminant par rapport aux modules 11, 13, 17, et comme, dans les trois cas, j'ai trouvé que la valeur de ce déterminant se divise par le module, je crois que l'exactitude de mes chiffres est parfaitement démontrée, et qu'on peut rester tout à fait convaincu que l'existence d'un covariant irréductible du degré-ordre 10.4 appartenant au quantic octavique est impossible. Les détails du calcul vont être fournis au prochain fascicule de l'*American Mathematical Journal* [p. 509, below].

59.

TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS OF THE BINARY DUODECIMIC, WITH SOME GENERAL REMARKS, AND TABLES OF THE IRREDUCIBLE SYZYGIES OF CERTAIN QUANTICS*.

[*American Journal of Mathematics*, IV. (1881), pp. 41—61.]

Generating Function for differentiants,

Denominator:

$$(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)$$
$$(1-a^{10})(1-a^{11}).$$

Numerator:

$$1+4a^2+17a^3+49a^4+125a^5+285a^6+594a^7+1143a^8+2063a^9$$
$$+3517a^{10}+5693a^{11}+8817a^{12}+13104a^{13}+18769a^{14}+25979a^{15}$$
$$+34830a^{16}+45317a^{17}+57327a^{18}+70595a^{19}+84730a^{20}+99214a^{21}$$
$$+113430a^{22}+126698a^{23}+138345a^{24}+147722a^{25}+154297a^{26}$$
$$+157689a^{27}+157689a^{28}+154297a^{29}+147722a^{30}+138345a^{31}$$
$$+126698a^{32}+113430a^{33}+99214a^{34}+84730a^{35}+70595a^{36}+57327a^{37}$$
$$+45317a^{38}+34830a^{39}+25979a^{40}+18769a^{41}+13104a^{42}+8817a^{43}$$
$$+5693a^{44}+3517a^{45}+2063a^{46}+1143a^{47}+594a^{48}+285a^{49}+125a^{50}$$
$$+49a^{51}+17a^{52}+4a^{53}+a^{55}.$$

Generating Function for covariants, reduced form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})$$
$$(1-a^{11})(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})(1-ax^{12}).$$

* The tables of the duodecimic have been calculated by Mr F. Franklin in accordance with Professor Sylvester's second method (see this *Journal*†, Vol. III. p. 146), in pursuance of a grant made by the British Association for the Advancement of Science. The corresponding tables for the binary quantics of the first ten orders are given in this *Journal*, Vol. II. p. 223 [p. 283, above]; those for systems of quantics of the first four orders, taken two and two together, are given at page 293 of the same volume [p. 392, above].

[† *On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics*, By F. Franklin, *American Journal of Mathematics*, III. (1880), pp. 128—153.]

Numerator: *

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^0	1																				
a^1		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$															
a^2			1	1	2	2	3	2	2	1	1										
a^3			1		1	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{1}$						
a^4	1		3	1	3		1			1	1	2	2	3	2	2	1	1			
a^5	1	1	4	2	2	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{2}$	$\overline{2}$		$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	
a^6	3	3	7	4	4		$\overline{1}$	$\overline{2}$	$\overline{3}$			3	1	2	1	1					1
a^7	4	7	10	7	4	$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{4}$	$\overline{3}$	1	$\overline{1}$	1		$\overline{1}$	$\overline{1}$	$\overline{1}$			
a^8	7	12	17	10	6	$\overline{2}$	$\overline{11}$	$\overline{13}$	$\overline{12}$	$\overline{5}$	$\overline{2}$	5	3	6	2	1			1		
a^9	9	23	25	18	7	$\overline{6}$	$\overline{21}$	$\overline{24}$	$\overline{22}$	$\overline{10}$	$\overline{3}$	7	7	8	2	$\overline{1}$	$\overline{1}$	$\overline{2}$		$\overline{1}$	—
a^{10}	17	36	39	25	5	$\overline{15}$	$\overline{39}$	$\overline{45}$	$\overline{37}$	$\overline{16}$	$\overline{1}$	16	16	15	5	1	$\overline{2}$	$\overline{3}$		$\overline{1}$	1
a^{11}	21	56	53	32	$\overline{1}$	$\overline{32}$	$\overline{67}$	$\overline{72}$	$\overline{54}$	$\overline{19}$	9	31	34	24	9	$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	$\overline{1}$	1
a^{12}	36	81	76	44	$\overline{7}$	$\overline{52}$	$\overline{100}$	$\overline{108}$	$\overline{72}$	$\overline{17}$	24	56	56	35	11	$\overline{5}$	$\overline{14}$	$\overline{13}$	$\overline{4}$		3
a^{13}	45	112	97	51	$\overline{27}$	$\overline{93}$	$\overline{158}$	$\overline{162}$	$\overline{101}$	$\overline{21}$	45	85	87	47	11	$\overline{14}$	$\overline{24}$	$\overline{22}$	$\overline{6}$	3	4
a^{14}	65	151	133	63	$\overline{45}$	$\overline{134}$	$\overline{216}$	$\overline{218}$	$\overline{120}$	$\overline{3}$	90	143	136	69	15	$\overline{22}$	$\overline{40}$	$\overline{32}$	$\overline{8}$	7	7
a^{15}	81	199	163	66	$\overline{83}$	$\overline{206}$	$\overline{309}$	$\overline{303}$	$\overline{157}$	5	139	199	187	83	4	$\overline{48}$	$\overline{66}$	$\overline{51}$	$\overline{14}$	13	9
a^{16}	110	251	206	69	$\overline{124}$	$\overline{282}$	$\overline{404}$	$\overline{386}$	$\overline{173}$	52	226	301	267	114		$\overline{69}$	$\overline{97}$	$\overline{68}$	$\overline{16}$	24	17
a^{17}	131	309	241	59	$\overline{188}$	$\overline{389}$	$\overline{532}$	$\overline{489}$	$\overline{196}$	97	323	403	345	127	$\overline{29}$	$\overline{120}$	$\overline{146}$	$\overline{100}$	$\overline{23}$	37	21
a^{18}	168	370	288	51	$\overline{253}$	$\overline{495}$	$\overline{653}$	$\overline{580}$	$\overline{193}$	188	460	550	446	150	$\overline{58}$	$\overline{169}$	$\overline{202}$	$\overline{127}$	$\overline{23}$	57	36
a^{19}	193	433	318	22	$\overline{347}$	$\overline{636}$	$\overline{808}$	$\overline{692}$	$\overline{196}$	274	604	691	539	149	$\overline{116}$	$\overline{251}$	$\overline{274}$	$\overline{168}$	$\overline{26}$	83	45
a^{20}	232	493	359	$\overline{6}$	$\overline{436}$	$\overline{759}$	$\overline{939}$	$\overline{770}$	$\overline{153}$	421	797	883	655	160	$\overline{176}$	$\overline{329}$	$\overline{352}$	$\overline{204}$	$\overline{19}$	115	65
a^{21}	256	551	377	$\overline{54}$	$\overline{546}$	$\overline{912}$	$\overline{1093}$	$\overline{861}$	$\overline{119}$	554	980	1045	742	128	$\overline{277}$	$\overline{448}$	$\overline{450}$	$\overline{254}$	$\overline{18}$	154	81
a^{22}	293	598	402	$\overline{97}$	$\overline{648}$	$\overline{1035}$	$\overline{1209}$	$\overline{909}$	$\overline{35}$	746	1201	1252	844	110	$\overline{372}$	$\overline{553}$	$\overline{543}$	$\overline{292}$	1	201	110
a^{23}	307	638	402	$\overline{161}$	$\overline{762}$	$\overline{1174}$	$\overline{1335}$	$\overline{956}$	43	914	1402	1410	907	45	$\overline{510}$	$\overline{697}$	$\overline{654}$	$\overline{343}$	14	251	131
a^{24}	336	667	407	$\overline{210}$	$\overline{852}$	$\overline{1266}$	$\overline{1404}$	$\overline{948}$	168	1131	1619	1594	972	$\overline{14}$	$\overline{637}$	$\overline{821}$	$\overline{752}$	$\overline{375}$	44	308	168
a^{25}	339	679	381	$\overline{280}$	$\overline{953}$	$\overline{1371}$	$\overline{1480}$	$\overline{944}$	274	1296	1791	1706	983	$\overline{119}$	$\overline{800}$	$\overline{972}$	$\overline{855}$	$\overline{417}$	69	369	193
a^{26}	351	678	367	$\overline{326}$	$\overline{1012}$	$\overline{1405}$	$\overline{1477}$	$\overline{867}$	430	1508	1972	1836	997	$\overline{209}$	$\overline{934}$	$\overline{1086}$	$\overline{937}$	$\overline{434}$	114	430	232
a^{27}	339	664	323	$\overline{382}$	$\overline{1070}$	$\overline{1446}$	$\overline{1479}$	$\overline{810}$	542	1632	2069	1860	937	$\overline{351}$	$\overline{1100}$	$\overline{1225}$	$\overline{1017}$	$\overline{457}$	148	490	256
a^{28}	336	635	291	$\overline{410}$	$\overline{1086}$	$\overline{1418}$	$\overline{1410}$	$\overline{692}$	688	1782	2164	1900	893	$\overline{453}$	$\overline{1211}$	$\overline{1301}$	$\overline{1056}$	$\overline{446}$	203	547	293
a^{29}	307	595	239	$\overline{445}$	$\overline{1093}$	$\overline{1393}$	$\overline{1342}$	$\overline{595}$	784	1837	2172	1837	784	$\overline{595}$	$\overline{1342}$	$\overline{1393}$	$\overline{1093}$	$\overline{445}$	239	595	307

* In the tabulated numerators, the *minus* sign is placed *over* the number which it affects.

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^{30}	293	547	203	$\overline{446}$	$\overline{1056}$	$\overline{1301}$	$\overline{1211}$	$\overline{453}$	893	1900	2164	1782	688	$\overline{692}$	$\overline{1410}$	$\overline{1418}$	$\overline{1086}$	$\overline{410}$	291	635	336
a^{31}	256	490	148	$\overline{457}$	$\overline{1017}$	$\overline{1225}$	$\overline{1100}$	$\overline{351}$	937	1860	2069	1632	542	$\overline{810}$	$\overline{1479}$	$\overline{1446}$	$\overline{1070}$	$\overline{382}$	323	664	339
a^{32}	232	430	114	$\overline{434}$	$\overline{937}$	$\overline{1086}$	$\overline{934}$	$\overline{209}$	997	1836	1972	1508	430	$\overline{867}$	$\overline{1477}$	$\overline{1405}$	$\overline{1012}$	$\overline{326}$	367	678	351
a^{33}	193	369	69	$\overline{417}$	$\overline{855}$	$\overline{972}$	$\overline{800}$	$\overline{119}$	983	1706	1791	1296	274	$\overline{944}$	$\overline{1480}$	$\overline{1371}$	$\overline{953}$	$\overline{280}$	381	679	339
a^{34}	168	308	44	$\overline{375}$	$\overline{752}$	$\overline{821}$	$\overline{637}$	$\overline{14}$	972	1594	1619	1131	168	$\overline{948}$	$\overline{1404}$	$\overline{1266}$	$\overline{852}$	$\overline{210}$	407	667	336
a^{35}	131	251	14	$\overline{343}$	$\overline{654}$	$\overline{697}$	$\overline{510}$	45	907	1410	1402	914	43	$\overline{956}$	$\overline{1335}$	$\overline{1174}$	$\overline{762}$	$\overline{161}$	402	638	307
a^{36}	110	201	1	$\overline{292}$	$\overline{543}$	$\overline{553}$	$\overline{372}$	110	844	1252	1201	746	$\overline{35}$	$\overline{909}$	$\overline{1209}$	$\overline{1035}$	$\overline{648}$	$\overline{97}$	402	598	293
a^{37}	81	154	$\overline{18}$	$\overline{254}$	$\overline{450}$	$\overline{448}$	$\overline{277}$	128	742	1045	980	554	$\overline{119}$	$\overline{861}$	$\overline{1093}$	$\overline{912}$	$\overline{546}$	$\overline{54}$	377	551	256
a^{38}	65	115	$\overline{19}$	$\overline{204}$	$\overline{352}$	$\overline{329}$	$\overline{176}$	160	655	883	797	421	$\overline{153}$	$\overline{770}$	$\overline{939}$	$\overline{759}$	$\overline{436}$	$\overline{6}$	359	493	232
a^{39}	45	83	$\overline{26}$	$\overline{168}$	$\overline{274}$	$\overline{251}$	$\overline{116}$	149	539	691	604	274	$\overline{196}$	$\overline{692}$	$\overline{808}$	$\overline{636}$	$\overline{347}$	22	318	433	193
a^{40}	36	57	$\overline{23}$	$\overline{127}$	$\overline{202}$	$\overline{169}$	$\overline{58}$	150	446	550	460	188	$\overline{193}$	$\overline{580}$	$\overline{653}$	$\overline{495}$	$\overline{253}$	51	288	370	168
a^{41}	21	37	$\overline{23}$	$\overline{100}$	$\overline{146}$	$\overline{120}$	$\overline{29}$	127	345	403	323	97	$\overline{196}$	$\overline{489}$	$\overline{532}$	$\overline{389}$	$\overline{188}$	59	241	309	131
a^{42}	17	24	$\overline{16}$	$\overline{68}$	$\overline{97}$	$\overline{69}$		114	267	301	226	52	$\overline{173}$	$\overline{386}$	$\overline{404}$	$\overline{282}$	$\overline{124}$	69	206	251	110
a^{43}	9	13	$\overline{14}$	$\overline{51}$	$\overline{66}$	$\overline{48}$	4	83	187	199	139	5	$\overline{157}$	$\overline{303}$	$\overline{309}$	$\overline{206}$	$\overline{83}$	66	163	199	81
a^{44}	7	7	$\overline{8}$	$\overline{32}$	$\overline{40}$	$\overline{22}$	15	69	136	143	90	$\overline{3}$	$\overline{120}$	$\overline{218}$	$\overline{216}$	$\overline{134}$	$\overline{45}$	63	133	151	65
a^{45}	4	3	$\overline{6}$	$\overline{22}$	$\overline{24}$	$\overline{14}$	11	47	87	85	45	$\overline{21}$	$\overline{101}$	$\overline{162}$	$\overline{158}$	$\overline{93}$	$\overline{27}$	51	97	112	45
a^{46}	3		$\overline{4}$	$\overline{13}$	$\overline{14}$	$\overline{5}$	11	35	56	56	24	$\overline{17}$	$\overline{72}$	$\overline{108}$	$\overline{100}$	$\overline{52}$	$\overline{7}$	44	76	81	36
a^{47}	1	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	9	24	34	31	9	$\overline{19}$	$\overline{54}$	$\overline{72}$	$\overline{67}$	$\overline{32}$	$\overline{1}$	32	53	56	21
a^{48}	1	$\overline{1}$		$\overline{3}$	$\overline{2}$	1	5	15	13	16	$\overline{1}$	$\overline{16}$	$\overline{37}$	$\overline{45}$	$\overline{39}$	$\overline{15}$	5	25	39	36	17
a^{49}		$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{1}$	2	8	7	7	$\overline{3}$	$\overline{10}$	$\overline{22}$	$\overline{24}$	$\overline{21}$	$\overline{6}$	7	18	25	23	9
a^{50}			1			1	2	6	3	5	$\overline{2}$	$\overline{5}$	$\overline{12}$	$\overline{13}$	$\overline{11}$	$\overline{2}$	6	10	17	12	7
a^{51}				$\overline{1}$	$\overline{1}$	$\overline{1}$		1	$\overline{1}$	1	$\overline{3}$	$\overline{4}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	4	7	10	7	4
a^{52}	1					1	1	2	1	3			$\overline{3}$	$\overline{2}$	$\overline{1}$		4	4	7	3	3
a^{53}		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$		$\overline{1}$		$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{2}$	$\overline{2}$	$\overline{1}$	2	2	4	1	1
a^{54}				1	1	2	2	3	2	2	1	1			1		3	1	3		1
a^{55}							$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	1		1		
a^{56}											1	1	2	2	3	2	2	1	1		
a^{57}																$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	
a^{58}																					1

Generating Function for covariants, representative form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})$$
$$(1-a^{11})(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-a^2x^{20})(1-ax^{12}).$$

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^0	1																	
a^3			1	1	2	1	2	2	1	2	1	1	1	1		1		
a^4	1		3	2	4	3	4	4	3	4	2	3	1	2	1	1		1
a^5	1	2	5	6	7	8	6	9	5	6	3	4	1	1		$\overline{1}$		$\overline{2}$
a^6	3	4	9	11	13	15	13	15	9	11	5	6	2	1		$\overline{1}$		$\overline{3}$
a^7	4	10	16	21	23	27	23	24	18	15	10	6	3	$\overline{2}$	1	$\overline{5}$		$\overline{4}$
a^8	7	16	28	34	40	46	40	37	27	22	12	6		$\overline{5}$	$\overline{5}$	$\overline{10}$	$\overline{5}$	$\overline{7}$
a^9	9	30	44	58	64	71	64	55	39	27	13	$\overline{3}$	$\overline{8}$	$\overline{18}$	$\overline{16}$	$\overline{22}$	$\overline{14}$	$\overline{13}$
a^{10}	17	45	71	89	99	110	97	77	51	29	5	$\overline{15}$	$\overline{30}$	$\overline{40}$	$\overline{39}$	$\overline{42}$	$\overline{30}$	$\overline{23}$
a^{11}	21	73	106	133	148	156	137	101	62	20	$\overline{10}$	$\overline{50}$	$\overline{65}$	$\overline{83}$	$\overline{76}$	$\overline{74}$	$\overline{54}$	$\overline{38}$
a^{12}	36	102	153	191	208	218	187	123	61	1	$\overline{53}$	$\overline{102}$	$\overline{129}$	$\overline{146}$	$\overline{136}$	$\overline{120}$	$\overline{89}$	$\overline{59}$
a^{13}	45	148	214	265	287	288	240	143	55	$\overline{44}$	$\overline{113}$	$\overline{190}$	$\overline{220}$	$\overline{239}$	$\overline{218}$	$\overline{181}$	$\overline{131}$	$\overline{75}$
a^{14}	65	196	290	353	377	373	299	152	21	$\overline{114}$	$\overline{218}$	$\overline{309}$	$\overline{352}$	$\overline{363}$	$\overline{327}$	$\overline{258}$	$\overline{178}$	$\overline{89}$
a^{15}	81	264	379	460	486	460	357	147	$\overline{33}$	$\overline{226}$	$\overline{357}$	$\overline{483}$	$\overline{524}$	$\overline{528}$	$\overline{467}$	$\overline{344}$	$\overline{227}$	$\overline{90}$
a^{16}	110	332	486	577	601	558	408	118	$\overline{129}$	$\overline{376}$	$\overline{558}$	$\overline{694}$	$\overline{750}$	$\overline{725}$	$\overline{627}$	$\overline{439}$	$\overline{266}$	$\overline{68}$
a^{17}	131	419	602	707	728	647	442	61	$\overline{261}$	$\overline{587}$	$\overline{805}$	$\overline{972}$	$\overline{1018}$	$\overline{960}$	$\overline{810}$	$\overline{529}$	$\overline{286}$	$\overline{12}$
a^{18}	168	501	728	842	848	734	457	$\overline{30}$	$\overline{452}$	$\overline{840}$	$\overline{1122}$	$\overline{1287}$	$\overline{1330}$	$\overline{1216}$	$\overline{996}$	$\overline{604}$	$\overline{266}$	91
a^{19}	193	601	856	979	970	800	443	$\overline{160}$	$\overline{677}$	$\overline{1160}$	$\overline{1476}$	$\overline{1654}$	$\overline{1664}$	$\overline{1489}$	$\overline{1171}$	$\overline{644}$	$\overline{194}$	265
a^{20}	232	686	985	1106	1068	854	397	$\overline{331}$	$\overline{964}$	$\overline{1516}$	$\overline{1887}$	$\overline{2038}$	$\overline{2017}$	$\overline{1751}$	$\overline{1318}$	$\overline{636}$	$\overline{48}$	507
a^{21}	256	783	1102	1223	1158	867	318	$\overline{541}$	$\overline{1279}$	$\overline{1920}$	$\overline{2310}$	$\overline{2451}$	$\overline{2356}$	$\overline{1995}$	$\overline{1422}$	$\overline{556}$	171	842
a^{22}	293	854	1209	1319	1207	865	203	$\overline{785}$	$\overline{1635}$	$\overline{2332}$	$\overline{2758}$	$\overline{2831}$	$\overline{2673}$	$\overline{2183}$	$\overline{1451}$	$\overline{403}$	491	1259
a^{23}	307	931	1293	1388	1241	814	54	$\overline{1050}$	$\overline{1993}$	$\overline{2763}$	$\overline{3171}$	$\overline{3200}$	$\overline{2930}$	$\overline{2310}$	$\overline{1408}$	$\overline{151}$	893	1768
a^{24}	336	974	1352	1430	1225	744	$\overline{118}$	$\overline{1330}$	$\overline{2364}$	$\overline{3155}$	$\overline{3567}$	$\overline{3490}$	$\overline{3115}$	$\overline{2352}$	$\overline{1263}$	182	1393	2335
a^{25}	339	1015	1384	1437	1192	631	$\overline{310}$	$\overline{1607}$	$\overline{2691}$	$\overline{3525}$	$\overline{3876}$	$\overline{3719}$	$\overline{3204}$	$\overline{2299}$	$\overline{1032}$	614	1953	2967
a^{26}	351	1017	1385	1409	1110	507	$\overline{511}$	$\overline{1867}$	$\overline{2998}$	$\overline{3812}$	$\overline{4123}$	$\overline{3834}$	$\overline{3192}$	$\overline{2144}$	$\overline{698}$	1105	2575	3605
a^{27}	339	1015	1352	1353	1018	350	$\overline{704}$	$\overline{2095}$	$\overline{3224}$	$\overline{4033}$	$\overline{4249}$	$\overline{3860}$	$\overline{3067}$	$\overline{1893}$	$\overline{301}$	1661	3199	4244
a^{28}	336	974	1294	1267	887	203	$\overline{883}$	$\overline{2274}$	$\overline{3389}$	$\overline{4139}$	$\overline{4281}$	$\overline{3752}$	$\overline{2839}$	$\overline{1550}$	175	2223	3824	4823
a^{29}	307	931	1210	1156	762	38	$\overline{1032}$	$\overline{2399}$	$\overline{3452}$	$\overline{4162}$	$\overline{4183}$	$\overline{3558}$	$\overline{2518}$	$\overline{1143}$	658	2788	4374	5329
a^{30}	293	854	1105	1031	611	$\overline{100}$	$\overline{1146}$	$\overline{2456}$	$\overline{3446}$	$\overline{4059}$	$\overline{3998}$	$\overline{3251}$	$\overline{2127}$	$\overline{692}$	1158	3290	4850	5697
a^{31}	256	783	988	893	482	$\overline{236}$	$\overline{1219}$	$\overline{2447}$	$\overline{3333}$	$\overline{3876}$	$\overline{3701}$	$\overline{2881}$	$\overline{1690}$	$\overline{226}$	1612	3730	5188	5942

Numerator—(*Continued.*)

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
																		a^0
																		a^3
																		a^4
	$\overline{2}$		$\overline{1}$		$\overline{1}$													a^5
	$\overline{2}$		$\overline{1}$		$\overline{1}$													a^6
	$\overline{1}$		1		2		2		1		1							a^7
$\overline{2}$	$\overline{2}$	$\overline{1}$			2		1		1									a^8
$\overline{7}$	$\overline{4}$	$\overline{3}$	2		3		1				$\overline{1}$		$\overline{1}$		$\overline{1}$			a^9
$\overline{13}$	$\overline{8}$	$\overline{2}$	2	3	5	2	3	1			$\overline{1}$							a^{10}
$\overline{22}$	$\overline{6}$		10	7	11	6	7	2	1		$\overline{1}$		$\overline{1}$				1	a^{11}
$\overline{28}$	$\overline{3}$	11	20	20	22	15	11	4	1		$\overline{1}$	$\overline{1}$	$\overline{1}$					a^{12}
$\overline{32}$	15	31	45	40	43	29	18	7		$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{3}$					a^{13}
$\overline{18}$	46	75	85	78	72	51	27	9	$\overline{1}$	$\overline{5}$	$\overline{9}$	$\overline{7}$	$\overline{4}$	$\overline{1}$				a^{14}
8	106	139	149	130	115	79	37	9	$\overline{10}$	$\overline{14}$	$\overline{19}$	$\overline{12}$	$\overline{8}$	$\overline{2}$			1	a^{15}
73	199	247	237	209	169	116	46	2	$\overline{23}$	$\overline{31}$	$\overline{34}$	$\overline{24}$	$\overline{13}$	$\overline{4}$	1	2	1	a^{16}
177	344	392	367	311	243	161	51	$\overline{11}$	$\overline{50}$	$\overline{58}$	$\overline{55}$	$\overline{37}$	$\overline{21}$	$\overline{5}$	3	4	3	a^{17}
348	540	602	526	445	326	206	48	$\overline{43}$	$\overline{90}$	$\overline{100}$	$\overline{88}$	$\overline{63}$	$\overline{28}$	$\overline{7}$	8	10	4	a^{18}
582	815	861	740	601	427	250	29	$\overline{94}$	$\overline{155}$	$\overline{163}$	$\overline{132}$	$\overline{89}$	$\overline{42}$	$\overline{7}$	14	16	7	a^{19}
906	1152	1192	980	780	525	279	$\overline{12}$	$\overline{179}$	$\overline{244}$	$\overline{250}$	$\overline{193}$	$\overline{132}$	$\overline{51}$	$\overline{5}$	27	30	9	a^{20}
1298	1574	1564	1264	962	621	287	$\overline{91}$	$\overline{298}$	$\overline{375}$	$\overline{365}$	$\overline{271}$	$\overline{174}$	$\overline{69}$	3	42	45	17	a^{21}
1788	2048	1998	1553	1149	692	260	$\overline{207}$	$\overline{468}$	$\overline{537}$	$\overline{510}$	$\overline{366}$	$\overline{232}$	$\overline{75}$	15	70	73	21	a^{22}
2334	2588	2443	1862	1312	734	191	$\overline{378}$	$\overline{680}$	$\overline{748}$	$\overline{684}$	$\overline{476}$	$\overline{283}$	$\overline{88}$	40	105	102	36	a^{23}
2947	3140	2910	2135	1441	723	57	$\overline{600}$	$\overline{954}$	$\overline{993}$	$\overline{890}$	$\overline{597}$	$\overline{348}$	$\overline{82}$	74	154	148	45	a^{24}
3568	3714	3338	2388	1510	656	$\overline{134}$	$\overline{886}$	$\overline{1269}$	$\overline{1285}$	$\overline{1116}$	$\overline{730}$	$\overline{394}$	$\overline{78}$	123	216	196	65	a^{25}
4201	4240	3733	2561	1511	513	$\overline{399}$	$\overline{1221}$	$\overline{1639}$	$\overline{1596}$	$\overline{1360}$	$\overline{860}$	$\overline{444}$	$\overline{48}$	189	293	264	81	a^{26}
4772	4721	4029	2667	1422	298	$\overline{723}$	$\overline{1617}$	$\overline{2029}$	$\overline{1942}$	$\overline{1605}$	$\overline{987}$	$\overline{466}$	$\overline{14}$	272	383	332	110	a^{27}
5285	5088	4239	2658	1250	2	$\overline{1115}$	$\overline{2036}$	$\overline{2454}$	$\overline{2276}$	$\overline{1844}$	$\overline{1098}$	$\overline{480}$	53	370	490	419	131	a^{28}
5665	5349	4301	2561	981	$\overline{359}$	$\overline{1538}$	$\overline{2489}$	$\overline{2858}$	$\overline{2612}$	$\overline{2062}$	$\overline{1183}$	$\overline{454}$	124	488	606	501	168	a^{29}
5920	5442	4250	2331	634	$\overline{780}$	$\overline{2003}$	$\overline{2925}$	$\overline{3257}$	$\overline{2905}$	$\overline{2242}$	$\overline{1237}$	$\overline{417}$	233	615	731	601	193	a^{30}
6003	5403	4043	2023	225	$\overline{1231}$	$\overline{2446}$	$\overline{3347}$	$\overline{3592}$	$\overline{3161}$	$\overline{2374}$	$\overline{1251}$	$\overline{330}$	340	753	860	686	232	a^{31}

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^{32}	232	686	860	753	340	$\overline{330}$	$\overline{1251}$	$\overline{2374}$	$\overline{3161}$	$\overline{3592}$	$\overline{3347}$	$\overline{2446}$	$\overline{1231}$	225	2023	4043	5403	6003
a^{33}	193	601	731	615	233	$\overline{417}$	$\overline{1237}$	$\overline{2242}$	$\overline{2905}$	$\overline{3257}$	$\overline{2925}$	$\overline{2003}$	780	634	2331	4250	5442	5920
a^{34}	168	501	606	488	124	$\overline{454}$	$\overline{1183}$	$\overline{2062}$	$\overline{2612}$	$\overline{2858}$	$\overline{2489}$	$\overline{1538}$	$\overline{359}$	981	2561	4301	5349	5665
a^{35}	131	419	490	370	53	$\overline{480}$	$\overline{1098}$	$\overline{1844}$	$\overline{2276}$	$\overline{2454}$	$\overline{2036}$	$\overline{1115}$	2	1250	2658	4239	5088	5285
a^{36}	110	332	383	272	$\overline{14}$	$\overline{466}$	$\overline{987}$	$\overline{1605}$	$\overline{1942}$	$\overline{2029}$	$\overline{1617}$	$\overline{723}$	298	1422	2667	4029	4721	4772
a^{37}	81	264	293	189	$\overline{48}$	$\overline{444}$	$\overline{860}$	$\overline{1360}$	$\overline{1596}$	$\overline{1639}$	$\overline{1221}$	$\overline{399}$	513	1511	2561	3733	4240	4201
a^{38}	65	196	216	123	$\overline{78}$	$\overline{394}$	$\overline{730}$	$\overline{1116}$	$\overline{1285}$	$\overline{1269}$	$\overline{886}$	$\overline{134}$	656	1510	2388	3338	3714	3568
a^{39}	45	148	154	74	$\overline{82}$	$\overline{348}$	$\overline{597}$	$\overline{890}$	$\overline{993}$	$\overline{954}$	$\overline{600}$	57	723	1441	2135	2910	3140	2947
a^{40}	36	102	105	40	$\overline{88}$	$\overline{283}$	$\overline{476}$	$\overline{684}$	$\overline{748}$	$\overline{680}$	378	191	734	1312	1862	2443	2588	2334
a^{41}	21	73	70	15	$\overline{75}$	$\overline{232}$	$\overline{366}$	$\overline{510}$	$\overline{537}$	$\overline{468}$	$\overline{207}$	260	692	1149	1553	1998	2048	1788
a^{42}	17	45	42	3	$\overline{69}$	$\overline{174}$	$\overline{271}$	$\overline{365}$	$\overline{375}$	$\overline{298}$	$\overline{91}$	287	621	962	1264	1564	1574	1298
a^{43}	9	30	27	$\overline{5}$	$\overline{51}$	$\overline{132}$	$\overline{193}$	$\overline{250}$	$\overline{244}$	$\overline{179}$	$\overline{12}$	279	525	780	980	1192	1152	906
a^{44}	7	16	14	$\overline{7}$	$\overline{42}$	$\overline{89}$	$\overline{132}$	$\overline{163}$	$\overline{155}$	$\overline{94}$	29	250	427	601	740	861	815	582
a^{45}	4	10	8	$\overline{7}$	$\overline{28}$	$\overline{63}$	$\overline{88}$	$\overline{100}$	$\overline{90}$	$\overline{43}$	48	206	326	445	526	602	540	348
a^{46}	3	4	3	$\overline{5}$	$\overline{21}$	$\overline{37}$	$\overline{55}$	$\overline{58}$	$\overline{50}$	$\overline{11}$	51	161	243	311	367	392	344	177
a^{47}	1	2	1	$\overline{4}$	$\overline{13}$	$\overline{24}$	$\overline{34}$	$\overline{31}$	$\overline{23}$	2	46	116	169	209	237	247	199	73
a^{48}	1			$\overline{2}$	$\overline{8}$	$\overline{12}$	$\overline{19}$	$\overline{14}$	$\overline{10}$	9	37	79	115	130	149	139	106	8
a^{49}				$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{9}$	$\overline{5}$	$\overline{1}$	9	27	51	72	78	85	75	46	$\overline{18}$
a^{50}					$\overline{3}$	$\overline{2}$	$\overline{5}$	$\overline{1}$		7	18	29	43	40	45	31	15	$\overline{32}$
a^{51}					$\overline{1}$	$\overline{1}$	$\overline{1}$		1	4	11	15	22	20	20	11	$\overline{3}$	$\overline{28}$
a^{52}	1				$\overline{1}$		$\overline{1}$		1	2	7	6	11	7	10		$\overline{6}$	$\overline{22}$
a^{53}							$\overline{1}$			1	3	2	5	3	2	$\overline{2}$	$\overline{8}$	$\overline{13}$
a^{54}			$\overline{1}$		$\overline{1}$		$\overline{1}$				1		3		2	$\overline{3}$	$\overline{4}$	$\overline{7}$
a^{55}									1		1		2			$\overline{1}$	$\overline{2}$	$\overline{2}$
a^{56}							1		1		2		2		1		$\overline{1}$	
a^{57}													$\overline{1}$		$\overline{1}$		$\overline{2}$	
a^{58}													$\overline{1}$		$\overline{1}$		$\overline{2}$	
a^{59}																		
a^{60}																		
a^{63}																		

Numerator—(*Continued.*)

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
5942	5188	3730	1612	$\overline{226}$	$\overline{1690}$	$\overline{2881}$	$\overline{3701}$	$\overline{3876}$	$\overline{3333}$	$\overline{2447}$	$\overline{1219}$	$\overline{236}$	482	893	988	783	256	a^{32}
5697	4850	3290	1158	$\overline{692}$	$\overline{2127}$	$\overline{3251}$	$\overline{3998}$	$\overline{4059}$	$\overline{3446}$	$\overline{2456}$	$\overline{1146}$	$\overline{100}$	611	1031	1105	854	293	a^{33}
5329	4374	2788	658	$\overline{1143}$	$\overline{2518}$	$\overline{3558}$	$\overline{4183}$	$\overline{4162}$	$\overline{3452}$	$\overline{2399}$	$\overline{1032}$	38	762	1156	1210	931	307	a^{34}
4823	3824	2223	175	$\overline{1550}$	$\overline{2839}$	$\overline{3752}$	$\overline{4281}$	$\overline{4139}$	$\overline{3389}$	$\overline{2274}$	$\overline{883}$	203	887	1267	1294	974	336	a^{35}
4244	3199	1661	$\overline{301}$	$\overline{1893}$	$\overline{3067}$	$\overline{3860}$	$\overline{4249}$	$\overline{4033}$	$\overline{3224}$	$\overline{2095}$	$\overline{704}$	350	1018	1353	1352	1015	339	a^{36}
3605	2575	1105	$\overline{698}$	$\overline{2144}$	$\overline{3192}$	$\overline{3834}$	$\overline{4123}$	$\overline{3812}$	$\overline{2998}$	$\overline{1867}$	$\overline{511}$	507	1110	1409	1385	1017	351	a^{37}
2967	1953	614	$\overline{1032}$	$\overline{2299}$	$\overline{3204}$	$\overline{3719}$	$\overline{3876}$	$\overline{3525}$	$\overline{2691}$	$\overline{1607}$	$\overline{310}$	631	1192	1437	1384	1015	339	a^{38}
2335	1393	182	$\overline{1263}$	$\overline{2352}$	$\overline{3115}$	$\overline{3490}$	$\overline{3567}$	$\overline{3155}$	$\overline{2364}$	$\overline{1330}$	$\overline{118}$	744	1225	1430	1352	974	336	a^{39}
1768	893	$\overline{151}$	$\overline{1408}$	$\overline{2310}$	$\overline{2930}$	$\overline{3200}$	$\overline{3171}$	$\overline{2763}$	$\overline{1993}$	$\overline{1050}$	54	814	1241	1388	1293	931	307	a^{40}
1259	491	$\overline{403}$	$\overline{1451}$	$\overline{2183}$	$\overline{2673}$	$\overline{2831}$	$\overline{2758}$	$\overline{2332}$	$\overline{1635}$	$\overline{785}$	203	865	1207	1319	1209	854	293	a^{41}
842	171	$\overline{556}$	$\overline{1422}$	$\overline{1995}$	$\overline{2356}$	$\overline{2451}$	$\overline{2310}$	$\overline{1920}$	$\overline{1279}$	$\overline{541}$	318	867	1158	1223	1102	783	256	a^{42}
507	$\overline{48}$	$\overline{636}$	$\overline{1318}$	$\overline{1751}$	$\overline{2017}$	$\overline{2038}$	$\overline{1887}$	$\overline{1516}$	$\overline{964}$	$\overline{331}$	397	854	1068	1106	985	686	232	a^{43}
265	$\overline{194}$	$\overline{644}$	$\overline{1171}$	$\overline{1489}$	$\overline{1664}$	$\overline{1654}$	$\overline{1476}$	$\overline{1160}$	$\overline{677}$	$\overline{160}$	443	800	970	979	856	601	193	a^{44}
91	$\overline{266}$	$\overline{604}$	$\overline{996}$	$\overline{1216}$	$\overline{1330}$	$\overline{1287}$	$\overline{1122}$	$\overline{840}$	$\overline{452}$	$\overline{30}$	457	734	848	842	728	501	168	a^{45}
$\overline{12}$	$\overline{286}$	$\overline{529}$	$\overline{810}$	$\overline{960}$	$\overline{1018}$	$\overline{972}$	$\overline{805}$	$\overline{587}$	$\overline{261}$	61	442	647	728	707	602	419	131	a^{46}
$\overline{68}$	$\overline{266}$	$\overline{439}$	$\overline{627}$	$\overline{725}$	$\overline{750}$	$\overline{694}$	$\overline{558}$	$\overline{376}$	$\overline{129}$	118	408	558	601	577	486	332	110	a^{47}
$\overline{90}$	$\overline{227}$	$\overline{344}$	$\overline{467}$	$\overline{528}$	$\overline{524}$	$\overline{483}$	$\overline{357}$	$\overline{226}$	$\overline{33}$	147	357	460	486	460	379	264	81	a^{48}
$\overline{89}$	$\overline{178}$	$\overline{258}$	$\overline{327}$	$\overline{363}$	$\overline{352}$	$\overline{309}$	$\overline{218}$	$\overline{114}$	21	152	299	373	377	353	290	196	65	a^{49}
$\overline{75}$	$\overline{131}$	$\overline{181}$	$\overline{218}$	$\overline{239}$	$\overline{220}$	$\overline{190}$	$\overline{113}$	$\overline{44}$	55	143	240	288	287	265	214	148	45	a^{50}
$\overline{59}$	$\overline{89}$	$\overline{120}$	$\overline{136}$	$\overline{146}$	$\overline{129}$	$\overline{102}$	$\overline{53}$	1	61	123	187	218	208	191	153	102	36	a^{51}
$\overline{38}$	$\overline{54}$	$\overline{74}$	$\overline{76}$	$\overline{83}$	$\overline{65}$	$\overline{50}$	$\overline{10}$	20	62	101	137	156	148	133	106	73	21	a^{52}
$\overline{23}$	$\overline{30}$	$\overline{42}$	$\overline{39}$	$\overline{40}$	$\overline{30}$	$\overline{15}$	5	29	51	77	97	110	99	89	71	45	17	a^{53}
$\overline{13}$	$\overline{14}$	$\overline{22}$	$\overline{16}$	$\overline{18}$	$\overline{8}$	$\overline{3}$	13	27	39	55	64	71	64	58	44	30	9	a^{54}
$\overline{7}$	$\overline{5}$	$\overline{10}$	$\overline{5}$	$\overline{5}$		6	12	22	27	37	40	46	40	34	28	16	7	a^{55}
$\overline{4}$		$\overline{5}$	1	$\overline{2}$	3	6	10	15	18	24	23	27	23	21	16	10	4	a^{56}
$\overline{3}$		$\overline{1}$		1	2	6	5	11	9	15	13	15	13	11	9	4	3	a^{57}
$\overline{2}$		$\overline{1}$		1	1	4	3	6	5	9	6	8	7	6	5	2	1	a^{58}
1		1	1	2	1	3	2	4	3	4	4	3	4	2	3		1	a^{59}
		1		1	1	1	1	2	1	2	2	1	2	1	1			a^{60}
																	1	a^{63}

Table of Groundforms.

Degree in the Coefficients. \ Order in the Variables.	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	34
1							1										
2	1		1		1		1		1		1						
3	1		1	1	2	1	2	2	1	2	1	1	1	1		1	
4	2		3	2	4	3	4	4	3	4	2	3	1	2	1	1	1
5	2	2	5	6	7	8	6	9	5	6	3	4	1	1			
6	4	4	9	11	12	14	10	12	3	5							
7	5	10	15	20	18	21	9	8									
8	7	16	24	29	21	21											
9	9	28	33	37	15												
10	14	39	41	30													
11	15	53	40														
12	19	56	7														
13	18	44															
14	12																

The total number of groundforms (counting in the absolute constant and the quantic itself) is 949.

The manuscript sheets containing the original calculations from which the preceding tables have been constructed (as is the case also with the calculations connected with all the similar tables which have appeared in this journal) are deposited in the iron safe of the Johns Hopkins University, Baltimore, where they can be seen and examined, or copied, by any one interested in the subject. From the manifold independent systematic tests*

* One of these tests depends upon the following property of the generating function, which has been disclosed by observation, and of which the significance is not yet known. On putting $a=1$ in the numerator of the generating function, the coefficients of the various powers of x are integer multiples of the coefficient of x^0. Thus in the case of the duodecimic, the numerator of the reduced form becomes, on putting $a=1$,

$$5663\,(1+2x^2+x^4-x^6-3x^8-4x^{10}-4x^{12}-2x^{14}+2x^{16}+5x^{18}+6x^{20}+5x^{22}+2x^{24}-2x^{26}-4x^{28}-4x^{30}-3x^{32}-x^{34}+x^{36}+2x^{38}+x^{40}).$$

Thus the numerical divisibility of the result of putting $a=1$ furnishes a test for the sums of the *columns*, while the algebraic divisibility of the result of putting $x=1$ (see this *Journal*†, Vol. III. p. 151) tests the sums of the *rows*; and the satisfaction of both tests makes the correctness of the result practically certain.

[† See footnote, above, p. 489.]

to which the work has been subjected, Mr Franklin estimates that the chance is far more than a million to one that the generating functions for the twelfthic as calculated do not contain a single numerical error. The highest order of any ground-covariant to the twelfthic it will be seen is 34, which is the superior limit of order given by M. Camille Jordan's formula for the ground-covariants to a system of an indefinite number of simultaneous binary forms of each of which the order is 12 or less: M. Jordan's "superior limit" in fact in this as in all the other calculated cases, being actually attained by one (and only one) ground-covariant to a single form*. It will also be noticed that for all orders of the primitive which have been calculated, namely, from 3 to 12 (with 11 omitted), the degree of the covariant of highest order is either 3 or 4. Looking at single quantics of the even orders 6, 8, 10, 12, it will be observed that the maximum order of their ground-covariants for any degree (from and after the 4th degree) diminishes, or, to speak more strictly, never increases as the degree increases. As regards quantics of the odd orders 5, 7, 9, the same rule applies for the maximum order of their groundforms of even degrees; and in respect to their groundforms of odd degrees, the maximum order from and after the 3rd degree diminishes or remains stationary as the degree increases. Also (alike for quantics of odd or even order) when (beginning with the 3rd degree) in passing from an odd to the next even or from an even to the next odd degree of the groundforms, an increase in the maximum order takes place, it is only to the extent of a single unit. These facts, which constitute a sort of *law of shrinkage*, assume practical importance when the successive tables of groundforms are compared together, with a view to track the ground-differentiants (or, in Mr Cayley's language, the ground-seminvariants or *sources* of covariants), as the order of the primitive quantic is increased. Some of these ground-sources retain their irreducible character permanently, others only up to a particular limit of order in the primitive. The former may be regarded as the irreducible differentiants to a quantic of an infinite order: such for instance are all the differentiants of the second and third degree. But when we consider differentiants of the 4th degree this is no longer true. Thus we have the well-known example of the discriminant to $(a, b, c, d \between x, y)^3$, namely, $a^2d^2 + 4ac^3 + 4df^3 - 3b^2c^2 - 6abcd$, which is irreducible for this quantic, but for the quantic $(a, b, c, d, e \between x, y)^4$ remains, it is obvious, a differentiant, but no longer a ground-differentiant, being expressible under the form of the difference of two products of lower differentiants, namely, as

$$(ac - b^2)(ae - 4bd + 3c^2) - a \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

* It is also particularly noticeable that the number of the successively positive and negative blocks in the table follows the law observed in the inferior cases, namely, for Quantics of orders 3 and 4 there is a single block, for Quantics of orders 5 and 6 two blocks, for order 8 three blocks, and for orders 9 and 10 four blocks, there being five distinct blocks alternately positive and negative in the instance before us of the Quantic of order 12.

Suppose a differentiant to be the source of a covariant of the deg-order $j.\epsilon$ considered as belonging to the quantic $(a_0, a_1, \dots a_i \between x, y)^i$; then it is easily seen that it will be the source of a covariant of the deg-order $j.j+\epsilon$ in respect to the quantic $(a_0, a_1, \dots a_{i+1} \between x, y)^{i+1}$. We can, therefore, in many cases by a mere inspection of successive tables of groundforms eliminate some at least of the transient ground-differentiants: that is, wherever there are K groundforms of deg-order $j.\epsilon$ to a quantic of the order i, but only $K-\Delta$ of the deg-order $j.\epsilon+\lambda j$ to the quantic of the order $i+\lambda$, we know that at least Δ of the sources to the K groundforms, that is, Δ ground-differentiants of degree j and weight $\frac{1}{2}(ij-\epsilon)$ are only transiently irreducible. Thus, for example, the table of groundforms for the quintic exhibits a groundform of deg-order 4.4, that is, of deg-weight 4.8; but the table of groundforms for the sextic contains no groundform of the same deg-weight, that is, of deg-order 4.8. Hence the differentiant of deg-weight 4.8, although irreducible when regarded as a function of 6 letters (the number of letters which actually appear in it), is reducible when regarded as a function potentially of 7 or more.

So, again, for a like reason, the ground-differentiants of 5 letters, of deg-orders (in respect to the quintic) 5.1 and 5.7, that is, of deg-weights 5.12, 5.9, are only transiently irreducible; and, what is very interesting, it will be seen at a glance (and here the law of shrinkage makes its importance felt) that the sources of all the groundforms to a quintic of a higher order than the 5th are only transitory (or provisional, so to say) ground-differentiants. So in like manner it will be recognized by comparing the tables of groundforms for the seventhic and eighthic, that of the 9 ground-sources of the degree 6 to the former, only two *can be* permanent, namely, one of the weight $\frac{1}{2}(6.7-2)$ and one of the weight $\frac{1}{2}(6.7-4)$, that is, of the deg-weights 6.20 and 6.19 respectively: all the others becoming resoluble when an additional letter is introduced into the quantic. Moreover, as the table for the eighthic contains no groundforms of deg-order 7.8, we see from the law of shrinkage that there can be no ground-source to the seventhic of a higher than the 6th degree which is permanently irreducible*.

A systematic weeding out of the transitory ground-sources from the published tables, which cannot in all cases for groundforms of earlier degrees be effected completely without an examination of a more searching kind than that illustrated by the above examples, must be reserved for a future occasion—after I shall have completed, as I hope soon to do, the study of a subject of higher interest and more pressing importance, which has for its object to determine not only the groundforms so called, but also the ground-syzygants, the ground-counter-syzygants, &c., of quantics from their

* For the 6th degree it will at once be seen that there can be no permanent differentiant to the seventhic except one of the 2nd and one of the 4th order.

generating functions by a purely arithmetical process, which I believe to be already substantially in my possession.

As the first fruits of this method, I may state that the invariantive ground-syzygants (or, if the expression is preferred, fundamental syzygies) to the octavian quantic $(x, y)^8$ are 5 in number, and of the degrees 16, 17, 18, 19, 20 respectively in the coefficients. As regards the ground-syzygants (invariantive and covariantive) of the quintic, my method furnishes the same list as that given in Professor Cayley's *Tenth Memoir on Quantics*. Their deg-orders may be found as follows.

By the supernumerary ground-types understand the deg-orders of the ground-covariants exclusive of those represented by the factors which appear in the denominator of the representative generating function*, which are therefore $23 - 6$, that is, 17 in number. Let these types be added each to itself and every other, thus giving rise to $\frac{17 \cdot 18}{2}$ types: out of these sums strike out the types

$$8.4 \quad 9.5 \quad 10.2 \quad 10.4 \quad 11.3 \quad 12.2 \quad 14.4 \quad 16.2$$

and replace them by

$$13.5 \quad 14.6 \quad 15.3 \quad 15.5 \quad 16.4 \quad 17.3 \quad 19.5 \quad 21.3$$

The 153 types thus formed, together with the types, 26 in number, furnished by the negative terms in the numerator to the generating function (see this *Journal*, vol. II. p. 224 [p. 284, above]), 179 in all, will be the deg-orders of the fundamental syzygants. Mr Cayley founds this rule on his theory of the so-called Real Generating Function, which essentially consists in what may be termed the Dialytic Presentation of the Representative G. F. for the Quintic—namely as a sum of 26 pairs, each pair containing one positive and one negative term of the numerator divided by the denominator, so selected for conjunction that the developed expression of each pair shall be seen to be omni-positive by an obvious dialytic process.

The method followed by the eminent author in singling out the fundamental syzygants does not appear (as far as I can make out) to be explicitly stated in his memoir. The dialytic form (supposing, as is probably the case, it always exists for *finite* representative generating functions) is not easy to arrive at: a serious additional obstacle to the use of the dialytic method would arise in the case where (as for the seventhic) the numerator of the representative form becomes an infinite series. The method I employ does not require the use of the dialytic method, nor even of the *representative* form of the G. F., although the practical process is much simplified by the use of the representative form when it has a finite numerator. The result

* In such denominator the number of factors for a Quantic of any odd order $2i-1$ is $3i-3$, and for any even order $2i$ is $3i-2$ (i in each case being supposed greater than unity).

I obtain for the fundamental syzygants of the sextic is as follows: Take the 19 supernumerary ground-types (see* vol. II. p. 225), and add them each to each and to every other, as in the preceding case. Then strike out of the sums so formed the types of the deg-orders 6 . 4, 9 . 6, 8 . 4, 11 . 6, 10 . 4, 7 . 8, 8 . 6, 11 . 4, as well as one of the two sums 13 . 4 obtained from the addition of 5 . 2 and 8 . 2 or of 3 . 2 and 10 . 2 and replace the nine types so omitted by the eight types 12 . 8, 14 . 8, 13 . 6, 15 . 6, 10 . 10, 11 . 8, 14 . 6, 16 . 6. There will thus arise $19 \cdot \frac{20}{2} - 9 + 8$, or 189 types: to these adjoin the 29 types given by the negative terms in the numerator of the Rep. G. F.: the total number of types 189 + 29 or 218 so obtained will be the deg-orders of the complete system of fundamental syzygants to the sextic. The two types of the deg-order 6 . 6 which appear among the supernumerary types, it will of course be understood, are to be treated as distinct types in forming the binary sums. It is just barely possible (but I think very unlikely) that I may have committed some oversight in the table of replacement in the above calculation, and that the true number of ground-syzygies may be $19 \cdot \frac{18}{2} + 29$ or 219 instead of 218†.

I subjoin a brief *aperçu* of the general theory.

A generating function (whatever its subject-matter) developed in a series consists of facients and coefficients, where any facient is a product of a finite set of letters each raised to a certain power. The totality of the exponents expressing these powers may be termed the type of the facient. In the generating functions to be referred to hereinunder, the letters employed are just as many in number as there are quantics in the system to be considered: namely, one letter corresponds to each quantic.

A generating function proper (with reference to the present theory) is defined to be one that is or can be developed into a series of facients whose coefficients and whose types are omni-positive integers, and where each such numerical coefficient is the number of linearly independent invariants whose degrees in the coefficients of the several quantics of the system are identical with the indices of the corresponding letters in the facient to which that numerical coefficient is attached‡. The type of the facient may be also styled the type of the connoted invariants. A binomial expression consisting

[* p. 285, above.]

† Nine binary sums of types are omitted, and are replaced by only eight other combinations. This is analogous to the loss of a unit in counting the irreducible syzygies to the invariants of an eighthic. The *supernumerary* invariants in this case are 3 in number; of degrees 8, 9, 10 respectively. Their binary combinations would give 6, but the true number of irreducible syzygies is only 5.

‡ I speak designedly (for greater facility of expression) of invariants only, which can be done for binary quantics without any loss of generality, inasmuch as covariants may be regarded as invariants of a given system of quantics with a linear quantic superadded.

of unity followed by a facient and separated from it by the negative sign may be termed a *generator**.

A proper generating function to a system of quantics may always by known methods (see this *Journal*, vol. III. p. 133)† be expressed by a fraction whose numerator is a finite series of facients with numerical coefficients and its denominator a finite product of generators.

It may also be expressed (according to a definite process), and in one way only, by a fraction whose numerator and denominator alike consist of a finite or infinite (except in a few trivial cases, an infinite) product of generators‡.

A finite product of generators (or powers of generators) may be termed a generator-group.

For greater uniformity of statement in regard to what follows, let us agree to understand by a syzygant of the grade zero, an irreducible invariant. Then the two infinite products above referred to (whose ratio is algebraically equal to the generating function) may each be resolved into a product (usually infinite) of collect-groups, such that the totality of the types of the 1st, 2nd, ... ith groups of the denominator shall respectively represent the totality of the types of irreducible syzygants of the grades $0, 2, \ldots (2i-2)$ and the totality of the types of the 1st, 2nd, ... ith groups of the numerator the totality of the types of irreducible syzygants of the grades $1, 3, 5, \ldots (2i-1)$, so that each group may be said to be related to or to represent a complete system of irreducible syzygants of a certain grade (invariants being regarded as zero-graded syzygants)—that is to say, as many times as any generator is repeated in a group so many (and no more) irreducible syzygants of that type will there be of the corresponding grade.

Let G be a proper generating function to a system of quantics, $\Gamma_0, \Gamma_1, \Gamma_2 \ldots$ generator-groups such that

$$G = \frac{1 \,.\, \Gamma_1 \,.\, \Gamma_3 \,.\, \Gamma_5 \ldots}{\Gamma_0 \,.\, \Gamma_2 \,.\, \Gamma_4 \,.\, \Gamma_6 \ldots};$$

then, as suggested to me by Mr Franklin, in order that the Γ series may be

* If $a, b, c, \ldots$ are facients, $1 - a^\alpha b^\beta c^\gamma \ldots$ is a *generator*, and $\alpha, \beta, \gamma \ldots$ (taken in a definite order) is its *type*.

[† See above, p. 489, footnote.]

‡ For instance let G be the generating function proper to the invariants of an eighthic.

$$\begin{aligned}\text{Then } G &= \frac{1 + a^8 + a^9 + a^{10} + a^{18}}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)} \\ &= [(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})]^{-1} \\ &\quad .\, (1-a^{16})(1-a^{17})(1-a^{18})(1-a^{19})(1-a^{20}) \\ &\quad .\, [(1-a^{25})(1-a^{26})(1-a^{27})(1-a^{28})(1-a^{29})]^{-1} \\ &\quad .\, (1-a^{33})(1-a^{34})(1-a^{35})^2(1-a^{36})^2(1-a^{37})^2(1-a^{38})(1-a^{39}) \\ &\quad .\, [(1-a^{41})(1-a^{42})^2(1-a^{43})^3(1-a^{44})^4(1-a^{45})^4(1-a^{46})^4(1-a^{47})^3(1-a^{48})^2]^{-1} \\ &\quad .\, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

representative of complete systems of irreducible syzygants of the successive grades, it is *necessary* that $\frac{1}{\Gamma_0} - G$; $\frac{\Gamma_1}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2} - G$; ... shall, when developed in series of facients with omni-positive indices, be alternately omni-positive and omni-negative. But the existence of these inequalities, although a *necessary*, is not a *sufficient* condition in order that the Γ's shall be so representative; for example, $\Gamma_0 . \Gamma_2$ and $\Gamma_1 . \Gamma_3$ might evidently be regarded as single groups and the inequalities would still be satisfied; but suppose we further limit the Γ's in succession by the following rule, namely, that on withdrawing any one of the generator-factors from Γ_0 and calling Γ_0' the group so reduced $\frac{1}{\Gamma_0'} - G$ is no longer omni-positive, this will serve to define Γ_0 absolutely; Γ_0 being so determined, Γ_1 may in like manner be limited by the condition that its quotient by any one of its generators being called Γ_1, $\frac{\Gamma_1'}{\Gamma_0} - G$ shall be no longer omni-negative; then Γ_1 is accurately determined, and, proceeding in like manner with each group in succession, the whole system of groups becomes exactly defined, and thus we obtain the necessary and sufficient condition of group-representation.

Calling $$\frac{1}{\Gamma_0},\ \frac{\Gamma_1}{\Gamma_0},\ \frac{\Gamma_1\Gamma_3}{\Gamma_0},\ \frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2},\ \ldots\ \nu_0,\ \nu_1,\ \nu_2,\ \nu_3\ \ldots$$

respectively, the ν series of quantities stand to G in somewhat the same relation as the complete quotients of a continued fraction to its complete value. Observe that $\nu_0 - 1$, $\nu_1 - 1$, $\nu_2 - 1$, ... each vanish when the variables in G are each zero, and become infinite when the variables in G are each unity.

When each such variable has any value intermediate between 0 and 1, I think it almost certain that no two of the ν's can become equal, so that for all values of the variables inside those limits the parabolic lines or surfaces or hyper-surfaces, &c., represented (after introducing a new variable ω) by the equations $\omega - \nu_0 = 0$, $\omega - \nu_1 = 0$, $\omega - \nu_2 = 0, \ldots$ (which coincide for the limiting values of the original variables at the origin and at a point at infinity) will never intersect, so that within the prescribed limits $\nu_0 - \nu_2$, $\nu_2 - \nu_4$, $\nu_4 - \nu_6, \ldots$ will be always positive and $\nu_1 - \nu_3$, $\nu_3 - \nu_5$, ... will be always negative, the limited boundaries represented by

$$\omega - G,\quad \omega - \nu_0,\quad \omega - \nu_2,\quad \omega - \nu_4,\ \ldots$$

being each external to the one that precedes it on one side of $\omega - G$, and

$$\omega - G,\quad \omega - \nu_1,\quad \omega - \nu_3,\quad \omega - \nu_5,\ \ldots$$

following the same law on the other side. It is possible, moreover, that a more stringent condition than the above may be verified, namely, that

$$\nu_0 - G,\quad \nu_2 - \nu_0,\quad \nu_4 - \nu_2 \ldots$$
$$G - \nu_1,\quad \nu_1 - \nu_3,\quad \nu_3 - \nu_5 \ldots$$

may each be developable into omni-negative functions, and again (to complete the analogy with the parallel theory of continued fractions or converging continued products) that

$$\nu_0 - G,\quad G - \nu_1,\quad \nu_2 - G,\quad G - \nu_3,\quad \nu_4 - G, \ldots$$

shall form a single series of continually decreasing quantities, or even in their developed state, of functions in which the corresponding coefficients to each facient form a continually decreasing (or, at least, never-increasing) series of numbers. Then in the case of a single quantic, within the limits defined by the facient a being 0 and 1 the curves $\omega - \nu_1$, $\omega - \nu_3, \ldots \omega - G, \ldots \omega - \nu_2$, $\omega - \nu_0$, will form an infinite series of loops having one common asymptote and one common point of intersection, and except at that one point keeping clear of each other.

I annex tables (pp. [506, 507, below]) of the fundamental syzygants* or (if one pleases so to say) irreducible syzygies for the quintic and sextic, rendered more complete by inserting entries corresponding to the fundamental in- and- covariants. The positive integers correspond to these latter, the negative integers (the negative sign being set over the figure) to the irreducible syzygants. Thus, for example, in the table to the sextic the positive integer 2 found in the 6th line and 6th column, indicates that there are 2 ground-covariants of deg-order 6 . 6. The negative integer $\overline{7}$ found in the 12th line and 12th column indicates that there are 7 irreducible syzygies of deg-order 12 . 12†. The negative sign is appropriate, inasmuch as every independent syzygy of any deg-order lowers by a unit the number of linearly independent in- or- covariants of that deg-order that can be produced out of the inferior groundforms, so that syzygants may be regarded as negative existences in regard to groundforms: carrying on the same idea, counter-syzygants might be numbered by integers carrying two negative signs contradicting each other, and so on indefinitely.

* N.B.—A syzygant to a Quantic is a rational integer function of its in- or- covariants which, expressed as a function of the coefficients, vanishes identically, but we may still understand its "degree in the coefficients" to mean the degree of any one of the terms of which it is the sum.

† If j or e exceed the highest degree or order respectively found in any table, or, if without that being the case there is a blank space in the jth line and eth column of the table, the meaning is that there is no irreducible groundform or syzygy of the deg-order $j \,.\, e$. In the tables exhibited it will be seen that the deg-order $j' \,.\, e'$ of each syzygant is superior to the deg-order $j \,.\, e$ of every groundform: that is, the differences $j' - j$, $e' - e$ are neither of them less and one of them is greater than zero. The same is true for all quantics which have a finite Rep. G. F., but not necessarily and probably never actually so in other cases; thus, for example, to the seventhic belongs an irreducible invariant of degree 22 and an irreducible syzygy of degree 20, so that here the $j' \,.\, e'$ (20.0) is inferior to the $j \,.\, e$ (22.0). The fact of every $j' \,.\, e'$ being superior to the $j \,.\, e$ can be expressed by saying that the invariantive syzygetic portions of a Rep. G. F. table are not intermingled but lie totally apart and may be divided from each other by a single continuous cut.

The method of partitions or generating functions, which leads to these surprising constructions, looks at invariants and their connexions solely with regard to their deg-order or type without taking any account of their content; in other words it deals only with the *idea* or *notion* of these beings and their relations, and may therefore, I think, suitably be termed the Idealistic method*. I cannot see the faintest possibility of the symbolic method serving to determine a complete system of syzygies in any but the trivial cases of quantics of the 3rd or 4th order—the only cases where the infinite procession of beings (syzygants, counter-syzygants, anti-counter-syzygants, &c.), rising out of each other, comes to a stop—there being for those cases no procession after the 1st step, as is also true of invariants (as distinguished from covariants) for quantics of the 6th order. This is how it came to pass in the infancy of the theory that the number of ground-covariants was supposed to become infinite for quantics beyond the fourth and their ground-invariants for quantics beyond the 6th order.

I think it may be interesting to some of the readers of the *Journal* to be put in possession of the complete system of irreducible syzygies to a system of two or more quantics, and I select as an easy example the case of a combined quadratic and cubic, reserving the other combinations of which the groundform tables have been published for a subsequent number of the *Journal*. The supernumerary groundforms for the quadri-cubic system (see

* My proof in the *Phil. Trans.*, founded on the canonical form of the Quintic, of its 4th, 8th, 12th and 18th-degreed invariants forming a complete system, the late Mr Boole's discovery of the cubinvariant to the Quartic, the various disproofs in the *Comptes Rendus* and in this *Journal* of the existence of supposed groundforms, are all exemplifications of the Realistic point of view. The Symbolic lies between this and the Idealistic aspect of the subject, in so far as the operations by which invariants are engendered constitute a new and so to say finer subject-matter, capable of being itself operated upon in all respects like ordinary algebraical substance. In Professor Cayley's *Tenth Memoir on Quantics* there is a sort of half return from the Idealistic to the Realistic view—a kind of substantiality being attributed to the groundforms themselves as primary elements in the study of their syzygetic interconnections. It may be well to notice, for the benefit of the readers of that memoir (*Phil. Trans.* 1878), that in the Representative Form given at p. 657 two terms are omitted by an oversight, namely, $-a^{17}x^4$ and a^3x^{12}. I need hardly add (since the publication of my tables in this *Journal*), with reference to a doubt expressed by Prof. Cayley (*loc. cit.*), that I *had* obtained the form referred to in the paragraph following the R. G. F. in question, though *not* by dividing out the common factors from the numerator and denominator of the R. G. F.; on the contrary, the N. G. F. is first obtained from the generating function in its crude form (which if left in that form would lead to a bivergent series), and then the R. G. F. is obtained from this, through multiplying its numerator and denominator by the factors needed to render the denominator a product of representative groundforms.

The Symbolic and the Idealistic (which I formerly called the fatalistic or peprotic) method alike, as far as is known, owe their conception to the same (unnecessary to be named) acute and capacious intellect. Whether very much that is essential remains to be added to the great discoveries of Gordan and Jordan in the direction of the former may reasonably be doubted, but no such misgiving can be entertained with respect to the latter, which already has given rise to many more questions than it has settled (of a kind, too, of which a solution sooner or later may reasonably be anticipated).

this *Journal**, vol. II. pp. 295, 296), are of the deg-deg-orders 3 . 4 . 0, 1 . 1 . 1, 2 . 1 . 1, 1 . 3 . 1, 2 . 3 . 1, 1 . 2 . 2, 1 . 1 . 3, 0 . 3 . 3, where the first and second numbers express the degrees in the coefficients of the quadric and cubic respectively, and the last number expresses the order in the variables. Adding each of these triads to itself and every other, rejecting the combinations 2 . 2 . 2, 3 . 2 . 2, 2 . 4 . 2, which appear in the numerator of the G. F. (and arise from the additions 1 . 1 . 1 + 1 . 1 . 1, 1 . 1 . 1 + 2 . 1 . 1, 1 . 1 . 1 + 1 . 3 . 1), replacing them by the higher combinations 1 . 1 . 1 + 1 . 1 . 1 + 1 . 1 . 1, 1 . 1 . 1 + 1 . 1 . 1 + 2 . 1 . 1, 1 . 1 . 1 + 1 . 1 . 1 + 1 . 3 . 1, that is, 3 . 3 . 3, 4 . 3 . 3, 3 . 5 . 3, and adding in the 12 types furnished by the negative terms in the numerator of the G. F., the totality of the irreducible syzygies (48 in number) to the binary quadri-cubic system is arrived at and exhibited in the annexed table, in which the exponents attached to any type signify the number of irreducible syzygies of the corresponding deg-deg-order.

Table of Irreducible Syzygies to the Quadri-cubic System.

6 . 8 . 0,	4 . 5 . 1,	4 . 7 . 1,	5 . 5 . 1,	5 . 7 . 1,	2 . 6 . 2,	$(3 . 4 . 2)^2$,
$(3 . 6 . 2)^2$,	4 . 2 . 2,	$(4 . 4 . 2)^2$,	$(4 . 6 . 2)^2$,	1 . 5 . 3,	2 . 3 . 3,	$(2 . 6 . 3)^2$,
$(3 . 3 . 3)^2$,	$(3 . 5 . 3)^2$,	3 . 6 . 3,	3 . 7 . 3,	4 . 3 . 3,	4 . 5 . 3,	1 . 4 . 4,
1 . 6 . 4,	2 . 2 . 4,	$(2 . 4 . 4)^4$,	2 . 6 . 4,	3 . 2 . 4,	3 . 4 . 4,	3 . 6 . 4,
$(1 . 5 . 5)^2$,	2 . 3 . 5,	2 . 5 . 5,	3 . 5 . 5,	0 . 6 . 6,	1 . 3 . 6,	1 . 4 . 6,
2 . 2 . 6,	4 . 7 . 6,					

there being thus one irreducible invariantive syzygy and 4, 10, 12, 11, 5, 5 covariantive syzygies of orders 1, 2, 3, 4, 5, 6 respectively.

It may be worth while just to notice that the types to the complete system of irreducible syzygies to a simultaneous linear and quartic form will consist simply of the sums of the 13 supernumerary types, (*A. M. J.* vol. II. p. 295†), 6 . 3 . 0, 3 . 1 . 1, 3 . 2 . 1, 5 . 3 . 1, 2 . 1 . 2, 2 . 2 . 2, 4 . 3 . 2, 1 . 1 . 3, 1 . 2 . 3, 3 . 3 . 3, 2 . 3 . 4, 1 . 3 . 5, 0 . 3 . 6, added each to itself and every other, together with the 14 types taken from the negative terms in the numerator of the G. F., namely, 7 . 3 . 1, 6 . 3 . 2, 5 . 3 . 3, 4 . 3 . 4, 6 . 4 . 4, 6 . 5 . 4, 3 . 3 . 5, 5 . 4 . 5, 5 . 5 . 5, 2 . 3 . 6, 4 . 4 . 6, 4 . 5 . 6, 1 . 3 . 7, 7 . 6 . 7, making $\frac{13 \cdot 14}{2} + 14$, that is, 105 in all. In this instance there is no rejection or substitution of sums called for.

A word or two seems necessary to leave unambiguous the meaning of the term syzygants of any specified grade in what precedes.

In- or- covariants may be termed syzygants of grade zero (as already stated). Syzygants of the first grade are defined to be rational integer

[* p. 394, above.] [† p. 393, above.]

ORDER IN THE VARIABLES.

DEGREE IN THE COEFFICIENTS.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	DEGREE IN THE COEFFICIENTS.
1						1														1
2			1				1													2
3				1		1				1										3
4	1				1		1													4
5		1		1				1				$\bar{1}$								5
6			1		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{2}$				$\bar{1}$	6
7		1				1		$\bar{1}$		$\bar{3}$		$\bar{1}$		$\bar{2}$		$\bar{2}$				7
8	1		1				$\bar{2}$		$\bar{2}$		$\bar{3}$		$\bar{3}$				$\bar{1}$			8
9				1		$\bar{1}$		$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{3}$						9
10					$\bar{1}$		$\bar{3}$		$\bar{4}$		$\bar{5}$				$\bar{2}$					10
11		1				$\bar{4}$		$\bar{3}$		$\bar{3}$		$\bar{3}$								11
12	1				$\bar{2}$		$\bar{4}$		$\bar{5}$		$\bar{1}$		$\bar{2}$							12
13		1		$\bar{2}$		$\bar{3}$		$\bar{3}$		$\bar{4}$										13
14			$\bar{1}$		$\bar{2}$		$\bar{6}$		$\bar{1}$		$\bar{3}$									14
15				$\bar{2}$		$\bar{3}$		$\bar{4}$		$\bar{1}$										15
16					$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{2}$									16
17				$\bar{2}$		$\bar{3}$		$\bar{2}$		$\bar{1}$										17
18	1		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$									18
19				$\bar{2}$		$\bar{3}$		$\bar{1}$												19
20			$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$											20
21				$\bar{3}$		$\bar{1}$				$\bar{1}$										21
22			$\bar{1}$		$\bar{2}$		$\bar{1}$													22
23		$\bar{1}$		$\bar{1}$				$\bar{1}$				$\bar{1}$								23
24			$\bar{2}$		$\bar{1}$															24
25		$\bar{1}$				$\bar{1}$														25
26			$\bar{2}$																	26
27				$\bar{1}$																27
28																				28
29		$\bar{1}$																		29
30																				30
31		$\bar{1}$																		31
32																				32
33																				33
34																				34
35																				35
36	$\bar{1}$																			36

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

ORDER IN THE VARIABLES.

ORDER IN THE VARIABLES.

DEGREE IN THE COEFFICIENTS.	0	2	4	6	8	10	12	14	16	18	20	22	24	DEGREE IN THE COEFFICIENTS.
1				1										1
2	1		1		1									2
3		1		1	1		1							3
4	1		1	1		1								4
·5		1	1		1				$\bar{1}$					5
6	1			2	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$		$\bar{1}$	6
7		1	1	$\bar{1}$		$\bar{1}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{1}$		7
8		1			$\bar{2}$	$\bar{4}$	$\bar{2}$	$\bar{4}$	$\bar{3}$		$\bar{2}$			8
9			1		$\bar{4}$	$\bar{2}$	$\bar{5}$	$\bar{4}$		$\bar{3}$				9
10	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$					10
11					$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$						11
12		1	$\bar{1}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{7}$	$\bar{1}$	$\bar{1}$					12
13			$\bar{1}$	$\bar{2}$	$\bar{5}$	$\bar{5}$	$\bar{1}$	$\bar{3}$						13
14			$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{3}$							14
15	1		$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{2}$						15
16			$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	$\bar{2}$							16
17			$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{2}$		$\bar{1}$						17
18		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{4}$		$\bar{1}$							18
19			$\bar{2}$	$\bar{3}$		$\bar{1}$								19
20		$\bar{1}$	$\bar{3}$		$\bar{1}$				$\bar{1}$					20
21				$\bar{3}$										21
22		$\bar{1}$	$\bar{2}$											22
23		$\bar{1}$												23
24			$\bar{2}$											24
25		$\bar{1}$												25
26														26
27		$\bar{1}$												27
28														28
29														29
30	$\bar{1}$													30
	0	2	4	6	8	10	12	14	16	18	20	22	24	

ORDER IN THE VARIABLES.

functions of those of grade zero which vanish when the latter are expressed in terms of the original coefficients. It is not *necessary* to define *these* syzygants as functions of *irreducible* ones of grade zero (which vanish under the condition aforesaid), because every in- or- covariant is a rational integer function of the irreducible in- or- covariants. But when we come to syzygants of the second grade (since those of the first grade are not necessarily functions of the irreducible ones of that grade, but may be so of the in- or- covariants as well), it becomes necessary to define syzygants of the second grade (*aliter* counter-syzygants) as rational integer functions of *irreducible* ones of the first grade which vanish when they are expressed in terms of the quantities (here the in- or- covariants) which immediately precede them in the scale of generation. And so, in general, following out the defining process step by step, by a syzygant of the $(i+1)$th grade for the purpose of this theory, is to be understood a rational integer function of the *irreducible* ones of the ith grade which vanishes when these latter are expressed in terms of those of the grade $i-1$. Such at least is my present impression; but, supposing that I am labouring under a misconception on this point, it will in nowise affect the validity of the theory in what regards the computation of the irreducible in- or- covariants and the syzygants of the first grade.

60.

A DEMONSTRATION OF THE IMPOSSIBILITY OF THE BINARY OCTAVIC POSSESSING ANY GROUNDFORM OF DEG-ORDER 10.4.

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DR VON GALL has rendered an inestimable service to algebraical science by working out, according to Gordan's method, the complete system of groundforms to the octavian binary quantic $[(x, y)^8]$. His results, published in the *Mathematische Annalen*, were at first widely discordant from those which have appeared in this *Journal*, but eventually have been brought by their author into perfect agreement with them, with the sole exception that his table includes a covariant of *deg-order* 10.4, not included in my list, which he states that he has not been able to decompose: it is the object of the present communication to bring the two tables into exact accord by demonstrating that no irreducible covariant to $(x, y)^8$ of that deg-order can exist. The total number of covariants of deg-order 10.4 obtained by multiplying together the irreducible covariants of an inferior deg-order (which appear equally in von Gall's table and in my own, and whose existence therefore may be taken for granted*) will be seen to be 32, which is the number of linearly independent covariants of that deg-order given by Cayley's law, [see p. 525, below]; hence, by the fundamental postulate, the 32 compounds in question must not be supposed subject to any linear relation, so that, according to that postulate, there exists no groundform of the deg-order in question; but my object is to use this instance as another exemplification of the validity of that same very reasonable postulate—as I have done on the three former occasions where the tables of Clebsch, Gordan and Gundelfinger comprised groundforms extraneous to the tables obtained by me on the assumption of its truth; the proof, however, on the present occasion, is much lengthier than any that has ever hitherto been employed, and involves arithmetical computations of considerable prolixity,

* As I have elsewhere remarked, since no groundforms can exist exterior to the tables furnished by Gordan's method, and no reducible forms can be contained in the tables furnished by the English method, it follows, even without assuming the truth of the fundamental postulate, that wherever the tables furnished by the two methods accord, they must, of logical necessity, be correct, *mere errors of calculation excepted.*

all necessity for which I had, in previous cases, been able to evade. It is, I may add, only after repeated trials and discomfitures, that I have succeeded at length in devising a special method adequate to prove the important point at issue.

The irreducible invariants and covariants of deg-order *inferior* to 10.4, (that is, whose degree in the coefficients and whose order in the variables are not each as great as 10 and 4 respectively), and which also can enter as factors of a covariant of deg-order 10.4 (this excludes the necessity of considering invariants of degrees 9 or 10) are as follows: the invariants are of degrees 2, 3, 4, 5, 6, 7, 8, one of each degree; the covariants are one of deg-orders 5.2, 2.4, 3.4 respectively, and two of deg-orders 4.4, 5.4, 6.4, 7.4, 8.4 respectively. We may denote the invariants by 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, and the covariants by 5.2, 2.4, 3.4, 4.4, $4.4*$, 5.4, $5.4*$, 6.4, $6.4*$, 7.4, $7.4*$, 8.4, $8.4*$, and it is an easy arithmetical calculation to show (see† *Comptes Rendus*, July 25, 1881) that there are (as already stated) 32 different ways in which these duads, by their combination, can give rise to the duad 10.4. Out of these 32 it is important, with a view to what follows, to isolate those in which neither 2.0 nor 3.0 appears; their number will easily be seen to be 10, as shown in the scheme below—

$4.0+6.4$	$4.0+6.4*$	$4.0+4.0+2.4$	$5.0+5.4$	$5.0+5.4*$
$6.0+4.4$	$6.0+4.4*$	$7.0+3.4$	$8.0+2.4$	$5.2+5.2$.

What I have to prove is, that no equation $\Omega=0$ exists, where Ω is a linear function of the 32 products in question, connected by numerical coefficients. Suppose it can be shown that Ω does not contain any of the 10 functions above indicated. Then Ω is either of the form $(2.0)U$ or $(3.0)V$, or is a linear function of $(2.0)U$ and $(3.0)V$. In the former two cases we should obtain $U=0$ or $V=0$ respectively; and in the third case the equation $\lambda(2.0)U+\mu(3.0)V=0$, since 2.0 and 3.0 have no common factor, implies the existence of an integral equation $\lambda\dfrac{U}{(3.0)}+\mu\dfrac{V}{(2.0)}=0$. Hence, in the three cases supposed, there would exist a syzygy of the deg-order 8.4, 7.4, 5.4 respectively between composite covariants of the inferior deg-orders; but if this were so, the number of irreducible covariants of one or another of these deg-orders would not be what it is at present, but, in order to satisfy Cayley's law, would have to be increased by a unit: or, in other words, results obtained by my method and coincident with those resulting, or capable of resulting, from the German method, would be erroneous, which never can be the case‡. Hence, the non-existence of $\Omega=0$ will be demonstrated if it can be shown that, for some particular form of the general primitive $(x, y)^8$

[† p. 481, above.]

‡ Towards the end of this paper I establish the same conclusion by a more direct method, in which nothing extraneous to Dr von Gall's own table is assumed, except the one fact of the linearly independent covariants of deg-order 10.4 being 32 in number.

which causes the invariants of the second and third degrees each to vanish, the particular values then assumed by the 10 compounds which remain in Ω are not subject to any linear relation. Of course the converse would not be true; the fact of the existence of a syzygy between these 10, or even between the whole 32 compounds for a special form of the primitive, would not establish the existence of a syzygy between them in the general case.

The great practical gain of making the first two invariants vanish is that it leads to a computation in which only 10 instead of 32 linear functions have to be handled—but it is not possible *à priori* before the calculations have been gone through, to feel at all assured that the particular form assumed may not be such as to lead only to nugatory results. Such happily, however, turns out not to be the case with the form I am about to employ which leads to the expression of the 10 compounds as homogeneous linear functions of 11 arguments*, giving rise to a rectangular matrix 11 places wide and 10 places deep of which it can be shown that the complete minors (determinants of the 10th order) do not all vanish, so that the 10 functions cannot be subject to any syzygy; and consequently, if $\Omega = 0$ were a really existing syzygy, Ω must consist exclusively of 22 terms, every one of which contains one or both of the two first invariants; but this has been shown to be impossible, so that the non-evanescence of the minors referred to at once establishes the non-existence of a syzygy of deg-order 10 . 4, and, therefore, the non-existence of a *groundform of that deg-order*.

I take for the primitive the special form $(0, b, 2c, d, 0, 0, 0, 0, 1 \between x, y)^8$, that is to say, $8bx^7y + 56cx^6y^2 + 56dx^5y^3 + y^8$, with the relation $bd = 3c^2$, and proceed to form the required derivatives in conformity with von Gall's scheme of derivation. I use, as the best practical method of obtaining the "alliance" of the ith order between any two forms ϕ, ψ (of the orders μ, ν) denoted by $(\phi, \psi)_i$, the lineo-linear quadrinvariant (with respect to the variables of emanation) of the ith emanant of ϕ combined into a system with the ith emanant of ψ, taking care to reduce the result to the *parenthetical* form $(\ldots\ldots \between x, y)^{\mu+\nu-2i}$, containing only integer coefficients free from any common numerical factor. For the sake of brevity, too, I omit in general the symbolical factor containing (x, y): so that $(a_0, a_1, a_2, \ldots, a_i)$ will indicate the same thing as $(a_0, a_1, a_2, \ldots, a_i \between x, y)^i$. I shall adhere, in what follows, to the notation employed by Dr von Gall.

We have then, according to this notation,

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1) \tag{1}$$

$$\begin{aligned} i = (f, f)_4 &= (4bx^3y + 12cx^2y^2 + 4dxy^3)\, y^4 - 4dx^4 (bx^4 + 8cx^3y + 6dx^2y^2) \\ &\qquad + 3\,(2cx^4 + 4dx^3y)^2 \\ &= [12c^2 - 4bd, 16cd, 24d^2, 0, 0, 4b, 12c, 4d, 0][x, y]^8, \end{aligned}$$

* One of these arguments is itself a linear function of 3 combinations of the coefficients and variables, the total number of such combinations which appear in the 10 compounds being 13.

where the square bracket is employed to signify the same thing as would be indicated by the use of the round clamp, with the exception that the binomial coefficients are suppressed. We have, therefore, introducing the multipliers

$$\frac{14}{1},\ \frac{14}{8},\ \frac{14}{28},\ \frac{14}{56},\ \frac{14}{70},\ \frac{14}{56},\ \frac{14}{28},\ \frac{14}{8},\ \frac{14}{1},$$

$$\begin{aligned}
i &= (0,\ 28cd,\ 12d^2,\ 0,\ 0,\ b,\ 6c,\ 7d,\ 0) && (2)\\
k = (f, f)_6 &= (2bxy + 2cy^2)\,y^2 - 10d^2x^4 &&\\
&\equiv (\overline{20}d^2,\ 0,\ 0,\ b,\ 4c)^* && (3)\\
\Delta = (k, k)_2 &= \overline{20}d^2x^2\,(2bxy + 4cy^2) - b^2y^4 &&\\
&\equiv (0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2) && (4)\\
C = (k, k)_4 &= 20d^2\,.\,4c \equiv cd^2 && (5)\\
f_4 = (f, k)_4 &= 4c\,(4bx^3y + 12cx^2y^2 + 4dxy^3) - 4b\,(bx^4 + 8cx^3y + 6d)\,x^2y^2 &&\\
&\qquad + \overline{20}d^2y^4 &&\\
&= -\,4b^2x^4 - 16bcx^3y + (48c^2 - 24bd)\,x^2y^2 + 16cdxy^3 - 20d^2y^4 &&\\
&\equiv (b^2,\ bc,\ c^2,\ \overline{cd},\ 5d^2) && (6)\\
f_{k,\,2} = (f_4, k)_2 &= (b^2x^2 + 2bcxy + c^2y^2)\,(2bxy + 4cy^2) - 2by^2\,(bcx^2 + 2c^2xy - cdy^2) &&\\
&\qquad + \overline{20}d^2x^2\,(c^2x^2 - 2cdxy + 5d^2y^2) &&\\
&= [\overline{20}c^2d^2,\ 2b^3 + 40cd^3,\ 6b^2c + \overline{100}d^4,\ 6bc^2,\ 4c^3 + 2bcd]\,[x, y]^4 &&\\
&\equiv (\overline{120}c^2d^2,\ 3b^3 + 60cd^3,\ 6b^2c + 100d^4,\ 9bc^2,\ 60c^3) && (7)\\
f_{k,\,3} = (f_4, k)_3 &= (b^2x + bcy)\,(bx + 4cy) - 3by\,(bcx + c^2y) - \overline{20}d^2x\,(\overline{cd}x + 5d^2y) &&\\
&= (b^3 + \overline{20}cd^3)\,x^2 + (2b^2c + 100d^4)\,xy + bc^2y^2 &&\\
(f_{k,\,3})^2 &= (b^6 + \overline{40}b^3cd^3 + 400c^2d^6)\,x^4 + (4b^5c\,[\overline{80}b^2c^2d^3 + 200b^3d^4] - 4000cd^7)\,x^3y &&\\
&\qquad + (6b^4c^2 + 400b^2cd^4 - 40bc^3d^3 + 10000d^8)\,x^2y^2 &&\\
&\qquad + (4b^3c^3 + 200bc^2d^4)\,xy^3 + b^2c^4y^4 &&\\
&= [b^6 + \overline{1080}c^7 + \overline{400}c^2d^6,\ 4b^5c + 4680c^6d + \overline{4000}cd^7, &&\\
&\qquad 6b^4c^2 + 3480c^5d^2 + 10000d^8,\ 4b^3c^3 + 600c^4d^3,\ b^2c^4]\,[x, y]^4 &&\\
&\equiv (3b^6 + \overline{3240}c^7 + 1200c^2d^6,\ 3b^5c + 3510c^6d + \overline{3000}cd^7, &&\\
&\qquad 3b^4c^2 + 1740c^5d^2 + 5000d^8,\ 3b^3c^3 + 450c^4d^3,\ 3b^2c^4) && (8)\\
(f_\Delta) = (f, \Delta)_4 &= 3b^2\,(4bx^3y + 12cx^2y^2 + 4dxy^3) + 6\,(40cd^2)\,(2cx^4 + 4dx^3y) &&\\
&\qquad - 120bd^2\,(dx^4) &&\\
&= [480c^2d^2 - 120bd^3,\ 12b^3 + 960cd^3,\ 36b^2c,\ 12b^2d,\ 0]\,[x, y]^4 &&\\
&\equiv (40c^2d^2,\ b^3 + 80cd^3,\ 2b^2c,\ 3bc^2,\ 0) && (9)\\
i_\Delta = (i, \Delta)_4 &= -\,120bd^2\,(6bx^2y^2 + 24cxy^3 + 7dy^4) &&\\
&\qquad + 240cd^2\,(12d^2x^4 + 4bxy^3 + 6cy^4) + 3b^2\,(112cdx^3y + 72d^2x^2y^2) &&\\
&= [2880cd^4,\ 336b^2cd,\ \overline{504}b^2d^2,\ \overline{1920}bcd^2,\ 1440c^2d^2 &&\\
&\qquad + \overline{840}bd^3]\,[x, y]^4 &&\\
&\equiv (240cd^4,\ 21bc^3,\ \overline{63}c^4,\ \overline{120}c^3d,\ \overline{90}c^2d^2) && (10)
\end{aligned}$$

* The sign of equivalence ($\equiv$) is used in the above and in what follows in the sense of "may be superseded by."

$$\begin{aligned} i_4 = (i, k)_4 &= \overline{20}d^2(4bx^3y + 36cx^2y^2 + 28dxy^3) - 4\,.\,b\,(28cdx^4 + 48d^2x^3y + by^4) \\ &\qquad + (4c)(112cdx^3y + 72d^2x^2y^2) \\ &= [\overline{112}bcd,\ 448c^2d + \overline{272}bd^2,\ \overline{432}cd^2,\ \overline{560}d^3,\ \overline{4}b^2][x, y]^4 \\ &\equiv (336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2) \qquad (11) \end{aligned}$$

$$\begin{aligned} i_{k,\,2} = (i_4, k)_2 &= \overline{20}d^2x^2(72cd^2x^2 + 280d^3xy + 4b^2y^2) \\ &\qquad - 2by^2(92c^2dx^2 + 144cd^2xy + 140d^3y^2) \\ &\qquad + (2bxy + 4cy^2)(336c^3x^2 + 184c^2dxy + 72cd^2y^2) \\ &= [\overline{1440}cd^4,\ 672bc^3 + \overline{5600}d^5,\ 184bc^2d - 80b^2d^2 + 1344c^4, \\ &\qquad 736c^3d + \overline{144}bcd^2,\ 288c^2d^2 + \overline{280}bd^3][x, y]^4 \\ &\equiv (\overline{360}cd^4,\ 42bc^3 + \overline{350}d^5,\ 49c^4,\ 19c^3d,\ 138c^2d^2) \qquad (12) \end{aligned}$$

$$\begin{aligned} f_{k,\,\Delta} = (f_4, \Delta)_4 &= -4\,(30bd^2)(\overline{cd}) + 6\,(40cd^2)\,c^2 + 3b^2\,.\,b^2 \\ &= 3b^4 + 120bcd^3 + 240c^3d^2 \\ &\equiv b^4 + 200c^3d^2 \qquad (13) \end{aligned}$$

$$\begin{aligned} i_{k,\,\Delta} = (i_4, \Delta)_4 &= -4\,(30bd^2)(140d^3) + 6\,(40cd^2)(72cd^2) + (3b^2)(336c^3) \\ &= 1008b^2c^3 + \overline{33120}c^2d^4 \equiv 7b^2c^3 - 230c^2d^4. \end{aligned}$$

The term involving c^2d^4 being a multiple of the square of C (the invariant of the 4th degree) may be neglected, and, instead of $i_{k,\,\Delta}$, we may write the irreducible invariant of the 8th degree (say)

$$I_8 = b^2c^3. \qquad (14)$$

That of the 7th degree we have just found $= b^4 + 200c^3d^2$; and obviously the quadrinvariant of f is identically zero, or say

$$I_2 = 0. \qquad (15)$$

Also the cubinvariant $I_3 = (f, i)_8$, where

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1)$$

and $\qquad i = (0, 28cd, 12d^2, 0, 0, b, 6c, 7d, 0).$

Hence $\qquad I_3 = -56bd + 336c^2 - 168bd = 504c^2 - 168bd = 0, \qquad (16)$

and we have found $I_4 \equiv cd^2$.

Also, $I_5 = (f, k^2)_8$ where

$$\begin{aligned} k^2 &= (10d^2x^4 - 2bxy^3 - 2cy^4)^2 \\ &= 100d^4x^8 - 40bd^2x^5y^3 - 40cd^2x^4y^4 + 4b^2x^2y^6 + 8bcxy^7 + 4c^2y^8. \end{aligned}$$

Hence $\qquad I_5 = 100d^4 + 2c\,.\,4b^2 - b\,.\,8bc \equiv d^4$*.

* It will of course be recognized that the lineo-linear quadrinvariant to the system

$$(a_0, a_1, a_2, \ldots a_i)(x, y)^i,\ [b_0, b_1, b_2, \ldots b_i][x, y]^i$$

is simply $\qquad a_0b_i - a_1b_{i-1} + a_2b_{i-2} \ldots \pm a_ib_0$:

the disappearance of the argument b^2c from companionship with d^4 in I_5 is rather remarkable, and could not have been predicted. This circumstance considerably simplifies the subsequent calculations.

The only remaining invariant required for present purposes is I_6, represented by $(i_4, k)_4$ where

$$k = [\overline{10}d^2, 0, 0, 2b, 2c][x, y]^4,$$

and $$i_4 = (336c^3, 92c^2d, 72cd^2, 140d^3, 4b^2 \between x, y)^4.$$

Hence $$I_6 = \overline{40}b^2d^2 - (2b)\, 92c^2d + 2c\,(336c^3)$$

$$= (-360 - 552 + 672)\, c^4 \equiv c^4.$$

On proceeding to form the 10 compound covariants of deg-order 10.4 obtained by suitable combinations of the invariants and covariants of inferior deg-order, it will be found that the following 13 arguments will make their appearance, in which, for greater brevity, x and y are each taken equal to unity, which in nowise affects (favourably or unfavourably) the course of the reasoning: these arguments are

$$b^6,\ c^7,\ c^2d^6;\ b^5c,\ c^6d,\ cd^7;\ b^4c^2,\ c^5d^2,\ d^8;\ b^3c^3,\ c^4d^3;\ b^2c^4,\ c^3d^4,$$

where the 5 groups of arguments, separated from one another by semicolons, are elements of the coefficients of x^4, x^3y, x^2y^2, xy^3, y^4, and when supplemented by such powers of k (of weight 8) as will bring their degrees up to the number 10, are of the respective weights 38, 39, 40, 41, 42, which is right, since the weight of the differentiant of deg-order 10.4 to $(x, y)^8$ is $\dfrac{10.8-4}{2}$, that is, 38; for greater brevity (in what precedes) k, the coefficient of y^8 in f, has been made unity, and it is worthy of notice that all the arguments that can appear consistently with the law of weight are represented by these 13, upon the understanding that any power of bd in an argument is replaceable by the like power of c^2.

But it is further noticeable that the 10 compounds in question, although apparently linear functions of 13 arguments, are virtually such of only 11; for it will be seen that $b^6 + 4b^5c + 6b^4c^2$ may be regarded as a single argument, none of the three simpler arguments which appear in it occurring except in two of the 10 compounds, and their coefficients in each of those two being in the ratio 1 : 4 : 6.

Had the contraction in the number of really independent arguments extended two steps further, so that the 10 compounds had been linear functions of only 9 quantities (as might, for anything that could be known *à priori*, have been the case), they would necessarily have been linearly connected, and no inference could have been drawn from the particular value assigned to f: moreover, had the 10 compounds been linear functions of only 10 quantities, although the particular form might have been sufficient for drawing a positive inference as to the non-existence of the general syzygy $\Omega = 0$, still there would have been no room for applying the all-important *test* of the correctness of the arithmetical computations upon which that inference would have reposed; and it would have been very

unsatisfactory and unphilosophical to have made so important a conclusion rest upon the negative fact of a determinant of the 10th order *not vanishing,* when the undisproved existence of a single error committed in the many hundreds (or even—it might be said—thousands) of arithmetical steps involved in the calculations of the elements of that determinant would have been sufficient to account for its value differing from zero.

Fortunately, as will be seen, the correctness of the calculations may be *verified* (thanks to the existence of elements one more than barely sufficient —namely, 11 instead of 10) by the *positive* fact of a certain determinant of the 11th order being found equal to zero. It has often seemed to me that a special providence or pre-established harmony in the intellectual world brings it about that honest labour, persevering in the pursuit of an important truth in the face of doubts and difficulties and repeated disappointments, shall not in the end lose its due reward†.

Let us now denote the quantities $b^6 + 4b^5c + 6b^4c^2$, c^7, c^2d^6; $4c^6d$, $4cd^7$; $6c^5d^2$, $6d^8$; $4b^3c^3$, $4c^4d^3$; b^2c^4, c^3d^4 by A, β, γ, δ, ϵ, ζ, η, θ, κ, λ, μ, respectively, and denote the covariants of the order 4 that have been calculated in what precedes according to their deg-order—namely, let us call

$(f_{k,3})^2$; $i_{k,2}$; i_Δ; $f_{k,2}$; f_Δ; Δ; i_4; f_4; k

10.4; 6.4; 6.4*; 5.4; 5.4*; 4.4; 4.4*; 3.4; 2.4 respectively,

then the values of 10.4, $I_4 \times 6.4$, $I_4 \times 6.4*$, $I_5 \times 5.4$, $I_5 \times 5.4*$, $I_6 \times 4.4$, $I_6 \times 4.4*$, $I_7 \times 3.4$, $I_4^2 \times 2.4$, $I_8 \times 2.4$, will be as shown in the table annexed

A	β	γ	δ	ϵ	ζ	η	θ	κ	λ	μ	
3	$\overline{3240}$	1200	3510	$\overline{3000}$	1740	5000	3	450	3	.	.........(1)
.	.	$\overline{360}$	126	$\overline{350}$	49	.	.	19	.	$\overline{138}$	.........(2)
.	.	240	63	.	$\overline{63}$	.	.	$\overline{120}$	.	$\overline{90}$	.........(3)
.	.	$\overline{120}$	81	60	54	$\overline{100}$	.	27	.	60	.........(4)
.	.	40	27	80	18	.	.	9	.	.	.........(5)
.	.	.	90	.	40	.	.	.	3	.	.........(6)
.	336	.	92	.	72	.	.	140	4	.	.........(7)
1	1800	.	600	.	200	.	$\overline{3}$	$\overline{200}$	45	1000	.........(8)
.	.	$\overline{20}$	.	.	.	.	.	3	.	4	.........(9)
.	$\overline{180}$	.	.	.	.	.	1	.	4	.	.......(10)

Line (1) of course signifies $3A - 3240\beta + \ldots + 3\lambda$,

(2) .... $- 360\gamma + 126\delta \ldots - 138\mu$,

† I began with taking as a special form $ax^8 + by^8 + cz^8$, with the relation $x + y + z = 0$ (which, like the form f, contains two arbitrary ratios), and went through the very considerable labour of calculating all its inferior derivatives capable of entering into the composition of a covariant of deg-order 10 . 4, but the result turned out altogether nugatory.

and so for all the other lines, each being a linear function of the 11 quantities $A, \beta, \ldots, \lambda, \mu$.

If these 10 linear functions are linearly connected, all the *complete* minors of the rectangular matrix (11 by 10) must vanish.

It is not so difficult as it might at first sight appear, to calculate the actual value of any one of these minors, convenient combinations of the lines and columns having been previously effected; this arises from the number of zeros which appear in the matrix. Mr Morgan Jenkins, of the London Mathematical Society, and myself actually calculated two of them in the course of an hour or two; but the same object may be reached more expeditiously and quite as satisfactorily by proving that the minors do not vanish in respect to some judiciously or fortunately chosen modulus. I find that the number 11, taken as modulus, will accomplish the end in view. It will be found convenient to change the order of sequence of the lines and columns; to take the lines in the order 1, 8, 4, 10, 7, 6, 9, 5, 3, 2, and the columns in the order $A, \eta, \theta, \beta, \lambda, \gamma, \delta, \epsilon, \zeta, \kappa, \mu$. These transpositions having been effected, and the least positive residue of each element in respect to 11 being substituted in place of the element, the rectangular matrix above given will be replaced by the following:

3	6	3	5	3	1	1	3	2	10	.
1	.	8	7	1	.	6	.	2	9	10
.	10	.	.	.	1	4	5	10	5	5
.	.	1	7	4	.	.	.	.	.	.
.	.	.	6	4	.	4	.	6	8	.
.	.	.	.	3	.	2	.	7	.	.
.	.	.	.	.	2	.	.	.	3	4
.	.	.	.	.	7	5	3	7	9	.
.	.	.	.	.	9	8	.	8	1	9
.	.	.	.	.	3	5	2	5	8	5

It is easy to see that by proceeding as if to eliminate A between the two first lines, then β between the new line so formed and the third line, then γ between the new line again so formed and the fourth line, and so on, (always substituting the remainders to modulus 11 in lieu of the numbers themselves that arise in the process,) the first six lines may be replaced successively by the six following:

3	6	3	5	3	1	1	3	2	10	.
	5	10	5	.	10	6	8	4	6	8
		10	5	.	4	4	.	10	9	.
			10	7	7	7	.	1	2	.
				9	2	9	.	10	2	.
					5	2	.	.	5	.

Consequently, it only remains to ascertain whether the complete minors all disappear in the matrix of the dimensions (6 × 5) given below, namely:

$$\begin{array}{cccccc} 5 & 2 & . & . & 5 & . \\ 2 & . & . & . & 3 & 4 \\ 7 & 5 & 3 & 7 & 9 & . \\ 9 & 8 & . & 3 & 1 & 9 \\ 3 & 5 & 2 & 5 & 8 & 5 \end{array}$$

If all the complete minors of this matrix contain 11, the same must be true of the determinant formed by subtracting the first column in the above from the fifth and substituting the difference in place of the fifth column, that is,

$$\begin{vmatrix} 2 & . & . & . & . \\ . & . & . & 1 & 4 \\ 5 & 3 & 7 & 2 & . \\ 8 & . & 3 & 3 & 9 \\ 5 & 2 & 5 & 5 & 5 \end{vmatrix} \text{ and therefore } \begin{vmatrix} . & . & 1 & 4 \\ 3 & 7 & 2 & . \\ . & 3 & 3 & 9 \\ 2 & 5 & 5 & 5 \end{vmatrix}$$

should contain 11, and (as we may see by substituting the excess of 4 times the 3rd column over the fourth in place of the 3rd) the same must be true of

the determinant $\begin{vmatrix} 3 & 7 & 8 \\ . & 3 & 3 \\ 2 & 5 & 4 \end{vmatrix}$ of which the value is $3(12-15)+2(21-24)$,

that is, -15, and as this does not contain 11, it follows that the complete minors of the matrix which expresses the 10 compounds as linear functions of the 11 arguments $A, \beta, \gamma \ldots \lambda, \mu$ are not all zero, and they are consequently not linearly connected*. But, obviously, the calculations on which this proof depends imperatively call for a verification, as nothing would be more easy than to bring out some or all of the minors different from zero by a single error of calculation or slip of the pen. To this end I calculate the

* In the *Comptes Rendus* for 22nd August of this year, I have given a brief *résumé* of the contents of this paper. At page 367 of that fascicule, (third line from foot†), in the last line but one of the matrix, I have written 9 8 . 8 1 9 in error for 9 8 . 3 1 9 (having mistaken a 3, *covered with a blot*, for 8); consequently, the calculations which follow page 368 of the *C. R.* are erroneous. Fortunately, I did not repeat the mistake in calculating the value of the determinant subsequently given of the 11th order, in proving that it contains the divisor 11. Moreover, this determinant, or rather its remainder to modulus 11, has been calculated by an entirely different process by Mr Morgan Jenkins (whose work is before my eyes), and with the same result of its being divisible by 11. This instance shows how unsafe it would have been to have trusted to the fact of the minors not vanishing, unsupported by the positive evidence which the determinant of the 11th order affords of the preceding calculations, as regards the values of the groundforms, being unaffected with *one single error* in spite of the vast number of processes of addition, subtraction, multiplication, division, transposition, transcription and change of sign employed in working them out.

[† p. 486, above.]

value of von Gall's undecomposed covariant for the assumed special form f, and shall show that the 10 compounds and this 11th function do become linearly connected, that is, subject to a syzygy, on the assumption that the arithmetical values of the coefficients have been correctly calculated.

The function in question, Dr von Gall's i_4'', is obtained as follows: $i'' = (i,\ \Delta)_2$ of deg-order 6 . 8 is equal to

$$\begin{aligned}
&(168cdx^5y + 180d^2x^4y^2 + 6bxy^5 + 6cy^6)\,(40cd^2x^2 + 3b^2y^2) \\
&- (180c^2dx^2 + 160cd^2xy)\,(28cdx^6 + 72d^2x^5y + 15bx^2y^4 + 36cxy^5 + 7dy^6) \\
&+ (12d^2x^6 + 20bx^3y^3 + 90cx^2y^4 + 42dxy^5)\,(180c^2dxy + 40cd^2y^2) \\
&= [\overline{5040}c^3d^2,\ \overline{8560}c^2d^3,\ \overline{3840}cd^4,\ 504b^2cd,\ 7560c^4,\ 5640c^3d,\ 4380c^2d^2, \\
&\qquad\qquad 18b^3 + 560cd^3,\ 18b^2c]\,.\,[x,\ y]^8
\end{aligned}$$

which, multiplied by 28, will be seen to be equivalent to

$$\begin{aligned}
&(\overline{141120}c^3d^2,\ \overline{29960}c^2d^3,\ \overline{3840}cd^4,\ 756bc^3,\ 3024c^4,\ 2820c^3d,\ 4380c^2d^2, \\
&\qquad\qquad 63b^3 + 1960cd^3,\ 504b^2c).
\end{aligned}$$

Finally,

$$\begin{aligned}
i_4'' = (i'',\ \Delta)_4 &= 3b^2\,(\overline{141120}c^3d^2x^4 + \overline{119840}c^2d^3x^3y + \overline{23040}cd^4x^2y^2 \\
&\qquad\qquad + 3024bc^3xy^3 + 3024c^4y^4) \\
&+ 6\,.\,40cd^2\,(\overline{3840}cd^4x^4 + 3024bc^3x^3y + 18144c^4x^2y^2 + 11280c^3dxy^3 + 4380c^2d^2y^4) \\
&- 4\,.\,90c^2d\,[756bc^3x^4 + 12096c^4x^3y + 16920c^3dx^2y^2 + 17520c^2d^2xy^3 \\
&\qquad\qquad + (63b^3 + 1960cd^3)]\,y^4
\end{aligned}$$

which, dividing out by 144,

$$\begin{aligned}
&\equiv (\overline{32130}c^7 + \overline{6400}c^2d^6)\,x^4 + \overline{37590}c^6dx^3y + \overline{16380}c^5d^2x^2y^2 \\
&\qquad\qquad + (63b^3c^3 + \overline{25000}c^4d^3)\,xy^3 + \left(\frac{\overline{819}}{2}\,b^2c^4 + 2400c^3d^4\right)y^4 \\
&\equiv (\overline{128520}c^7 + \overline{25600}c^2d^6,\ \overline{37590}c^6d,\ \overline{10920}c^5d^2,\ 63b^3c^3 + \overline{25000}c^4d^3, \\
&\qquad\qquad \overline{1638}b^2c^4 + 9600c^3d^4).
\end{aligned}$$

Here it will be noticed that the arguments collected in what I have designated by A, namely, b^6, b^5c, b^4c^2, do not appear at all in i_4''. Had they made their appearance with other than coefficients bearing to each other the ratios of $1:4:6$, i_4'' could not have been a linear function of the 10 compounds which are linear functions of A and of 10 other arguments. This is in itself, to some extent, a verification of a portion at least of the preceding calculations: i_4'', as it turns out, is a linear function of only 8 out of the 11 arguments which appear in the other 10 compound covariants, namely, of β, γ, δ, ζ, θ, κ, λ, μ, neither A, ϵ nor η appearing in i_4''.

If the figuring throughout is correct, the determinant represented by the matrix constituted of the coefficients of the 11 compounds, ought to vanish identically; but it will be sufficient for all reasonable purposes (that is, to

satisfy any reasonable doubts on the subject) if I show that this is the case for the value of that determinant in respect to three consecutive prime numbers 11, 13, 17 taken almost at hazard.

It must be understood that the vanishing of the determinant in question adds *no additional strength whatever* to the proof—which, by Cayley's law, is perfect without it—provided that the figures in the coefficients of the 10 compounds (excluding i_4'') have been correctly calculated. It is to authenticate these figures, and not to verify the legitimacy of the argument, that the 11th compound is calculated, and the determinant formed by all the eleven shown to contain any number taken at will. It must be remembered that the calculations have been most carefully conducted and verified at each step: consequently, if any person, after the evidence that will be given, entertains any doubt of the correctness of the result, the duty is incumbent on him to put his finger upon some one of the coefficients of the 10 first compounds and prove it to be incorrectly stated.

First, for the modulus 11. In respect to this modulus, the coefficients in i_4'' of

$$\begin{array}{ll} & A,\ \eta,\ \theta,\ \beta,\ \lambda,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \kappa,\ \mu \\ \text{are congruous to} & 0,\ 0,\ 8,\ 4,\ 1,\ 8,\ 8,\ 0,\ 3,\ 3,\ 8. \end{array}$$

Hence, (making use of the transformations already calculated of the upper half of the rectangular matrix), it has to be shown that 11 is a divisor of the determinant of the 9th order

$$\begin{array}{ccccccccc}
8 & 4 & 1 & 8 & 8 & . & 3 & 3 & 8 \\
10 & 5 & . & 4 & 4 & . & 10 & 9 & . \\
 & 10 & 7 & 7 & 7 & . & 1 & 2 & . \\
 & & 9 & 2 & 9 & . & 10 & 2 & . \\
 & & & 5 & 2 & . & . & 5 & . \\
 & & & 2 & . & . & . & 3 & 4 \\
 & & & 7 & 5 & 3 & 7 & 9 & . \\
 & & & 9 & 8 & . & 3 & 1 & 9 \\
 & & & 3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

The first and second lines of this matrix combined give rise to the line $1\ \ 7\ \ 7\ \ .\ \ 6\ \ 9\ \ 8$, and this, combined with the 4th, to the line $5\ \ 1\ \ .\ \ .\ \ 9\ \ 5$ under which last, writing the 5 remaining lines

$$\begin{array}{cccccc}
5 & 1 & . & . & 9 & 5 \\
5 & 2 & . & . & 5 & . \\
2 & . & . & . & 3 & 4 \\
7 & 5 & 3 & 7 & 9 & . \\
9 & 8 & . & 3 & 1 & 9 \\
3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

it has to be shown that the determinant to the above matrix of the 6th order contains 11.

Let the fourth line be replaced by 3 times itself + the last line, which, to the modulus 11, reduces the third column to the form of five zeros followed by 2. This shows that we may use, instead of the above, the determinant

$$\begin{array}{ccccc} 5 & 1 & . & 9 & 5 \\ 5 & 2 & . & 5 & . \\ 2 & . & . & 3 & 4 \\ 2 & 9 & 4 & 2 & 5 \\ 9 & 8 & 3 & 1 & 9 \end{array};$$

and again, replacing the fourth line of the new matrix by its double + the last line, we fall upon the matrix

$$\begin{array}{cccc} 5 & 1 & 9 & 5 \\ 5 & 2 & 5 & . \\ 2 & . & 3 & 4 \\ 2 & 4 & 5 & 8 \end{array},$$

for which we may substitute

$$\begin{array}{cccc} 5 & 1 & 4 & 5 \\ 5 & 2 & . & . \\ 2 & . & 1 & 4 \\ 2 & 4 & 3 & 8 \end{array},$$

or (as may be seen by replacing the second column by 3 times itself + the first column) $\begin{vmatrix} 8 & 4 & 5 \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$, in which (to modulus 11) the first line is 4 times the second. Hence, the test is satisfied as regards the modulus 11.

I will next take the modulus 13.

The residues to modulus 13 of the coefficients in i_4'' of

$$\begin{array}{ccccccccc} \theta & \beta & \lambda & \gamma & \delta & \epsilon & \zeta & \kappa & \mu \\ 11 & 11 & . & 10 & 6 & . & . & 12 & 6 \end{array}$$

will be seen to be

and the matrix corresponding to the one of the same dimensions (11×10), previously calculated for modulus 11, will, in respect to modulus 13, become

$$\begin{array}{ccccccccccc} 3 & 8 & 3 & 10 & 3 & 4 & . & 3 & 11 & 8 & . \\ 1 & . & 10 & 6 & 6 & . & 2 & . & 5 & 8 & 12 \\ & 4 & . & . & . & 10 & 3 & 8 & 2 & 1 & . \\ & & 1 & 2 & 4 & . & . & . & . & . & . \\ & & & 11 & 4 & . & 1 & . & 7 & 10 & . \\ & & & & 3 & . & 12 & . & 1 & 1 & . \\ & & & & & 6 & . & . & . & 3 & 4 \\ & & & & & 1 & 1 & 2 & 5 & 9 & . \\ & & & & & 6 & 11 & . & 2 & 10 & 1 \\ & & & & & 4 & 9 & 1 & 10 & 6 & 5 \end{array}.$$

In place of the first six of the above lines, applying the same process as before, we may substitute

$$\begin{array}{ccccccccccc}
3 & 8 & 3 & 10 & 3 & 4 & . & 3 & 11 & 8 & . \\
 & 5 & 1 & 8 & 2 & 9 & 5 & 10 & 4 & 3 & 10 \\
 & & 9 & 7 & 5 & 1 & 8 & . & 7 & 6 & 12 \\
 & & & 11 & 5 & 12 & 5 & 12 & 6 & 7 & 1 \\
 & & & & 2 & 11 & 8 & 11 & 11 & 7 & 2 \\
 & & & & & 6 & . & 7 & 8 & 7 & 7.
\end{array}$$

Combining the i_4'' line (that is, the coefficients of $\theta\ \beta\ \lambda \ldots \mu$ in i_4'' above given) with the third of these, we obtain the line

$$\begin{array}{ccccccccc} . & 4 & 3 & 12 & 8 & . & 12 & 10 & . \end{array}$$

which, again combined with the fourth of the same, gives rise to the line

$$\begin{array}{cccccc} 7 & 10 & 9 & 9 & 9 & 4. \end{array}$$

Adding on the sixth line, namely $\begin{array}{cccccc} 6 & . & 7 & 8 & 7 & 7 \end{array}$ and the four last lines of the first matrix, namely, the lines marked with an asterisk,

$$\begin{array}{cccccc}
*6 & . & . & . & 3 & 4 \\
*1 & 1 & 2 & 5 & 9 & . \\
*6 & 11 & . & 2 & 10 & 1 \\
*4 & 9 & 1 & 10 & 6 & 5,
\end{array}$$

the arithmetical problem to be solved reduces itself to showing that the above determinant vanishes to modulus 13.

Substituting for the 1st column twice the 1st less three times the 6th, and for the 5th column twice the 5th less the 1st, and neglecting the factor 3, we fall upon the determinant

$$\begin{vmatrix}
2 & 10 & 9 & 9 & 11 \\
4 & . & 7 & 8 & 8 \\
2 & 1 & 2 & 5 & 4 \\
9 & 11 & . & 2 & 1 \\
6 & 9 & 1 & 10 & 8
\end{vmatrix}
\quad \text{or} \quad
\begin{vmatrix}
2 & 10 & 9 & 9 & 2 \\
4 & . & 7 & 8 & . \\
2 & 1 & 2 & 5 & 12 \\
9 & 11 & . & 2 & 12 \\
6 & 9 & 1 & 10 & 11
\end{vmatrix}.$$

Then in this last, substituting for the 4th column the 4th less twice the 1st, say M, and for the 3rd column 5 times the 1st less the 3rd, say N, we descend in like manner upon the determinant

$$\begin{array}{cccc}
5 & 2 & 2 & 1 \\
1 & 2 & 12 & 8 \\
10 & 9 & 12 & 6 \\
11 & 6 & 11 & 3
\end{array}$$

where the 1st column is the M with the zero in it left out, and the 4th column the N with the zero in it left out.

This, by elimination (so to say) of the first variable to the left between the successive pairs of lines, gives rise to the determinant

$$\begin{array}{ccc} 8 & 6 & . \\ 2 & 9 & 4 \\ . & 4 & 3 \end{array}$$

which (to modulus 13) $\equiv 8 \,.\, 1 - 8 \,.\, 3 - 6 \,.\, 6 \equiv 8 - 11 - 10 \equiv 0$.

It remains only to apply the 3rd proposed test, using 17 as the modulus.

The i_4'' line here becomes

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \end{array}$$

and the grand rectangular matrix becomes

$$\begin{array}{ccccccccccc} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . \\ 1 & . & 14 & 15 & 11 & . & 5 & . & 13 & 4 & 14 \\ & 2 & . & . & . & 16 & 13 & 9 & 3 & 10 & 9 \\ & & 1 & 7 & 4 & . & . & . & . & . & . \\ & & & 13 & 4 & . & 7 & . & 4 & 4 & . \\ & & & & 3 & . & 5 & . & 6 & . & . \end{array}$$

with 4 more lines, which will be presently supplied in their proper place. For those above written may be substituted

$$\begin{array}{ccccccccccc} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . \\ & 15 & 5 & 4 & 13 & 7 & 7 & 8 & 16 & 4 & 8 \\ & & 7 & 9 & 8 & 5 & 11 & . & 13 & 6 & . \\ & & & 6 & 3 & 12 & 6 & . & 4 & 11 & . \\ & & & & 2 & 14 & 15 & . & 6 & . & . \\ & & & & & 9 & 16 & . & 11 & . & . \end{array}$$

Rejecting the first two lines, and writing over the remaining ones the i_4'' line, there results

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \\ 7 & 9 & 8 & 5 & 11 & . & 13 & 6 & . \\ & 6 & 3 & 12 & 6 & . & 4 & 11 & . \\ & & 2 & 14 & 15 & . & 6 & . & . \\ & & & 9 & 16 & . & 11 & . & . \end{array}$$

which may be replaced by

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \\ & 6 & 2 & 12 & . & . & 11 & 6 & 1 \\ & & 6 & . & 2 & . & 9 & 13 & 11 \\ & & & 16 & 1 & . & 1 & 8 & 12 \\ & & & 9 & 16 & . & 11 & . & . \end{array}$$

to which I add in the 4 pretermitted lines distinguished by asterisks,

$$\begin{array}{cccccc} {}^{*}14 & . & . & . & 3 & 4 \\ {}^{*}\ 6 & 10 & 12 & 1 & 9 & . \\ {}^{*}\ 2 & 12 & . & 5 & 16 & 12 \\ {}^{*}14 & 7 & 7 & 15 & 2 & 15 \end{array}$$

and the determinant, represented by the square matrix (6×6) exhibited by the 6 lines last appearing above, ought to contain the modulus 17 as a divisor. Instead of the 3rd line from the bottom we may substitute its double less the last line, and thus, neglecting the factor 7, fall upon the matrix

$$\begin{array}{rrrrr} 16 & 1 & 1 & 8 & 12 \\ 9 & 16 & 11 & . & . \\ 14 & . & . & 3 & 4 \\ 15 & 13 & 4 & 16 & 2 \\ 2 & 12 & 5 & 16 & 12. \end{array}$$

Substituting for the 4th column the sum of itself and the 1st, and for the 5th column 5 times itself + the 1st, and neglecting the factor 14, we obtain the determinant

$$\begin{array}{rrrr} 1 & 1 & 7 & 8 \\ 16 & 11 & 9 & 9 \\ 13 & 4 & 14 & 8 \\ 12 & 5 & 1 & 11. \end{array}$$

Subtracting the 2nd column from the 1st and the 4th from the 2nd + the 3rd, we obtain the matrix

$$\begin{array}{rrrr} 0 & 0 & 1 & 7 \\ 5 & 11 & 11 & 9 \\ 9 & 10 & 4 & 14 \\ 7 & 12 & 5 & 1, \end{array}$$

and replacing the 3rd column by 7 times the 3rd less the 4th, we descend upon the determinant

$$\begin{array}{rrr} 5 & 11 & . \\ 9 & 10 & 14 \\ 7 & 12 & . \end{array}$$ where the 1st line to modulus 17 equals 8 times the 3rd.

Hence the determinant in question contains 17, as was to be shown.

It seems needless to multiply these tests—the object being, as before stated, not a confirmation of the argument, which is wholly unnecessary, but a verification of the accuracy of the arithmetic: for this reason it has seemed to me essential that the calculations, authenticating the figures previously obtained, should be set out in considerable detail.

Instead of founding anything upon the concordance (as far as it extends) between Dr von Gall's table and my own, the proof of the non-existence of the 10 . 4 irreducible covariant may be inferred exclusively from the former and completed as follows.

I have proved that the syzygetic function Ω of the deg-order 10 . 4, if it exists, must be a consequence of the existence of a like function of the deg-

order 8.4, 7.4, or 5.4. The last hypothesis may at once be rejected as implying an equation of the form $\frac{2.4}{3.4}$ = a numerical multiple of $\frac{2.0}{3.0}$.

Next, for the deg-order 7.4, again using for the primitive the same special form f, which causes 2.0 and 3.0 to vanish, the only non-vanishing arguments in the supposed syzygetic function Ω' for the particular form f will be 4.0×3.4 and 5.0×2.4, that is, $cd^2(b^2, bc, c^2, -cd, 5d^2)$, and $d^4(20d^2, 0, 0, b, 4c)$, between which obviously no syzygy is possible, so that neither of them can appear in the general form of Ω'. Hence the terms in the general form of Ω' must be divisible all by 2.0 or all by 3.0, or some by 2.0 and some by 3.0, and consequently there must exist a syzygy of the deg-order 5.4, 4.4, or 2.4. The first of these hypotheses has already been shown to be impossible, and the remaining two need not even have been mentioned, as there is only a single compound of the deg-order 4.4, namely, 2.0×2.4, and none of the deg-order 2.4. Lastly, for the deg-order 8.4, still using the same special form of f, the arguments in the supposed syzygetic functions which do not vanish are 4.0×4.4, $4.0 \times 4.4*$, 5.0×3.4, and 6.0×2.4, that is,

$$cd^2(0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2)$$
$$cd^2(336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2)$$
$$d^4(b^2,\ bc,\ c^2,\ -cd,\ 5d^2)$$
and
$$c^4(-20d^2,\ 0,\ 0,\ b,\ 4c).$$

The argument d^6 in the 3rd of these quantities has no equivalent in any of the other 3. Hence the 3rd quantity does not appear in the syzygy: moreover, the 4th compound contains one argument, namely, bc^4, which does not rationally contain d^2c $\left(\text{for } \frac{bc^3}{d^2} = \frac{b^2c}{3d}\right)$. Hence this compound also disappears, and obviously no syzygy connects together the first two. Hence in the supposed general syzygy there exist no compounds containing neither 2.0 nor 3.0, and by the same reasoning as before, this supposed syzygetic function must imply the existence of one of the deg-order 6.4 or 5.4 or 4.4. The two last of the three suppositions have already been seen to be impossible, and the first would imply a linear relation between 2.0×4.4, $2.0 \times 4.4*$, 3.0×3.4, 4.0×2.4, the last of which we see, by taking f for the primitive, cannot appear in the general syzygy, and the remaining 3 arguments would imply that the general covariant 3.4 would contain the invariant 2.0, which is absurd. Hence it follows from Dr von Gall's own results that the existence of a groundform of deg-order 10.4 is impossible. The only principle extraneous to his results made use of is Cayley's all-important rule, of which an irrefragable demonstration has been given by the author of this paper, but which still, as far as he is aware, remains unutilized, and is almost passed over in silence by invariantists of the German school.

It may be as well to make this article self-contained by showing that the number of compound irreducible groundforms of deg-order 10 . 4 is, as stated, 32, namely the same as the number of linearly-independent covariants of that deg-order requisitioned by Cayley's rule.

Using then, for brevity's sake, i to represent the invariant i . 0, it is easy to see that the following is an exhaustive enumeration of all the compounded irreducibles of deg-order 10 . 4:

$(5\,.\,2)^2$; $8 \times 2\,.\,4$; $7 \times 3\,.\,4$; $6 \times 4\,.\,4$; $6 \times 4\,.\,4*$; $5 \times 5\,.\,4$; $5 \times 5\,.\,4*$; $4 \times 6\,.\,4$; $4 \times 6\,.\,4*$; $4 \times 4 \times 2\,.\,4$; $3 \times 7\,.\,4$; $3 \times 7\,.\,4*$; $3^2 \times 4\,.\,4$; $3^2 \times 4\,.\,4*$; $3 \times 4 \times 3\,.\,4$; $3 \times 5 \times 2\,.\,4$; $2 \times 8\,.\,4$; $2 \times 8\,.\,4*$; $2 \times 3 \times 5\,.\,4$; $2 \times 3 \times 5\,.\,4*$; $2 \times 4 \times 4\,.\,4$; $2 \times 4 \times 4\,.\,4*$; $2 \times 5 \times 3\,.\,4$; $2 \times 6 \times 2\,.\,4$; $2 \times 3^2 \times 2\,.\,4$; $2^2 \times 6\,.\,4$; $2^2 \times 6\,.\,4*$; $2^2 \times 3 \times 3\,.\,4$; $2^2 \times 4 \times 2\,.\,4$; $2^3 \times 4\,.\,4$; $2^3 \times 4\,.\,4*$; $2^4 \times 2\,.\,4$.

The same number 32, it is all-important to bear in mind, is also the number of linearly independent covariants of deg-order 10 . 4 given by Cayley's law. For this number is represented by $(w:8,\ 10)-(w':8,\ 10)$ where $w=\dfrac{10\,.\,8-4}{2}=38$, $w'=w-1=37$; that is, (by Euler's Theorem) is the coefficient of t^{38} in the development of

$$\frac{(1-t^{11})(1-t^{12})(1-t^{13})(1-t^{14})(1-t^{15})(1-t^{16})(1-t^{17})(1-t^{18})}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)(1-t^8)},$$

which may be calculated as follows: The numerator is

$$1-t^{11}-t^{12}-t^{13}-t^{14}-t^{15}-t^{16}-t^{17}-t^{18}+t^{23}+t^{24}+2t^{25}+2t^{26}+3t^{27}+3t^{28}$$
$$+4t^{29}+3t^{30}+3t^{31}+2t^{32}+2t^{33}+t^{34}+t^{35}-t^{36}-t^{37}-2t^{38}\ldots.$$

Dividing this by $1-t^8$, the quotient by $1-t^7$, and so on for $1-t^6, \ldots 1-t^2$, we have for the numerator and the successive quotients so obtained the following values respectively:

t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}	t^{18}	t^{19}	t^{20}	t^{21}	t^{22}	t^{23}	t^{24}	t^{25}	t^{26}	t^{27}	t^{28}	t^{29}	t^{30}	t^{31}	t^{32}	t^{33}	t^{34}	t^{35}	t^{36}	t^{37}	t^{38}
1	0	0	0	0	0	0	0	0	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	0	1	1	2	2	3	3	4	3	3	2	2	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$
1	0	0	0	0	0	0	0	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	1	1	1	2	2	3	2	3	3	3	2	3	1	2	0
1	0	0	0	0	0	0	1	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	0	0	$\bar{1}$	$\bar{1}$	0	1	2	2	3	2	2	2	4	3	4	3
1	0	0	0	0	0	1	1	1	0	0	$\bar{1}$	0	0	1	0	0	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	0	0	0	0	0	1	2	4	3	4	3
1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	1	0	1	1
1	0	0	0	1	1	1	1	2	1	2	1	3	2	3	2	3	1	2	1	3	0	0	$\bar{2}$	0	$\bar{3}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{8}$	$\bar{6}$	$\bar{8}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{7}$	$\bar{6}$
1	0	0	1	1	1	2	2	3	3	4	4	6	6	7	8	9	8	10	10	11	10	10	9	10	7	6	5	4	0	$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{9}$	$\overline{11}$	$\overline{13}$	$\overline{15}$	$\overline{18}$	$\overline{19}$
1	0	1	1	2	2	4	4	7	7	11	11	17	17	24	25	33	33	43	43	54	53	64	62	74	69	80	74	84	74	83	70	77	61	66	48	51	30	32

Hence the required coefficient is 32.

It is obvious that the particular method adopted in treating the grand determinant made up of 11^2 places employed in the foregoing investigation furnishes or indicates a good practical process for determining 10 out of the 32 numerical coefficients which enter into the expression of Dr von Gall's covariant i_4'' as a linear function of the 32 linearly independent covariants of its own deg-order; but, as this calculation possesses no point either of intrinsic theoretical interest or practical importance, I leave it to those who may feel any curiosity on the subject, to go through the calculations necessary to attain that end.

It may be supposed that the long calculations rendered necessary by the quadrinomial form f, attributed to the primitive in the preceding investigation, might have been evaded by using a trinomial form (of which several exist) possessing the same property of causing the two first invariants to vanish, and not less general, inasmuch as containing three independent coefficients in place of four connected by a homogeneous equation; for example, we might assume for the primitive $(0, b, 0, 0, 0, f, 0, 0, i \between x, y)^8$, where the weights of b, f, i are respectively 1, 5, 8.

The quadrinvariant vanishes because no binary combination of 1, 5, 8, with or without repetitions, will make up the required weight 8, and the cubinvariant because no ternary combination of the same will make up the weight 12. It may, however, easily be shown that such form will lead only to a nugatory conclusion, as not supplying the necessary number of arguments (10 at least are wanted) to support the independence of the 10 surviving compound covariants of deg-order 10 . 4. This may be seen as follows.

The weights of the coefficients of $x^4, x^3y, x^2y^2, xy^3, y^4$ in a 10 . 4 covariant are respectively 38, 39, 40, 41, 42. Let us ascertain in how many ways 10 numbers, consisting exclusively of the numbers 1, 5, 8, can be put together to make up these totals. I use the notation $a^\alpha . b^\beta . c^\gamma$ to indicate a sum of α numbers a, β numbers b, and γ numbers c.

Then the sole admissible representations of 38 are $8^4 . 1^6$, $5^7 . 1^3$,

„ 39 „ $8^3 . 5^2 . 1^5$,

„ 40 „ $8^2 . 5^4 . 1^4$,

„ 41 „ $8 . 5^6 . 1^3$,

„ 42 „ $8^4 . 5 . 1^5$, $5^8 . 1^2$,

that is, there are only at utmost 7 arguments contained in the expressions for the 10 compounds.

So, in like manner, if we assumed for the primitive

$$(0, b, 0, 0, 0, 0, g, 0, i \between x, y)^8$$

to find the number of independent arguments possible in a 10 . 4 covariant, we must ascertain the sum of the numbers of similar representations to the foregoing of the same integers 38, 39, 40, 41, 42, with 10 integers confined to be 1, 6 or 8, and we shall find that the sole representations of that kind are $8^4 . 1^6$; $8^2 . 6^3 . 1^5$; $6^6 . 1^4$; $8^4 . 6 . 1^3$; $8^3 . 6^2 . 1^5$; $8 . 6^5 . 1^4$, that is, 6 representations in all. In like manner it will be found that all the other trinomial forms of the primitive so taken that the first two invariants are null, will be incapable of yielding as many as 10 arguments to any covariant of deg-order 10 . 4†, so that the 10 compounds appurtenant to such special form will be bound to be linearly related, and no inference can be drawn from any such assumption. I have reason for believing that the quadrinomial form employed in the foregoing investigation is the most convenient and economical, as leading to the simplest calculations of any that could have been employed for the same purpose.

† On an exhaustive examination, it will be found that the only trinomial forms of the primitive which will cause the first two invariants to disappear, are those in which the surviving coefficients are

$$b, f, i; \ b, g, i$$
$$a, b, c; \ a, b, d; \ a, c, d; \ b, c, d,$$

or the complementary ones

$$g, d, a; \ h, c, a$$
$$i, h, g; \ i, h, f; \ i, g, f; \ h, g, f,$$

which, of course, are substantially equivalent to the former.

Confining our attention, then, to the upper group, it will readily be seen that the four last will cause not only the quadrinvariant and the cubinvariant, but all the other invariants as well, to vanish. Since, then, it has been shown that the b, f, i; b, g, i forms are insufficient to support the independence of the 10 compound covariants with which the reasoning is concerned, it follows that *no trinomial form* will be adequate to do so.

It may be asked what would have been the effect of using the form in which b, c, d, i are the surviving coefficients, but b, c, d are supposed mutually independent, instead of being subject to the condition employed in the refutation above: on this supposition the quadrinvariant, but not the cubinvariant, will vanish; and an easy calculation will show that of the 32 representations of the covariant of deg-order 10 . 4 as a product of inferior groundforms there will be only 16 in which the quadrinvariant does not appear as a factor. And, again, it will be found that the number of ways of representing 38, 39, 40, 41, 42, as the sum of 4 numbers, each of which is either 1, 2, 3 or 8 is 29. Hence there would arise a matrix of 16 lines and 29 columns, and to disprove the existence of the 10 . 4 groundform it would be sufficient to prove that some one of the *complete* minor determinants of this matrix differs from zero. The work involved in dealing with this and the subsequent verificatory matrix of 17 lines and 29 columns would evidently be vastly greater and more liable to error than when (as in the text) we assign the relation between b, c, d so as to make the cubinvariant vanish.

In the absence of the information as to the number of linearly independent 10 . 4's given by Cayley's rule, the direct mode of refutation would have required the calculation of the 32 compound 10 . 4's and the problematical one of von Gall for the general form of the Octavic, subject only to the simplification of taking two of the coefficients zero. There would then have remained to show that the leading terms of these 33 forms were linearly connected, which would necessarily imply that the same was true of the 33 entire forms themselves; a colossal task, probably transcending the sphere of human ability to execute.

It may be well (by way of confirmation) to determine *à priori* the number of possible arguments that can belong to the $10 \,.\, 4$ covariants of the quadrinomial form of $(x, y)^8$ employed in the antecedent investigation. Since c^2 may be replaced by a numerical multiple of bd, it follows that each argument may be brought to a form in which c does not enter at all, or in which it enters only in the first degree. The total possible number (which turns out to be the actual number) of arguments is, consequently, the number of ways in which 38, 39, 40, 41, 42 can be composed with 10 parts each of them 1, 3 or 8 + the number of ways in which 36, 37, 38, 39, 40 can be composed with 9 parts, each of them also 1, 3 or 8. All the possible different compositions of these kinds are exhibited in the annexed table.

$$38 = 4\,.\,8 + 6\,.\,1 = 2\,.\,8 + 7\,.\,3 + 1\,.\,1 \qquad 36 = 3\,.\,8 + 3\,.\,3 + 3\,.\,1$$

$$39 = 3\,.\,8 + 4\,.\,3 + 3\,.\,1 \qquad 37 = 4\,.\,8 + 5\,.\,1 = 2\,.\,8 + 7\,.\,3$$

$$40 = 4\,.\,8 + 5\,.\,1 + 1\,.\,3 = 2\,.\,8 + 8\,.\,3 \qquad 38 = 3\,.\,8 + 4\,.\,3 + 2\,.\,1$$

$$41 = 3\,.\,8 + 5\,.\,3 + 2\,.\,1 \qquad 39 = 4\,.\,8 + 1\,.\,3 + 4\,.\,1$$

$$42 = 4\,.\,8 + 4\,.\,1 + 2\,.\,3 \qquad 40 = 3\,.\,8 + 5\,.\,3 + 1\,.\,1.$$

There are thus $7 + 6$, that is, 13 distinct arguments, that is, the number which actually appear distributed among the 10 surviving covariants of deg-order $10 \,.\, 4$ as previously shown—it being at the same time remembered that three of the 13 enter as elements of a fixed linear combination into the 10 functions, which are thus virtually functions of only 11 independent arguments.

The method employed in what precedes suggests a mode of calculating in part at least the discriminant of the eighthic in terms of the subordinate groundforms. Thus, suppose we take for our special form,

$$(0,\ b,\ c,\ d,\ 0,\ 0,\ 0,\ 0,\ i \between x,\ y)^8$$

with b, c, d independent.

Then the quadrinvariant will vanish, and there will be no very great effort of calculation required to express the 8 remaining invariants as functions of b, c, d, i.

The discriminant is of the 14th degree and 14 may be made up in 10 (and no more than 10) ways as a sum of numbers each limited to be 3, 4, 5, 6, 7, 8, 9, or 10; as exhibited in the exhaustive table

$$14 = 10 + 4 = 9 + 5 = 8 + 6 = 8 + 3 + 3 = 7 + 7 = 7 + 4 + 3 = 6 + 4 + 4$$
$$= 6 + 5 + 3 = 5 + 5 + 4 = 4 + 4 + 3 + 3.$$

Again the weight of the discriminant is 56, and the number of ways of compounding 56 with 14 numbers each limited to be 1, 2, 3 or 8 is 11, as shown in the exhaustive table

$56 = 6 . 8 + 8 . 1 = 5 . 8 + 7 . 2 + 2 . 1 = 5 . 8 + 3 . 3 + 1 . 2 + 5 . 1 = 5 . 8 + 2 . 3$
$+ 3 . 2 + 4 . 1 = 5 . 8 + 1 . 3 + 5 . 2 + 3 . 1 = 5 . 8 + 7 . 2 + 2 . 1 = 4 . 8 + 7 . 3$
$+ 3 . 1 = 4 . 8 + 6 . 3 + 2 . 2 + 2 . 1 = 4 . 8 + 5 . 3 + 4 . 2 + 1 . 1 = 4 . 8 + 4 . 3$
$+ 6 . 2 = 3 . 8 + 10 . 3 + 1 . 2.$

Now there will be no difficulty at all in finding by substitution and multiplication the discriminant of the assumed quantic, say Q, which is in fact the same as the resultant of $\dfrac{dQ}{dy}$ and $bx^6y + 3cx^5y^2 + 5dx^4y^3$. Hence there will be 11 equations for determining the coefficients of the 10 invariants of the 14th degree which are products of the inferior invariants (the quadrinvariant excepted); consequently there will be sufficient or more than sufficient equations for the purpose, unless it should (unfortunately and contrary to probability) turn out to be the case that the 10 products, although linear functions of 11 arguments, are expressible as linear functions of only 9 linear functions of those arguments.

61.

ON TCHEBYCHEFF'S THEORY OF THE TOTALITY OF THE PRIME NUMBERS COMPRISED WITHIN GIVEN LIMITS.

[*American Journal of Mathematics*, IV. (1881), pp. 230—247.]

IF it be admitted that Legendre's approximate formula for the number of prime numbers inferior to a given number, which has been confirmed by direct enumeration of the number of prime numbers contained in the first few millions, can be extended to those remote regions of number which transcend the limits and even the possibilities of human experience, it will follow as a consequence that the average density of the distribution of prime numbers in the neighbourhood of a large quantity x approximates to $\frac{1}{\log x}$, and consequently that the number of primes included between x and $(1+\epsilon)x$, or if we like to say so, between $x+A$ and $(1+\epsilon)x+B$, will be approximately equal to $\frac{\epsilon x}{\log x}$, and therefore will become indefinitely great, however small ϵ may be taken. Although there can hardly be a doubt that such is the fact, no step had been taken previous to Tchebycheff's researches towards establishing this proposition demonstratively. Tchebycheff has succeeded in proving it, not, it is true, in an absolute sense, but for all values of ϵ exceeding the fraction $\frac{1}{5}$. He has done more, inasmuch as he has given formulae for actually ascertaining a number x for all values superior to which there will be at least any specified number K of primes included between $x+A$ and $(1+\epsilon)x+B$ when ϵ has any positive value superior to $\frac{1}{5}$, and A and B are any quantities positive or negative. He may not perhaps have actually stated this proposition in so many words, but it is an immediate inference from the limits (expressed in terms of x, $x^{\frac{1}{2}}$ and $\log x$) which he has obtained to the number of prime numbers not exceeding x. The object of what follows is to make a little further advance in the same

direction, and to show upon Tchebycheff's own principles that the proposition remains true when ϵ is conditioned no longer to be inferior to the fraction $\frac{1}{5}$, but to the fraction $\frac{1}{6}+\frac{1}{4642\frac{10}{11}}$, so that the excess above unity (the region so to say of darkness) is scarcely more than five-sixths of what it is for the first named fraction. This conclusion is arrived at by aid exclusively of Tchebycheff's own formulae.

Tchebycheff's method may be regarded as the *first* approximation to the inferior and superior limits of a quantity ψx subject to the conditions

$$Vx > Ax + F \log x,$$
$$Vx < Ax + F_1 \log x,$$

where
$$Vx = \psi x - \psi \frac{x}{6} + \psi \frac{x}{7} - \psi \frac{x}{10} \text{ etc.},$$

(see Serret's *Cours d'Algèbre supérieure*, 4th Ed., Vol. II., pp. 230—233), and to the further conditions that ψx is not less than $\psi x'$ if $x > x'$, and that $\psi x = 0$ when $x < 1$.

The limits obtained for ψx depend exclusively on these definitions, and would be applicable to any function ψx whatever that satisfied them.

The advance made in this article consists in pursuing the approximation through an indefinite number of steps, so as to bring the superior and inferior limits to ψx continually nearer and nearer to each other as regards the *principal* term (a multiple of x) which enters into each of them: the remaining terms over and above this multiple of x in the expressions for the limits always continue to be positive integer powers of $\log x$, and consequently the ratio of the limits becomes as nearly as we please identical with the ratio of the principal terms (that is of their coefficients) when x is taken sufficiently great: this ratio as given in the first approximation is $\frac{6}{5}$, but as the approximation is continued continually converges to but never reaches the fraction

$$\frac{7}{6}+\frac{1}{4642\frac{10}{11}}.$$

Such, and such only, is the small but not unimportant contribution here supplied to Tchebycheff's remarkable theory. As no allusion is made to the possibility of this contraction of the limits in a work published so recently as 1879, by an author so competent as M. Serret, I presume that it has hitherto remained unnoticed; but of this I cannot speak with certainty, inasmuch as it was enough for M. Serret's purpose to obtain for the ratio of the principal terms a number less than 2; that being sufficient for the object he had in view, which was to prove M. Bertrand's celebrated postulate that at least one prime number must be included (for all values of x greater than $\frac{7}{2}$) between x and $2x-2$.

Although I might confine myself exclusively to the determination of the limits to ψx which flow from the conditions above given, it is, I think, desirable to supply a brief summary of M. Tchebycheff's method, so as to point out the connexion between the determination of these limits and the limits to "the totality of the prime numbers comprised within a given range." In so doing I shall adopt for the convenience of reference the notation which I find in M. Serret's able exposition of the subject (*Alg. sup.*, Vol. II. pp. 225—239).

θx stands for the sum of the logarithms of all the *prime* numbers not exceeding x.

$$\psi x = \theta x + \theta x^{\frac{1}{2}} + \theta x^{\frac{1}{3}} + \theta x^{\frac{1}{4}} + \theta x^{\frac{1}{5}} + \ldots,$$

$$Tx = \psi x + \psi \frac{x}{2} + \psi \frac{x}{3} + \psi \frac{x}{4} + \psi \frac{x}{5} + \ldots,$$

and, as a consequence founded on purely arithmetical considerations, Tx is the sum of the logarithms of *all* the numbers not exceeding x, and therefore, as an easy deduction from Stirling's theorem, it follows that for all values of x superior to unity,

$$Tx < x \log x - x + \frac{1}{2} \log x + \left\{\log \sqrt{(2\pi)} + \frac{1}{12}\right\}$$

$$Tx > x \log x - x - \frac{1}{2} \log x + \log \sqrt{(2\pi)}.$$

If then Vx (a notation not in Serret) be used to denote

$$Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{5} + T\frac{x}{30}$$

(where it should be noticed that $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30} = 0$), limits for Vx can be found in which $x \log x$ will not appear, and expressed solely in terms of x and $\log x$: it may in fact be shown that for all values of x superior to unity,

$$Vx > A\,(x - 1) - \frac{5}{2} \log x$$

$$Vx < A\,(x - 1) + \frac{5}{2} \log x,$$

where $$A = \frac{1}{2} \log 2 + \frac{1}{3} \log 3 + \frac{1}{5} \log 5 - \frac{1}{30} \log 30 = \cdot 92129202\ldots.$$

The limits actually employed, however, are the slightly wider ones,

$$Vx > Ax - \frac{5}{2} \log x - 1$$

$$Vx < Ax + \frac{5}{2} \log x.$$

If now we take an infinite succession of numbers separable into batches of sixteen, such that every $(i+1)$th batch may be got by adding $30i$ to each of the numbers in the first batch, those numbers being

1, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 23, 24, 29, 30

(where it is perhaps worth noticing that leaving out the last number 30, the remaining 15 consist of a middle term 15 and pairs of numbers whose sum is always 30, disposed symmetrically about that middle term), it will readily be seen to follow from the expression for V in terms of the T's and of T in terms of the ψ's, that

$$\begin{aligned} Vx = \psi x - \psi\frac{x}{6} + \psi\frac{x}{7} - \psi\frac{x}{10} + \psi\frac{x}{11} - \psi\frac{x}{12} + \psi\frac{x}{13} - \psi\frac{x}{15} & \\ + \psi\frac{x}{17} - \psi\frac{x}{18} + \psi\frac{x}{19} - \psi\frac{x}{20} + \psi\frac{x}{23} - \psi\frac{x}{24} + \psi\frac{x}{29} - \psi\frac{x}{30} & \\ + \psi\frac{x}{31} - \psi\frac{x}{36} \cdots\cdots\cdots\cdots\cdots\cdots - \psi\frac{x}{45} & \\ + \psi\frac{x}{47} - \psi\frac{x}{48} \cdots\cdots\cdots\cdots\cdots\cdots - \psi\frac{x}{60} & \\ + \psi\frac{x}{61} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{aligned}$$

just in the same way as if supposing $\omega x = \psi x - 2\psi\frac{x}{2}$ we should find

$$\omega x = \psi x - \psi\frac{x}{2} + \psi\frac{x}{3} - \psi\frac{x}{4} + \psi\frac{x}{5} \cdots;$$

or as if supposing $\Omega x = \psi x - \psi\frac{x}{2} - \psi\frac{x}{3} - \psi\frac{x}{6}$ we should find

$$\Omega x = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \cdots.$$

From the limits to which Vx is subject (Vx being now regarded as representing the series of ψ's above written) limits can be found to ψx of the form $mx + R_1(\log x)$, $nx + R_2(\log x)$, where the R's signify rational integer forms of function. In the first approximation, for the inferior and superior limits respectively, $m = A$, $n = 6\frac{A}{5}$; R_1 is a linear and R_2 a quadratic form of function. In the approximation of the ith order m and n will become functions of i, and R_1, R_2 will be of the ith and $(i+1)$th orders respectively in $\log x$.

The limits of ψx being supposed to be given (say $\psi' x$ the superior and $\psi_1 x$ the inferior limit), $\psi' x$ will serve as a superior and $\psi_1 x - 2\psi' x^{\frac{1}{2}}$ as an inferior limit to θx. But instead of $\psi' x$ we may use (although not at all

necessary for the object in view) the slightly closer limit $\psi' x - \psi_1 x^{\frac{1}{2}}$, which is what M. Serret employs, and equally instead of $\psi_1 x - 2\psi' x^{\frac{1}{2}}$ we might use the slightly closer limit

$$\psi_1 x - \psi' x^{\frac{1}{2}} - \psi' x^{\frac{1}{3}} - \psi' x^{\frac{1}{5}} + \psi_1 x^{\frac{1}{30}},$$

which, probably as leading to calculations needlessly complicated (as regards the object in view), M. Serret does not employ. In any case, following the same notation as before to distinguish the two limits, we shall obtain

$$\theta' x = nx + F(x^{\frac{1}{2}}, \log x),$$
$$\theta_1 x = mx + F'(\ldots, \log x),$$

where F, F' are rational integer forms of function, and the dots in the F' may be filled in either with $x^{\frac{1}{2}}$ or with $x^{\frac{1}{2}}, x^{\frac{1}{3}}, x^{\frac{1}{5}}, x^{\frac{1}{30}}$; and we shall have

$$\theta' x = nx(1 + \epsilon_x), \quad \theta_1 x = mx(1 + \eta_x),$$

where ϵ_x and η_x vanish when $x = \infty$.

To come to our ultimate object, it is obvious that the number of primes between x and $(1+\rho)x$ will be greater than $[\theta_1(1+\rho)x - \theta' x] \div \log x$. It will therefore be greater than $\dfrac{[m(1+\rho) - n]x + \delta_x}{\log x}$, where $\delta_x = 0$ when $x = \infty$. Hence we may find a value of x so great that the number of primes shall be at least K by finding a number x sufficiently large to make

$$\theta_1(1+\rho)x - \theta' x - (K-1)\log x > 0,$$

which it must always be possible to do provided that $m(1+\rho) > n$, that is, that $\rho > \left(\dfrac{n}{m} - 1\right)$. Hence the importance of diminishing what I call the asymptotic ratio $\dfrac{n}{m}$, that is, the ratio of the coefficients in the principal terms of the superior and inferior limits to ψx. That is what I shall now proceed to accomplish, but first it is necessary to establish a certain easy lemma.

Suppose the equation $fx - f\dfrac{x}{c} = Ax^m$ is to be satisfied; this can be done by writing $fx = A\dfrac{c^m}{c^m - 1}x$, and in particular if $m = 1$, the only case that the present theory demands, $fx = \dfrac{c}{c-1}Ax$. Again if the equation

$$fx - f\frac{x}{c} = P(\log x)^\mu$$

is to be satisfied, this may be done by making

$$fx = P_0(\log x)^{\mu+1} + P_1(\log x)^\mu + P_2(\log x)^{\mu-1} + \ldots + P_\mu \log x,$$

for since $\log\dfrac{x}{c} = (\log x - \log c)$, $fx - f\dfrac{x}{c}$ will then obviously become a function

of $\log x$ of the μth order, which may be identified with $P(\log x)^\mu$ by properly assigning the values of the $(\mu+1)$ disposable constants $P_0, P_1, P_2, \ldots P_\mu$. In fact the equation might easily (if it were worth while to do so) be turned into an equation of differences, and the general values of the P's be expressed once for all in terms of Bernoulli's numbers for any value of μ. Hence it follows that the equation

$$fx - f\frac{x}{c} = Nx + R_\mu{}^* \log x,$$

where R_μ is a rational integer form of function of the μth order, may be satisfied by making

$$fx = \frac{c}{c-1} Nx + R_{\mu+1} \log x,$$

where the second term on the right hand side of the equation is a *known* function of $\log x$ of the $(\mu+1)$th order.

Suppose now that the inequality $\psi x - \psi\frac{x}{c} < Nx + R_\mu \log x$, where $c > 1$, is given, and it is desired to extract from this inequality an inferior limit to ψx. It is only necessary to get a solution of the equation

$$fx - f\frac{x}{c} = Nx + R_\mu \log x.$$

We shall then have

$$\psi x - \psi\frac{x}{c} < fx - f\frac{x}{c},$$

$$\psi\frac{x}{c} - \psi\frac{x}{c^2} < f\frac{x}{c} - f\frac{x}{c^2},$$

$$\psi\frac{x}{c^2} - \psi\frac{x}{c^3} < f\frac{x}{c^2} - f\frac{x}{c^3},$$

....

....

and consequently

$$fx - f\frac{x}{c^q} > \psi x - \psi\frac{x}{c^q}.$$

If then q be supposed to be taken such that $\frac{x}{c^q}$, say z, lies between 0 and 1, we shall have

$$fx - \psi x > fz,$$

and *à fortiori* $> R_{\mu+1} \log z$ (if N be positive, as is the case throughout the present investigation), where the right hand side of the inequality is a known rational integer function of $\log z$. If then M be a number less than the least value that $R_{\mu+1}\xi$ can assume between the limits $\xi = 0$, $\xi = -\log c$, we shall have $\psi x < fx - M$, and an inferior limit will have been obtained to ψx.

* The reader's attention is called to the fact that R_μ is used throughout to denote a *form of function*, and not, like P_μ, a *coefficient*.

In the first approximation (Serret, p. 234), where $\mu = 1$ and $c = 6$,

$$R_2\xi = \frac{5}{4\log 6}\,\xi^2 + \frac{5}{4}\,\xi,$$

the minimum value of which is got by taking $2\xi = -\log 6$ or $\xi = -\log\sqrt{6}$ (which happens to lie between the limits of $\log 1$ and $-\log 6$) and gives $M = \dfrac{-5\log 6}{16}$, so that $\psi x < fx + \dfrac{5\log 6}{16}$. The actual value employed for the superior limit, as sufficiently near and more convenient for use, is $fx + 1$.

So in the general case we shall have $fx - \psi x > M$ where M is any number less than the least value of $R_{i+1}\xi$ for values of ξ lying between 0 and $-\log c$. It may or may not be the absolute minimum of $R_{i+1}\xi$ that has to be taken according as the value of ξ which gives this absolute minimum does or does not lie between 0 and $-\log c$. In the latter case it may be either some other minimum, or one of the values of $R_{i+1}\xi$ corresponding to the extreme values $\xi = 0$ and $\xi = -\log c$, which might be found by trial. But a method practically better and sufficient for the demands made by the present investigation, would be to substitute zero in place of any term in the function of ξ of the form $+K\xi^{2m}$ or $-K\xi^{2m+1}$, and for any term of the form $-K\xi^{2m}$ or $+K\xi^{2m+1}$ to substitute $-K(\log c)^{2m}$ and $-K(\log c)^{2m+1}$ respectively.

For instance, in the case just considered we might have written

$$M = -\frac{5}{4}\log 6,$$

and the superior limit instead of being $fx + 1$ would have been $fx + \frac{5}{4}\log 6$, which would practically have been just as good. With a view to a remark which will subsequently be made it is well to notice that the inequality

$$\psi x - \psi\frac{x}{c} > Nx + R_\mu \log x$$

may also be solved precisely in the same manner, and will give for an inferior limit to ψx (using fx to signify the very same function as before) $fx - M_1$, where (N being supposed positive) $M_1 = -N\log c +$ any quantity not less than the greatest value of a known rational integer function of a variable conditioned to lie between 0 and $-\log c$, which may either be found by an exact algebraical process or by substituting 0 in those two cases where previously $-\log c$, and $-\log c$ in those other two cases where previously 0 was to be substituted for the variable.

The lemma needful for our purposes may now accordingly be stated in the following terms: *If* $\psi x - \psi\dfrac{x}{c}$ *is less or greater than* $Nx +$ *a given rational integer function of* $\log x$ *of any given order,* ψx is less or greater than $\dfrac{c}{c-1}Nx +$ a known (and easily determinable) rational integer function of $\log x$ of the order next superior.

If the coefficients of x in the superior and inferior limits to ψx at any stage of the investigation be called u and v, I shall show that these values will serve to give (step by step) other superior and inferior limits where u and v are replaced by quantities u', v', such that $u' < u$, $v' > v$; u', v' being known linear functions of u, v. We shall thus be led to a system of two simultaneous linear equations of differences in order to obtain the effect of those changes repeated any number, finite or infinite, of times: but for greater clearness I shall begin with supposing that one of the two expressions u, v, namely, v (which undergoes far less modification than the other) is kept constant. There will then result a single scheme of successive substitutions leading to the construction of a single linear equation in differences.

The first step will then be as follows:

$$\psi x - \psi \frac{x}{6} < Ax + \frac{5}{2}\log x - \psi \frac{x}{7} + \psi \frac{x}{10}$$

$$< Ax + \frac{5}{2}\log x - \left(A\frac{x}{7} - \frac{5}{2}\log\frac{x}{7} - 1\right) + \frac{6}{50}Ax + \frac{5}{4\log 6}\left(\log\frac{x}{10}\right)^2 + \frac{5}{4}\log\frac{x}{10}$$

or writing $\qquad \lambda = \log 6, \quad \mu = \log 7, \quad \nu = \log 10,$

$$\psi x - \psi\frac{x}{6} < \frac{171}{175}Ax + \frac{5}{4\lambda}(\log x)^2 + \left\{\left(\frac{25}{4} - \frac{5\nu}{2\lambda}\right)\log x + \frac{5\nu^2}{4\lambda} - \frac{5}{2}\mu - \frac{5}{4}\nu + 1\right\}.$$

Hence $\qquad \psi x < \dfrac{1026}{875}Ax + P(\log x)^3 + Q(\log x)^2 + Rx - M,$

where first to find P, Q, R, we have the three equations

$$3P\lambda = \frac{5}{4\lambda}$$

$$-3P\lambda^2 + 2Q\lambda = \frac{25}{4} - \frac{5\nu}{2\lambda}$$

$$P\lambda^3 - Q\lambda^2 + R\lambda = \frac{5\nu^2}{4\lambda} - \frac{5}{2}\mu - \frac{5}{4}\nu + 1,$$

that is, $\quad P = \dfrac{5}{12\lambda^2}; \quad Q = \dfrac{15}{4\lambda} - \dfrac{5\nu}{4\lambda^2}; \quad R = -\dfrac{5}{12} + \dfrac{15}{4} - \dfrac{5\nu}{2\lambda} + \dfrac{5\nu^2}{4\lambda^2} - \dfrac{5\mu}{2\lambda} + \dfrac{1}{\lambda}.$

Here P is positive; Q, whose sign depends on that of $3 - \dfrac{\log 10}{\log 6}$, is also positive; and

$$R = \frac{10}{3} + 5\left(\frac{\nu}{2\lambda} - \frac{1}{2}\right)^2 - \frac{5\mu - 2}{2\lambda} - \frac{5}{4}$$
$$= 3{\cdot}33333\ldots + {\cdot}10160\ldots - 2{\cdot}1570\ldots - 1{\cdot}25$$
$$= 3{\cdot}43493\ldots - 3{\cdot}4070\ldots, \text{ which is also positive.}$$

Hence we may make

$$M = -P\lambda^3 - R\lambda,$$

or $$-M = 1 + \frac{15}{4}\lambda - \frac{5(\mu+\nu)}{2} + \frac{5\nu^2}{4\lambda} = 1{\cdot}2947.$$

It is quite possible, and even most likely, that the minimum of

$$P\lambda^3 - Q\lambda^2 + R\lambda$$

(within the prescribed limits) would be found to exceed -1 were it worth while to go through the arithmetical calculations necessary to obtain it, but it is quite sufficiently near for all practical purposes to use the value above determined, or even to take $-M$ as great as 2 and to adopt for our new superior limit

$$\frac{171}{175}Ax + P(\log x)^3 + Q(\log x)^2 + R\log x + 2.$$

In like manner this new limit will enable us to find another, and it is obvious that the general form of the limit obtained after i of these steps have been gone through will be $u_i Ax + R_{i+2}\log x$, where

$$u_i = \frac{6}{5}\left(1 - \frac{1}{7} + \frac{u_{i-1}}{10}\right), \quad \text{that is,} \quad u_i - \frac{3u_{i-1}}{25} = \frac{36}{35}.$$

Putting $$u_i = \omega_i + h$$

and making $$\frac{22}{25}h = \frac{36}{35}, \quad \text{that is,} \quad h = \frac{90}{77},$$

we have $$\omega_i - \frac{3}{25}\omega_{i-1} = 0.$$

Hence $$u_i = C\left(\frac{3}{25}\right)^i + \frac{90}{77}.$$

The ultimate value of u_i is therefore $\frac{90}{77}$, and accordingly, by repeating the process indicated a sufficient number of times, we shall have for a superior limit $\left(\frac{90}{77} - \epsilon_i\right)Ax + R_{i+2}\log x$, where ϵ_i may be made as small as we please by taking i sufficiently great, and thus the ultimate asymptotic ratio of the two limits is $\frac{90}{77}$ instead of $\frac{6}{5}$.

Another mode of approximation may be used, as shown in what follows.

Since $$\psi x - \psi\frac{x}{10} < Ax - \psi\frac{x}{6} + \psi\frac{x}{7};$$

if we have found $$\psi x < u_i' Ax + R_{i+2}\log x$$

we shall have $$\psi x - \psi\frac{x}{10} < Ax + u_i' A\frac{x}{6} - A\frac{x}{7} + R'_{i+2}\log x,$$

and therefore $$\psi x < u'_{i+1}Ax + R_{i+3}\log x,$$

where $$u'_{i+1} = \frac{10}{9}\left\{1 - \frac{1}{7} + \frac{1}{6}u_i'\right\}$$

that is, $$u'_{i+1} - \frac{5}{27}u_i' = \frac{20}{21};$$

or
$$u_i' = K\left(\frac{5}{27}\right)^i + h'$$
where
$$h' = \frac{27}{22}\cdot\frac{20}{21} = \frac{90}{77}.$$

Thus $h' = h$ and consequently also, if we suppose each of the two sorts of approximation to start from the same point, $K = C$.

Hence the ultimate value of u_i and u_i' is the same, but the former method of approximation is to be preferred, as the same number of steps, that is, the same value of i, makes $C\left(\frac{5}{27}\right)^i + h$ always $> C\left(\frac{3}{25}\right)^i + h$. The corresponding values of u_i, u_i' have the same initial and final values, but for every intermediate value of i, $u_i < u_i'$. In fact u_i, u_i' are ordinates to the same abscissa of two non-intersecting curves, having a common starting point and a common asymptote.

The maximum value of $u_i' - u_i$ is found by making $\left(\frac{5}{27}\right)^i - \left(\frac{3}{25}\right)^i$ a maximum, which takes place when i is the integer next above or next below the value

$$\frac{\log\log\frac{25}{3} - \log\log\frac{27}{5}}{\log\frac{25}{3} - \log\frac{27}{5}},$$ which is obviously less than unity.

Hence after the *first* approximation u_i and u_i' are always drawing closer together.

We may now proceed to the more (but only very slightly more) advantageous method of approximation, namely, that in which the principal terms in both limits are simultaneously varied, decreasing as before in the superior, and now at the same time increasing in the inferior limit.

Suppose then that we have found
$$\psi x < u_i Ax + R_{i+2}\log x$$
$$\psi x > v_i Ax + R_{i+1}\log x;$$
observing that $\frac{v_i}{24} - \frac{u_i}{29}$ is always positive, we shall succeed in increasing the principal term of the inferior limit by writing
$$\psi x > Ax + v_i A\frac{x}{24} - u_i A\frac{x}{29} + R_{i+2}\log x,$$
and slightly more than previously diminishing the principal term in the superior limit by writing
$$\psi x - \psi\frac{x}{6} < Ax - v_i A\frac{x}{7} + u_i A\frac{x}{10} + R'_{i+2}\log x.$$

We shall thus easily derive

$$\psi x > v_{i+1} Ax + R_{i+2} \log x$$
$$\psi x < u_{i+1} Ax + R_{i+3} \log x$$

where
$$v_{i+1} = 1 + \frac{v_i}{24} - \frac{u_i}{29}$$

$$u_{i+1} = \frac{6}{5}\left(1 - \frac{v_i}{7} + \frac{u_i}{10}\right) = \frac{6}{5} - \frac{6}{35} v_i + \frac{3}{25} u_i$$

or, making
$$v_i = v_i' + f, \quad u_i = u_i' + e,$$

$$v'_{i+1} - \frac{1}{24} v_i' + \frac{1}{29} u_i' = 0$$

$$u'_{i+1} - \frac{3}{25} u_i' + \frac{6}{35} v_i' = 0,$$

if
$$\frac{23}{24} f + \frac{1}{29} e = 1, \quad \frac{6}{35} f + \frac{22}{25} e = \frac{6}{5}.$$

So that, calling ρ_1, ρ_2 the roots of
$$\begin{vmatrix} \rho - \frac{1}{24}; & \frac{1}{29} \\ \frac{6}{35}; & \rho - \frac{3}{25} \end{vmatrix} = 0,$$

$$u_i = C_1 \rho_1^i + C_2 \rho_2^i + e$$
$$v_i = C_1' \rho_1^i + C_2' \rho_2^i + f.$$

The equation for finding ρ_1, ρ_2 is

$$\rho^2 - \frac{97}{600} \rho - \frac{37}{40600} = 0,$$

whence
$$\rho_1 = \cdot 167253\ldots, \qquad \rho_2 = \cdot 005637\ldots.$$

Also the equations in e, f give e, f (the values of u_∞, v_∞) as follows:

$$e = \frac{59595}{50999}, \quad f = \frac{51072}{50999}.$$

If there were any use in obtaining the values of the disposable constants they could of course be obtained from the equations

$$C_1 + C_2 + e = u_0 = \frac{6}{5}, \qquad C_1 \rho_1 + C_2 \rho_2 + e = u_1 = \frac{1026}{875},$$

$$C_1' + C_2' + f = v_0 = 1, \qquad C_1' \rho_1 + C_2' \rho_2 + f = v_1 = \frac{3481}{3480}.$$

The asymptotic ratio of the two limits is

$$\frac{e}{f} = \frac{59595}{51072} = \frac{6}{7} + \frac{11}{51072}.$$

Various other modes of approximation may be adopted, but it will be found that no smaller value can be obtained for the asymptotic ratio than that above given: the value of u_∞ cannot be made less than $\frac{59595}{50999}$, nor the value of v_∞ greater than $\frac{51072}{50999}$.

Thus for example, making use of the inequality

$$\psi x - \psi\frac{x}{6} > Ax - \psi\frac{x}{7} + \psi\frac{x}{24} - \psi\frac{x}{29} + R(\log x),$$

we might by the lemma obtain

$$\psi x > \frac{6}{5}A\left(1 - \frac{u_i}{7} - \frac{u_i}{29} + \frac{v_i}{24}\right) + R_{i+3}\log x,$$

and consequently $$v_{i+1} = \frac{6}{5}\left(1 - \frac{u_i}{7} - \frac{u_i}{29} + \frac{v_i}{24}\right);$$

combining which with the previous equation for u_{i+1} we should have for finding u_∞, v_∞, say e', f', the two equations,

$$\frac{19}{24}f' + \frac{36}{203}e' = 1,$$

$$\frac{6}{35}f' + \frac{22}{25}e' = \frac{6}{5},$$

and consequently $$e' = \frac{331905}{284029}, \qquad f' = \frac{284424}{284029}.$$

Reduced to decimals

$$e = 1{\cdot}16855\ldots, \quad f = 1{\cdot}00143\ldots,$$
$$e' = 1{\cdot}16856\ldots, \quad f' = 1{\cdot}00125\ldots.$$

It may be noticed that $eA = 1{\cdot}006774\ldots$, $fA = {\cdot}992619\ldots$ of which the sum is nearly $1{\cdot}999394$, and their mean nearly ${\cdot}999697$, whereas the mean of A and $\frac{6A}{5}$ (the original coefficients of x in the limits) is nearly $1{\cdot}01342$. Thus the new mean is more than 44 times nearer than the latter to the true asymptotic value deducible from the empirical formula.

Were it desired merely to find superior and inferior limits to ψx in the form obtained in Tchebycheff's method, it would (as already indicated) have been sufficient to have taken for Vx, $Tx - 2T\frac{x}{2}$, which would have led to the inequalities

$$\psi x > (\log 2)\,x + R_1 \log x,$$
$$\psi x < 2(\log 2)\,x + R_2 \log x,$$

but the asymptotic ratio being here 2, these limits could not have conducted

to a proof of M. Bertrand's postulate. If, however, we were to take $Vx = Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{6}$ we should obtain

$$Vx > Bx + R_1 \log x,$$
$$Vx < Bx + R_1' \log x,$$

where $$B = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{6}\log 6 = 1{\cdot}0114043,$$

and $$Vx = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \ldots,$$

when we should obtain

$$\psi x - \psi\frac{x}{6} < Bx + R_1' \log x,$$

$$\psi x < \frac{6}{5}Bx + R_2 \log x,$$

and again $$\psi x + \psi\frac{x}{5} > Bx + R_1 \log x,$$

$$\psi x > B\left(1 - \frac{6}{25}\right)x + R_2' \log x.$$

Here the asymptotic ratio of the two limits is $\frac{30}{19}$, which being less than 2, the formulae above indicated would suffice to prove M. Bertrand's postulate, and would lead to an equation somewhat simpler in form than that led to by M. Tchebycheff's process, but whose greatest root would be considerably larger than that found by the established method; so that there would be a larger number of verifications of the postulate to be made for the lower numbers: this, however, is really a matter of very trifling importance, as the needful verifications could be made even up to 100,000 if necessary, by throwing a rapid glance over a few leaves of Burckhardt's tables.

It is noticeable that the limits above found by giving Vx the form $Tx - 2T\frac{x}{2}$ are the *only* limits that can be got in such case; no process of successive approximation being here possible, on account of the too close contiguity of the successive denominators in $\psi x - \psi\frac{x}{2} + \psi\frac{x}{3} \ldots$.

Such, however, would not be the case were we to use Vx to signify $Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{6}$, and consequently

$$Vx = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \psi\frac{x}{13} \ldots$$

The limits expressed by the inequalities

$$\psi x < u_i\, Bx + \dots,$$
$$\psi x > v_i\, Bx + \dots,$$

would lead to the narrower limits

$$\psi x < u_{i+1}\, Bx + \dots,$$
$$\psi x > v_{i+1}\, Bx + \dots,$$

where
$$u_{i+1} = \frac{6}{5}\left(1 - \frac{v_i}{7} + \frac{u_i}{12}\right),$$

$$v_{i+1} = 1 - \frac{u_i}{5} + \frac{v_i}{6} - \frac{u_i}{11},$$

that is to say
$$u_{i+1} - \frac{u_i}{10} + \frac{6v_i}{35} - \frac{6}{5} = 0,$$

$$\frac{u_i}{5} + \frac{u_i}{11} + v_{i+1} - \frac{v_i}{6} - 1 = 0.$$

Hence, using as before e, f to indicate the ultimate values of u_i, v_i, we should have

$$21e + 4f - 28 = 0,$$
$$96e + 275f - 330 = 0,$$

and consequently
$$e = \frac{6380}{5391}, \quad f = \frac{4242}{5391},$$

and
$$\frac{e}{f} = \frac{6380}{4242} = \frac{3}{2} + \frac{1}{249\frac{9}{17}},$$

which is the ultimate value of the asymptotic ratio, of which the initial value was $\frac{30}{19}$, that could be found by this method.

In every such kind of series as I have denoted by Vx, it is obvious that the sum of the multiples of x under the sign of ψ in Vx is equal to the coefficient of x in either limit to Vx. Thus, for example, in Tchebycheff's series, if we take n a multiple of 30, and make $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, the sum of n terms of $1 - \frac{1}{6} + \frac{1}{7} - \frac{1}{10} + \frac{1}{11} \dots$

$$= \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30}\right) S_n + \frac{1}{2}\left(\frac{1}{\frac{1}{2}n + 1} + \frac{1}{\frac{1}{2}n + 2} + \dots + \frac{1}{n}\right)$$
$$+ \frac{1}{3}\left(\frac{1}{\frac{1}{3}n + 1} + \frac{1}{\frac{1}{3}n + 2} + \dots + \frac{1}{n}\right) + \frac{1}{5}\left(\frac{1}{\frac{1}{5}n + 1} + \frac{1}{\frac{1}{5}n + 2} + \dots + \frac{1}{n}\right)$$
$$- \frac{1}{30}\left(\frac{1}{\frac{1}{30}n + 1} + \frac{1}{\frac{1}{30}n + 2} + \dots + \frac{1}{n}\right);$$

and the multiplier of S_n being always 0, it follows that the sum of an infinite number of the consecutive terms

$$=\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5-\frac{1}{30}\log 30=A.$$

It may not unreasonably be conjectured that whilst nothing more can be done with the Tchebycheffian Vx, it may be possible to find such other form of function in lieu of it, or such infinite succession of different forms of function, as may either directly or by successive approximation bring the coefficients of x in the two limits as near as we please to one another, at the expense, of course, of proportionally lengthening out the residues, or tails as they might be termed, of the two limits. Could this be done, it is easy to demonstrate that the limit thus continually approached from opposite sides must be unity, as indicated in advance by Legendre's empirical formula. For this purpose it will be sufficient to use the simplest form of Vx, namely, $Tx-2T\frac{x}{2}$, whence we obtain

$$\psi x-\psi\frac{x}{2}+\psi\frac{x}{3}\ldots>\log 2\,.\,x(1+\epsilon_x),$$

$$\psi x-\psi\frac{x}{2}+\psi\frac{x}{3}\ldots<\log 2\,.\,x(1+\eta_x),$$

ϵ_x, η_x being known logarithmic quantities which vanish when $x=\infty$.

For suppose it possible to prove that, with a value of h capable of being made less than any assignable quantity,

$$\psi x>Q(1-h)\,x+Gx,$$

$$\psi x<Q(1+h)\,x+Fx,$$

where $\frac{Fx}{x}$, $\frac{Gx}{x}$ may be made as small as we please by taking x sufficiently large, (I mean by taking x greater than some certain value ξ). Then

$$\begin{aligned}(1+\epsilon_x)\log 2\,.\,x&<\psi x-\psi\frac{x}{2}+\psi\frac{x}{3}\ldots-\psi\frac{x}{2m}\\&<Q(1+h)\,x\left(1-\frac{1}{2}+\frac{1}{3}\ldots-\frac{1}{2m}\right)\\&\qquad+Fx-F\frac{x}{2}+F\frac{x}{3}\ldots-F\frac{x}{2m}.\end{aligned}$$

Let ξ be taken so great that for all values of x greater than $\frac{\xi}{2m}$, $\frac{Fx}{x}$ shall be less in absolute numerical value than $\frac{k}{2m}$, where k is an *arbitrary* positive quantity: then, if we take $x>\xi$, the sum of the absolute values of Fx, $F\frac{x}{2}$, $F\frac{x}{3}$, ... $F\frac{x}{2m}$, is less than kx; and *à fortiori*

$$Fx-F\frac{x}{2}+F\frac{x}{3}\ldots-F\frac{x}{2m}<kx.$$

Therefore $$Q(1+h)\log 2 \,.\, x > (1+\epsilon_x)\log 2 \,.\, x - kx.$$

Hence, Q being greater than $\dfrac{1+\epsilon_x}{1+h} - \dfrac{k}{(1+h)\log 2}$, and ϵ_x, h, k being all three capable of becoming indefinitely small, $1-Q$ cannot be a finite positive quantity; which amounts to saying that $1-Q$ cannot be positive.

In precisely the same manner, dealing with the other limit to Vx and stopping in its development at the term $\psi\dfrac{x}{2m+1}$ $\left(\text{instead of stopping at the term } -\psi\dfrac{x}{2m}\right)$ it may be proved that $1-Q$ cannot be negative. Hence $1-Q$ must be zero, that is, $Q=1$. Q. E. D.

We have thus determined what is the common limit to which the principal terms in the superior and in the inferior limits of ψx are bound to approximate, on the supposition of the possibility of formulae being discoverable admitting of the interval between these principal terms being capable of being made as small as we please. But to pronounce with certainty upon the existence of such possibility, we shall probably have to wait until some one is born into the world as far surpassing Tchebycheff in insight and penetration as Tchebycheff has proved himself superior in these qualities to the ordinary run of mankind.

62.

ON THE SOLUTION OF A CERTAIN CLASS OF DIFFERENCE OR DIFFERENTIAL EQUATIONS.

[*American Journal of Mathematics*, IV. (1881), pp. 260—265.]

CASTING my eye over Mr Moulton's valuable edition of Boole's *Treatise on Finite Differences* (see pp. 229—231), I was gratified to find that he had embalmed in it a solution that I had given* many years ago, of an equation in differences, of the simple but very general form expressed by equating to zero or to Pm^x the persymmetrical determinant

$$\begin{vmatrix} u_x & u_{x+1} & \dots\, u_{x+i} \\ u_{x+1} & u_{x+2} & \dots\, u_{x+i+1} \\ u_{x+2} & u_{x+3} & \dots\, u_{x+i+2} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ u_{x+i} & u_{x+i+1} & \dots\, u_{x+2i} \end{vmatrix}$$

which is of the ith degree and $2i$th order.

To fix the ideas, let us consider the simple case

$$u_x u_{x+2} - u^2_{x+1} = Pm^x,$$

of which, when $P = 0$, the solution is $u_x = A\alpha^x$, A and α being both arbitrary, but for P not zero is expressed by $u_x = \pm (A\alpha^x + B\beta^x)$ with the conditions

$$\alpha\beta = m, \quad AB(\alpha - \beta)^2 = P$$

which solution as an *obiter dictum* I may remark may easily be converted into the simpler and more explicit form

$$(\sin\beta)^2 u_x^2 + P\{\sin(\alpha + \beta x)\}^2 m^{x-1} = 0$$

where α, β are arbitrary constants.

If we proceed now to verify the solution in its original form, we shall immediately be led to perceive a certain generalization which the given equation may be made to undergo without ceasing to be soluble—the solution however becoming narrowed from a general to a special one: whether particular or singular I shall not discuss.

[* This Reprint, Vol. II., pp. 308, 313.]

If we write $u_x = A\alpha^x + B\beta^x$, the determinant becomes

$$\begin{vmatrix} A\alpha^x \quad + B\beta^x, & A\alpha^{x+1} + B\beta^{x+1} \\ A\alpha^{x+1} + B\beta^{x+1}, & A\alpha^{x+2} + B\beta^{x+2} \end{vmatrix}$$

which is equal to $AB(\alpha-\beta)^2(\alpha\beta)^x$; this is the verification spoken of: but, as a consequence, it is apparent that we must have

$$\begin{vmatrix} A\alpha^x \quad + B\beta^x \quad + C\gamma^x, & A\alpha^{x+1} + B\beta^{x+1} + C\gamma^{x+1} \\ A\alpha^{x+1} + B\beta^{x+1} + C\gamma^{x+1}, & A\alpha^{x+2} + B\beta^{x+2} + C\gamma^{x+2} \end{vmatrix}$$

$$= AB(\alpha-\beta)^2(\alpha\beta)^x + BC(\beta-\gamma)^2(\beta\gamma)^x + CA(\gamma-\alpha)^2(\gamma\alpha)^x.$$

Hence we can solve the equation

$$u_x u_{x+2} - u^2_{x+1} = Pl^x + Qm^x + Rn^x,$$

namely, we may write $\quad u_x + A\alpha^x + B\beta^x + C\gamma^x = 0,$

where $$\beta\gamma = l, \quad \gamma\alpha = m, \quad \alpha\beta = n,$$

$$AB(\alpha-\beta)^2 = R, \quad BC(\beta-\gamma)^2 = P, \quad CA(\gamma-\alpha)^2 = Q,$$

that is to say $\quad \alpha = \sqrt{\left(\frac{mn}{l}\right)}, \quad \beta = \sqrt{\left(\frac{nl}{m}\right)}, \quad \gamma = \sqrt{\left(\frac{lm}{n}\right)},$

$$A = \sqrt{\left(\frac{QR}{P}\right)}\frac{(\beta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad B = \sqrt{\left(\frac{RP}{Q}\right)}\frac{(\gamma-\alpha)}{(\beta-\alpha)(\beta-\gamma)},$$

$$C = \sqrt{\left(\frac{PQ}{R}\right)}\frac{(\alpha-\beta)}{(\gamma-\alpha)(\gamma-\beta)};$$

or calling $$\sqrt{(lmn)} = g, \quad \sqrt{(PQR)} = G,$$

$$\alpha = \frac{g}{l}, \quad \beta = \frac{g}{m}, \quad \gamma = \frac{g}{n},$$

$$A = \frac{G}{g}\cdot\frac{(n-m)\,l^2}{(l-m)(l-n)P}, \quad B = \frac{G}{g}\cdot\frac{(l-n)\,m^2}{(m-n)(m-l)Q},$$

$$C = \frac{G}{g}\cdot\frac{(m-l)\,n^2}{(n-l)(n-m)R}.$$

The result therefore in its rational unambiguous form is

$$PQR(lm-mn)^2(mn-nl)^2(nl-lm)^2(lmn)^{2x-1}u_x^2 = \{\Sigma\,(lm-ln)\,QR\,(mn)^x\}^2.$$

When any of the quantities P, Q, R vanish, or any of the quantities l, m, n vanish or become equal to one another, the solution fails.

We shall, however, easily obtain a compensatory form of equation supplying the place of two of the exponentials, and another supplying the place of all three becoming identical, and the solution of these substituted forms may be deduced from that of the original form of the equation.

Thus, first, let

$$\left.\begin{aligned} &m=(1+\epsilon)\mu, \qquad n=(1-\epsilon)\mu \\ &Q=\frac{1}{2}\left(S+\frac{T}{\epsilon}\right), \quad R=\frac{1}{2}\left(S-\frac{T}{\epsilon}\right) \end{aligned}\right\} \text{ where } \epsilon \text{ is an infinitesimal.}$$

Then the equation becomes

$$u_x u_{x+2} - u^2_{x+1} = Pl^x + S\mu^x + Tx\mu^x$$

and the solution in its unreduced form is

$$u_x = A\alpha^x + B\beta^x + C\gamma^x,$$

where $\alpha = \sqrt{\dfrac{\mu^2}{l}}, \quad \beta = (1-\epsilon)\sqrt{l}, \quad \gamma = (1+\epsilon)\sqrt{l},$

and

$$A = \sqrt{\left(\frac{PR}{Q}\right)}\frac{\beta-\gamma}{(\alpha-\beta)(\alpha-\gamma)} = \frac{T}{\sqrt{(-P)}}\frac{\sqrt{l}}{\left\{\sqrt{\left(\dfrac{\mu^2}{l}\right)}-\sqrt{l}\right\}^2} = \frac{T}{\sqrt{(-P)}}\cdot\frac{l^{\frac{3}{2}}}{(\mu-l)^2}$$

$$B = \sqrt{\left(\frac{PR}{Q}\right)}\frac{\gamma-\alpha}{(\beta-\alpha)(\beta-\gamma)}$$

$$= \sqrt{(-P)}\left(1-2\epsilon\frac{S}{T}\right)\frac{\sqrt{l}-\sqrt{\left(\dfrac{\mu^2}{l}\right)}+\sqrt{(l)}\,\epsilon}{\sqrt{l}-\sqrt{\left(\dfrac{\mu^2}{l}\right)}-\sqrt{(l)}\,\epsilon} \div (-2\epsilon\sqrt{l})$$

$$= \sqrt{(-P)}\left\{\frac{-1}{2\sqrt{(l)}\,\epsilon}+\frac{S}{T\sqrt{l}}-\frac{\sqrt{l}}{l-\mu}\right\}$$

$$C = \sqrt{(-P)}\left\{\frac{1}{2\sqrt{(l)}\,\epsilon}+\frac{S}{T\sqrt{l}}-\frac{\sqrt{l}}{l-\mu}\right\}.$$

Hence $$B\beta^x + C\gamma^x = \sqrt{(-P)}\, l^{\frac{x-1}{2}}\left\{\begin{aligned} &(1+x\epsilon)\left(\frac{1}{2\epsilon}+\frac{S}{T}-\frac{l}{l-\mu}\right) \\ &+(1-x\epsilon)\left(-\frac{1}{2\epsilon}+\frac{S}{T}-\frac{l}{l-\mu}\right) \end{aligned}\right\}$$

$$= \sqrt{(-P)}\left\{2\left(\frac{S}{T}-\frac{l}{l-\mu}\right)+x\right\} l^{\frac{x-1}{2}};$$

so that $$\sqrt{(-lP)}\, u_x = T\left(\frac{l}{\mu-l}\right)^2\left(\frac{\mu^2}{l}\right)^{\frac{x}{2}} - P\left(\frac{2S}{T}-\frac{2l}{l-\mu}+x\right) l^{\frac{x}{2}}$$

or $$Pl^{x+1}u_x^2 + \left\{T\left(\frac{l}{\mu-l}\right)^2\mu^x - P\left(\frac{2S}{T}+\frac{2l}{\mu-l}+x\right)l^x\right\}^2 = 0$$

will satisfy the given equation

$$u_x u_{x+2} - u^2_{x+1} = Pl^x + S\mu^x + Tx\mu^x.$$

When $T=0$ the solution fails, as we know *à priori* it ought to do.

When $S=0$ it takes the form

$$Pl^{x+1}u_x^2+\left\{T\left(\frac{l}{\mu-l}\right)^2\mu^x-P\left(\frac{2l}{\mu-l}+x\right)l^x\right\}^2=0.$$

We might, by an analogous process, writing $(1+\epsilon)$, $(1+\rho\epsilon)$, $(1+\rho^2\epsilon)$ in lieu of l, m, n, and giving P, Q, R appropriate values involving ϵ^2 as well as ϵ, render ΣPl^x a finite function of the form $(S+Tx+Ux^2)\lambda^x$, and deduce the solution of $u_xu_{x+2}-u^2_{x+1}=(S+Tx+Ux^2)\lambda^x$ as a particular case of the solution of the general equation. But as we can easily see that the unreduced form of the solution must be $u_x=\lambda^{\frac{x}{2}}(A+Bx+Cx^2)$, it will be easier to find A, B, C immediately from the equation

$$\begin{vmatrix} A+Bx+Cx^2 & A+B(x+1)+C(x+1)^2 \\ A+B(x+1)+C(x+1)^2 & A+B(x+2)+C(x+2)^2 \end{vmatrix}$$

or

$$\begin{vmatrix} A+Bx+Cx^2 & A+B(x+1)+C(x+1)^2 \\ B+C+2Cx & B+3C+2Cx \end{vmatrix}$$

or

$$\begin{vmatrix} A+Bx+Cx^2 & B+C+2Cx \\ B+C+2Cx & 2C \end{vmatrix}=S+Tx+Ux^2.$$

Hence $\quad -2C^2=U,\quad -2BC-4C^2=T,\quad 2AC-(B+C)^2=S.$

Hence

$$C=\sqrt{\left(\frac{-U}{2}\right)},\quad B=-2C-\frac{T}{2C}=-\sqrt{(-2U)}-\frac{T}{\sqrt{(-2U)}}=\frac{2U+T}{\sqrt{(-2U)}},$$

$$A=\frac{S+(B+C)^2}{2C}=\frac{S+\left\{\sqrt{\left(\frac{-U}{2}\right)}+\frac{T}{\sqrt{(-2U)}}\right\}^2}{\sqrt{(-2U)}}=\frac{-2SU+(T+U)^2}{-2U\sqrt{(-2U)}},$$

or $$8U^3u_x^2+\{2U^2x^2+(4U^2-2UT)x+2SU-(T+U)^2\}\lambda^x=0$$

is the required primitive of the given equation.

The method may obviously be extended to any equation of the given form: that is to say when the persymmetrical determinant which it contains is of the degree i and is equated to $(i+1)$ multiples of exponentials each of the form Pl^x an integral of it can be found, and if these i exponentials be subdivided into partial groups of ϵ, ϵ', ϵ''... terms in a group, then instead of the ϵ multiples of exponentials belonging to any group may be substituted

$$(P_1+P_2x+P_3x^2+\ldots+P_\epsilon x^{\epsilon-1})\,l^x,$$

and the solution of the equation so modified may be deduced from the solution first mentioned as a particular case thereof.

It will be sufficient for all reasonable purposes of illustration briefly to consider the case of

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} \\ u_{x+1} & u_{x+2} & u_{x+3} \\ u_{x+2} & u_{x+3} & u_{x+4} \end{vmatrix}=Pl^x+Qm^x+Rn^x+Sp^x.$$

An integral of this may be found by writing

$$u_x = A\alpha^x + B\beta^x + C\gamma^x + D\delta^x,$$

where $\beta\gamma\delta = l, \quad \alpha\gamma\delta = m, \quad \alpha\beta\delta = n, \quad \alpha\beta\gamma = p,$

$$BCD\zeta(\beta, \gamma, \delta) = P, \quad ACD\zeta(\alpha, \gamma, \delta) = Q, \quad ABD\zeta(\alpha, \beta, \delta) = R,$$

$$ABC\zeta(\alpha, \beta, \gamma) = S,$$

ζ meaning the product of the squared differences of the letters which it governs. We have thus

$$\alpha = \frac{g}{l}, \quad \beta = \frac{g}{m}, \quad \gamma = \frac{g}{n}, \quad \delta = \frac{g}{p},$$

where $g = \sqrt[3]{(lmnp)}$

and $A^3B^3C^3D^3[\zeta(\alpha, \beta, \gamma, \delta)]^2 = PQRS,$

so that writing

$$G = \left\{\frac{PQRS}{[\zeta(\alpha, \beta, \gamma, \delta)]^2}\right\}^{\frac{1}{3}} = \frac{1}{g^8} \frac{(PQRS)^{\frac{1}{3}}}{\left\{\zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)\right\}^{\frac{2}{3}}},$$

$$A = \zeta(\beta, \gamma, \delta)\frac{G}{P}; \quad B = \zeta(\alpha, \gamma, \delta)\frac{G}{Q}; \quad C = \zeta(\alpha, \beta, \delta)\frac{G}{R}; \quad D = \zeta(\alpha, \beta, \gamma)\frac{G}{S};$$

and thus

$$(PQRS)^2(lmnp)^8\left\{\zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)\right\}^2 u_x^3 = (lmnp)^x\{\Sigma QRS\zeta(\beta, \gamma, \delta)\, l^{-x}\}^3$$

or $$\left\{PQRS\zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)\right\}^2 u_x^3 = (lmnp)^{x-2}\left\{\Sigma\, QRS\zeta\left(\frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right) l^{-x}\right\}^3.$$

It is scarcely necessary to add that all the above conclusions continue to hold, when, on the left hand side of the equation for u_{x+h} we write $\left(\frac{d}{dx}\right)^h y$ and at the same time for any exponential l^x on the right hand side substitute e^{lx}.

Thus for instance we may in general find an integral of

$$yy'' - y'^2 = Ae^{hx} + Be^{kx}\cos(\alpha x + \beta)$$

or again of

$$(yy'' - y'^2)\, y'''' - y\,(y''')^2 + 2y'y''y''' - y''^3 = Ae^{hx}\cos(\alpha x + \beta) + Be^{kx}\cos(\gamma x + \delta).$$

63.

NOTE ON THE THEORY OF SIMULTANEOUS LINEAR DIFFERENTIAL OR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS.

[*American Journal of Mathematics*, IV. (1881), pp. 321—326.]

THIS theory is virtually the same for differential as for finite-difference equations. The mere verbal part of the exposition being somewhat easier for the former of the two, I shall prefer in the first instance to deal with them, although the applications are more interesting when made to bear on the latter. Simple to the last degree as are the method of solution and the nature of the result, I do not find the one or the other set out, or even indicated, except in the most perfunctory manner, in the ordinary text-books. This brief notice, designed for the junior readers of the *Journal*, is intended to supply the lacuna.

Let $u_{j,k}$ denote a linear function, with constant coefficients, of ω_k and of its first ϵ_j derivatives in respect to t.

Let

$$\begin{gathered} u_{1,1} + u_{1,2} + \ldots + u_{1,i} = 0, \\ u_{2,1} + u_{2,2} + \ldots + u_{2,i} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{i,1} + u_{i,2} + \ldots + u_{i,i} = 0, \end{gathered}$$

be the system of differential equations proposed for integration.

Call

$$\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i = \sigma.$$

The process of arriving at the reducing equation for any one of the variables is after the manner of the dialytic method of elimination, namely:

Along with the first equation take each of its $(\sigma - \epsilon_1)$th derivatives, with the second equation each of its $(\sigma - \epsilon_2)$th derivatives, ... and with the ith equation each of its $(\sigma - \epsilon_i)$th derivatives.

There will thus come into existence $(\sigma + 1)\, i - (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i)$, that is, $i\,(\sigma + 1) - \sigma$ equations between the $i\,(\sigma + 1)$ quantities

$$\begin{array}{l} \omega_1, \quad \delta_t \omega_1, \ldots \delta_t{}^{\sigma} \omega_1, \\ \omega_2, \quad \delta_t \omega_2, \ldots \delta_t{}^{\sigma} \omega_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_i, \quad \delta_t \omega_i, \ldots \delta_t{}^{\sigma} \omega_i. \end{array}$$

If we omit those which appear in any one of the lines above written, there will remain $(\sigma + 1)(i - 1)$ or $i\,(\sigma + 1) - \sigma - 1$ which might be eliminated between the $i\,(\sigma + 1) - \sigma$ equations, and there would thus result an equation between the quantities contained in the omitted line. This elimination, it will presently be seen, there is no occasion to perform; the noticeable algebraical fact about it is, that supposing it were performed, the form of the equation resulting between ω_k, $\delta_t \omega_k$, ... $\delta_t{}^{\sigma} \omega_k$ is invariable, whichever of the numbers 1, 2, 3, ... i be the value assigned to k.

Let the order of the highest derivative of each ω be reduced by one unit below the highest order previously taken, then there will be $i\sigma - \sigma$ or $(i - 1)\,\sigma$ equations connecting the $i\sigma$ quantities

$$\begin{array}{l} \omega_1, \quad \delta_t \omega_1, \ldots \delta_t{}^{\sigma-1} \omega_1, \\ \omega_2, \quad \delta_t \omega_2, \ldots \delta_t{}^{\sigma-1} \omega_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_i, \quad \delta_t \omega_i, \ldots \delta_t{}^{\sigma-1} \omega_i, \end{array}$$

and accordingly, if we omit the σ quantities which appear in any one (say the first) of the above lines, the remaining $(i - 1)\,\sigma$ quantities may each of them be expressed as linear functions of ω_1 and its $(\sigma - 1)$ derivatives: but the elimination previously indicated would lead to a homogeneous linear equation between ω_1 and its σ derivatives, and if in that, each argument $\delta_t{}^{\lambda} \omega_1$ be replaced by h^{λ} and λ_1, λ_2, ... λ_{σ} be the σ roots of the algebraical equation so formed, it follows from the ordinary theory for a single equation that ω_1 (provided the given equations, and consequently the resulting ones, be left in their general form) will be of the form

$${}^1C_1 e^{h_1 t} + {}^1C_2 e^{h_2 t} + \ldots + {}^1C_{\sigma} e^{h_{\sigma} t},$$

and consequently by virtue of the previous remark ω_2, ω_3, ... ω_k will be of the same form as ω_1 (but, of course, with different coefficients), that is to say, the σ roots h_1, h_2, ... h_{σ} are the same for the equation in σ_k as for the equation in σ_1, so that the coefficients in the equation between ω_k and its σ derivatives are, as premised, independent of the value of k.

Finally, to determine the equation whose roots are $h_1, h_2, \dots h_\sigma$, let ${}^1Ce^{ht}$, one of the terms in the general value, be taken as a particular value of ω_1, which with corresponding values of the other ω's will serve to satisfy the given equations; $\omega_2, \omega_3, \dots \omega_i$ being each of them linear functions of ω and derivatives of ω, must be of the forms ${}^2Ce^{ht}, {}^3Ce^{ht}, \dots {}^iCe^{ht}$, so that $\omega_1, \omega_2, \dots \omega_i$ and the derivatives of each of them will contain the common factor e^{ht}, and by substitution in the original equations we shall obtain a system of simultaneous algebraical equations leading to the equation

$$\begin{vmatrix} R_{1,1}, & R_{1,2} \dots R_{1,i} \\ R_{2,1}, & R_{2,2} \dots R_{2,i} \\ \dots & \dots \\ R_{i,1}, & R_{i,2} \dots R_{i,i} \end{vmatrix} = 0,$$

where in general $R_{p,q}$ is what $u_{p,q}$ becomes on writing h^μ in place of $\delta_t{}^\mu \omega_q$.

The above determinant of the ith order will be of degree $\epsilon_1 + \epsilon_2 + \dots + \epsilon_i$, that is, of the degree σ (for the general case) in h, and the roots of the equation will give the σ values $h_1, h_2, \dots h_\sigma$.

It follows, therefore, that the result of the hypothetical elimination in the first instance referred to will be a linear function of $\delta_t{}^\sigma \omega_k, \delta_t{}^{\sigma-1}\omega_k, \dots \delta_t \omega_k, \omega_k$ of which the coefficients will be identical with the coefficients of $h^\sigma, h^{\sigma-1}, \dots h, 1$ in the above determinant. Hence no matter now what special values may be attributed to the coefficients of the given equations, the result last obtained remains of *universal* validity—without excepting those cases in which the result of the hypothetical elimination would be such that the corresponding algebraical equation possess equal roots, although in those cases the form assumed in the course of the argument for the value of ω_1 (namely, a linear function of exponentials) ceases to hold good. Neither for the same reason need any exception be made for those cases where the number of terms in the equation to ω_k falls below σ on account of one or more of the leading coefficients in the result of the hypothetical elimination becoming zero: the degree to which h rises in the determinant will be in all cases the right degree, whether it reaches the extreme possible limit σ or falls below it.

The result obtained may be briefly summarized as follows.

If
$$(\phi_1 \delta_t)\, x + (\phi_2 \delta_t)\, y + \dots + (\phi_i \delta_t)\, z = 0,$$
$$(\psi_1 \delta_t)\, x + (\psi_2 \delta_t)\, y + \dots + (\psi_i \delta_t)\, z = 0,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$(\omega_1 \delta_t)\, x + (\omega_2 \delta_t)\, y + \dots + (\omega_i \delta_t)\, z = 0,$$

(each $\phi, \psi, \dots, \omega$ standing for a rational-integral functional form) then will

$$(R\delta_t)\, x = 0, \quad (R\delta_t)\, y = 0, \dots (R\delta_t)\, z = 0,$$

where $R(\delta_t)$ is the resultant in respect to $x, y, \dots z$ of what the above equations become when δ_t is treated as an ordinary algebraical quantity; under which form the proposition (by virtue of Euler's method of multipliers) becomes so nearly intuitive as to abrogate all necessity for any other demonstration*.

To pass to the parallel and more important theory in finite differences, it is only necessary to interpret $u_{j,k}$ to signify a linear function, with constant coefficients, of $(\omega_k)_t$, $(\omega_k)_{t+1}$, ... $(\omega_k)_{t+\epsilon_j}$, where t is the integer independent variable, (*say* $(\omega_k)_t$ and its ϵ_j difference-augmentatives), and instead of taking the differential derivatives of any one of the given equations, to take the corresponding difference-augmentatives. Then by precisely the same reasoning as before we shall have

$$\omega_{t+\sigma} + B\omega_{t+\sigma-1} + \dots + L\omega_t = 0,$$

$B, C, \dots L$ being so taken as that $h^\sigma + Bh^{\sigma-1} + \dots + L$ shall be the determinant represented by the same form of matrix expressed by R's as before, but where $R_{p,q}$ is obtained from $u_{p,q}$ by writing h^θ in lieu of any argument $\omega_t + \theta$ which occurs in it.

The simplest example that can be given is where $i = 2$, $\epsilon_1 = \epsilon_2 = 1$,

$$u_{1,1} = -\eta_{t+1} + a\eta_t, \qquad u_{1,2} = b\theta_t,$$
$$u_{2,1} = c\eta_t \qquad u_{2,2} = -\theta_{t+1} + d\theta_t;$$

this was the case which occurred in the article on the extension of Tchebycheff's theorem, in the last number of the *Journal* [p. 530, above], leading to the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0,$$

and to expressions for η_t, θ_t as linear functions of λ_1^t, λ_2^t.

It may also be remarked that this same case gives an instantaneous solution of the problem proposed and successfully treated by Babbage in his *Calculus of Functions*, more than half a century ago, and since revived in connection with the theory of substitutions (Serret, *Alg. Sup.* 4 ed., tom. 2, pp. 256—262). The problem is to find $\phi x = \dfrac{ax+\alpha}{\beta x + b}$ so that $\phi^i x$, say $\dfrac{a_i x + \alpha_i}{\beta_i x + b_i}$, shall equal x for a given value of i.

* I regret that this simple reflection did not present itself to my mind before the preceding investigation, the necessity for which it does away with, had been set up in print. It of course applies equally well to the analogous proposition for finite-difference equations (u_i, v_i, ... being substituted for x, y ..., and $1+\Delta$ for δ_t). This last named proposition, limited to the case of equations of the first order, is the foundation-stone of my new theory of Matrices regarded as Quantities, that is, as subject to every kind of functional operation which ordinary arithmetical or algebraical quantities are or can be subject to: but though so important and so easily established, I know not where it can be found explicitly stated.

To find in general $\phi^i x$ it is only necessary to solve the difference equations

$$u_i = au_{i-1} + \alpha v_{i-1},$$
$$v_i = \beta u_{i-1} + bv_{i-1},$$

and then u_i, v_i will, if $u_0 = 1$, $v_0 = 0$, coincide with a_i, β_i, and if $u_0 = 0$, $v_0 = 1$ with α_i, b_i.

Thus calling ρ_1, ρ_2 the two roots of

$$\begin{vmatrix} -\rho + a & \alpha \\ \beta & -\rho + b \end{vmatrix} = 0,$$

α_i will be of the form $C(\rho_1^i - \rho_2^i)$ and β_i of the same form except as to C, say $\Gamma(\rho_1^i - \rho_2^i)$. Also a_i, b_i will be of the forms $C_1\rho_1^i + C_2\rho_2^i$, $\Gamma_1\rho_1^i + \Gamma_2\rho_2^i$, where $C_1 + C_2 = 1$, $\Gamma_1 + \Gamma_2 = 1$, and the required condition will be fulfilled, provided only that $\rho_1^i = \rho_2^i$, or say

$$\rho_1 = K\left(\cos\frac{\lambda\pi}{i} + \sqrt{(-1)}\sin\frac{\lambda\pi}{i}\right)$$
$$\rho_2 = K\left(\cos\frac{\lambda\pi}{i} - \sqrt{(-1)}\sin\frac{\lambda\pi}{i}\right)$$

that is, if $(a+b)^2 - 4(ab - \alpha\beta)\left(\cos\frac{\lambda\pi}{i}\right)^2 = 1$, λ having any integer value (which without loss of generality may be taken inferior to i) except zero*.

If $\lambda = 0$, the two roots of the equation in ρ become equal and the form of the solution changes into

$$u_i = (C_1 + C_2 i)\rho^i, \qquad v_i = (C_1' + C_2' i)\rho^i.$$

When $u_0 = 1$ and $v_0 = 0$ then $u_1 = a$, $v_0 = \beta$,

$$C_1 = 1,\ C_1' = 0, \qquad C_2 = \frac{a}{\rho} - 1,\ C_2' = \frac{\beta}{\rho},$$

and when $u_0 = 0$, $v_0 = 1$, $u_1 = \alpha$, $v_0 = b$,

$$C_1 = \frac{\alpha}{\rho},\ C_1' = \frac{b}{\rho} - 1, \qquad C_2 = 0,\ C_2' = 1,$$

and $\phi^i x = \dfrac{[\rho + i(a-\rho)]\,x + i\alpha}{i\beta x + \rho + (b-\rho)\,i}$, which cannot be periodic for any value of i, and when $i = \infty$ becomes

$$\frac{(a-\rho)\,x + \alpha}{\beta x + b - \rho} = \frac{a-\rho}{\beta} = \frac{\alpha}{b-\rho}, \text{ that is, } = \frac{a-b}{2\beta} \text{ or } \frac{2\alpha}{a-b},$$

so that $\phi^i x$ in this case continually converges to a constant limit.

I may add that $\phi^i x$ converges to a constant limit not merely when the roots ρ_1, ρ_2 of

$$\begin{vmatrix} a-\rho & \alpha \\ \beta & b-\rho \end{vmatrix}$$

* There will thus be $(i-1)$ values of λ which will each give a distinct admissible solution of the problem of periodicity, but of course only those values of λ which are relatively prime to i will give primitive solutions. If $i = i'\delta$ the effect of making $\lambda = \lambda'\delta$ will be to make $\phi^i x = x$ by virtue o its making $\phi^{i'} x = 0$.

are equal, but whenever they are real. For the general form of $\phi^i x$, it may easily be found, is

$$\frac{[(\rho_2-a)\,\rho_1^i+(\rho_1-a)\,\rho_2^i]\,x+\alpha\,(\rho_1^i-\rho_2^i)}{\beta\,(\rho_1^i-\rho_2^i)\,x+[(\rho_2-b)\,\rho_1^i+(\rho_1-b)\,\rho_2^i]},$$

which if $\rho_2>\rho_1$ when $i=\infty$ becomes $\dfrac{(\rho_1-a)\,x-\alpha}{-\beta x+\rho_1-b}=\dfrac{a-\rho_1}{\beta}$ or $\dfrac{\alpha}{b-\rho_1}$ where ρ_1 signifies the smaller of the two roots ρ_1, ρ_2; or in other words when $a-b>2\sqrt{(\alpha\beta)}$, the limiting value to $\phi^i x$, when ϕx represents $\dfrac{ax+\alpha}{\beta x+b}$, is $\dfrac{(a-b)+\sqrt{\{(a-b)^2-4\alpha\beta\}}}{2\beta}$, with the understanding that the quantity under the radical sign is to be taken positive.

So, if

$$x_{i+1}:y_{i+1}:z_{i+1}=ax_i+by_i+cz_i:a'x_i+b'y_i+c'z_i:a''x_i+b''y_i+c''z_i,$$

when all the roots of the determinant

$$\begin{vmatrix} a-\lambda & b & c \\ a' & b'-\lambda & c' \\ a'' & b'' & c''-\lambda \end{vmatrix}$$

are real, the point x_i, y_i, z_i, as i increases, will be found to approach indefinitely near to a fixed straight line; and if all the roots are equal, to a fixed point.

The condition of the system of ratios $x_i:y_i:z_i$ being periodic and having a period m is tantamount to the condition that the mth power of the matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$$

shall be the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

The complete solution of this problem, and of the more general one of extracting the mth root of any unit-matrix (that is, a matrix in which each element in the principal diagonal is unity, and the rest zero), which constitutes the ultimate generalization of Babbage's problem and is soluble by the same method, will probably appear in a memoir on matrices, in the forthcoming number of the *Journal*.

In general, for a matrix of the order ω, the number of mth roots is m^ω and each of them is perfectly determinate. But when the matrix is a unit-matrix or a zero-matrix (the latter meaning one in which every element is zero) there are distinct genera and species of such roots, and every species contains its own appropriate number of arbitrary constants.

64.

NOTE ON MECHANICAL INVOLUTION.

[*American Journal of Mathematics,* IV. (1881), pp. 336—340.]

MECHANICAL involution is the name invented by me to signify the relation between six lines in space, so situated that forces may be made to act along them whose statical sum is zero. The definition may be extended to comprise an indefinite number of lines, any six of which have this property.

I shall use $[p, q]$ for the present to denote the moment of a unit of force acting along the directed line p about the directed line q, taken positive or negative according as to a spectator looking in the given direction (or sense) of q, a force in the given direction (or sense) of p tends to produce a right-handed or a left-handed rotation, which tendency, by a property of our mental constitution, we know is not affected in kind by the lines p and q becoming interchanged—a fact which might also be anticipated with a high degree of probability from the circumstance that the unit-moment is measured by the product of the perpendicular distance from each other, of the two lines, multiplied by the sine of the angle between them, so that *each* factor of this product changes its sign when the relation or aspect of the two lines to each other is reversed. Hence it follows that $[p, q] = [q, p]$.

Three lines in a plane, it may be noticed, are in involution when they intersect in the same point, or, as a particular case, are parallel to each other.

Let a, b, c, d, e, f be any six lines in space, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ six forces capable of balancing when acting along the lines 1, 2, 3, 4, 5, 6 supposed to be in involution.

Then by the equation of moments in regard to each of the first series of lines taken successively as axes of rotation, we must have

$$\lambda_1[1, a]+\lambda_2[2, a]+\lambda_3[3, a]+\lambda_4[4, a]+\lambda_5[5, a]+\lambda_6[6, a]=0$$
$$\lambda_1[1, b]+\ldots\ldots\ldots\ldots\ldots\ldots +\lambda_6[6, b]=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_1[1,f]+\lambda_2[2,f]+\lambda_3[3,f]+\lambda_4[4,f]+\lambda_5[5,f]+\lambda_6[6,f]=0$$

and consequently the determinant

$$\begin{vmatrix} [1, a] \ldots\ldots\ldots\ldots\ldots\ldots [6, a] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [1,f] \ldots\ldots\ldots\ldots\ldots\ldots [6,f] \end{vmatrix} = 0.$$

Consequently we may find quantities μ_a, μ_b, μ_c, μ_d, μ_e, μ_f such that

$$\mu_a[a, 1]+\mu_b[b, 1]+\mu_c[c, 1]+\mu_d[d, 1]+\mu_e[e, 1]+\mu_f[f, 1]=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mu_a[a, 6]+\mu_b[b, 6]+\mu_c[c, 6]+\mu_d[d, 6]+\mu_e[e, 6]+\mu_f[f, 6]=0.$$

Thus it becomes evident by regarding μ_a, μ_b, μ_c, μ_d, μ_e, μ_f as the magnitudes of forces acting along the lines a, b, c, d, e, f, that the equations of moments of a given set of forces about six lines which are in general independent, become linearly related when the six axes are in involution—a conclusion which springs also immediately from the consideration that the law of statical composition of directed lengths is the same whether they be regarded as representing forces or as representing the axes of couples. So much by way of introduction.

I now pass to the formation of the intrinsic equation of condition to be satisfied in the case of involution.

To obtain this, let the lines a, b, c, d, e, f be made identical with 1, 2, 3, 4, 5, 6.

In each of these latter lines (say in i) let two points be taken at the distance $\frac{1}{l_i}$ apart, whose quadriplanar coordinates are respectively $\dot{i}_x$, $\dot{i}_y$, $\dot{i}_z$, $\dot{i}_t$, $\dot{i}_x'$, $\dot{i}_y'$, $\dot{i}_z'$, $\dot{i}_t'$, and let (i, j)—where j is another of the lines in involution—denote the determinant

$$\begin{vmatrix} \dot{i}_x & \dot{i}_y & \dot{i}_z & \dot{i}_t \\ \dot{i}_x' & \dot{i}_y' & \dot{i}_z' & \dot{i}_t' \\ \dot{j}_x & \dot{j}_y & \dot{j}_z & \dot{j}_t \\ \dot{j}_x' & \dot{j}_y' & \dot{j}_z' & \dot{j}_t' \end{vmatrix}$$

This determinant will represent (enlarged six-fold) a tetrahedron, two of whose opposite edges are the lengths intercepted between the pairs of points on i, j respectively, and consequently $l_i l_j (i, j)$ will serve to represent (on the same scale) the quantities previously represented by $[i, j]$.

Hence the determinant of the sixth order above written becomes

$$\begin{vmatrix} 0\ , & l_1 l_2 (1, 2), & l_1 l_3 (1, 3), & l_1 l_4 (1, 4), & l_1 l_5 (1, 5), & l_1 l_6 (1, 6) \\ l_2 l_1 (2, 1), & 0\ , & l_2 l_3 (2, 3), & l_2 l_4 (2, 4), & l_2 l_5 (2, 5), & l_2 l_6 (2, 6) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_6 l_1 (6, 1), & l_6 l_2 (6, 2), & l_6 l_3 (6, 3), & l_6 l_4 (6, 4), & l_6 l_5 (6, 5), & 0 \end{vmatrix}$$

and this equated to zero gives the intrinsic condition of involution.

Imagining this equation to be formed, the terms in each line and also the terms in each column will have some common factor, removing which, by a two-fold scheme of division, all the quantities l will disappear, so that now regarding each of the pairs of points on the lines 1, 2, 3, 4, 5, 6 respectively as *any two non-coincident points whatever*, the intrinsic condition is represented by the evanescence of the following symmetrical invertebrate (that is, zero-axial) compound determinant

$$\begin{vmatrix} 0 & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & 0 & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & 0 & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & 0 & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & 0 & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & 0 \end{vmatrix}$$

where each pair of numbers within a parenthesis represents a determinant of the fourth order*.

Just as the equations of moments of a system of forces about six lines in space are in general independent, but cease to be so if (and only if) these lines are in involution, so the equations of moments of a system of forces in a plane about three points are in general independent, and only cease to be so when the three points lie in a right line. Thus under the two-fold aspect of a system of force-directions and a system of axes of moments, six lines in involution in space are on the one hand the analogues of three force-directions in a plane in involution, that is, meeting in a point, and on the other hand of three points (centres of moments) lying in a right line; and as *concurrence* is the polar correlative to *collineation* we ought to expect to

* This determinant (which is sufficiently obvious, I have found since going to press) has been given by Professor Cayley in his memoir on line-coordinates, *Camb. Phil. Trans.*, 1861, which is avowedly based upon my constructions connected with the problem of Involution.

find involution in space to be its own polar correlative; that is, that the polar reciprocal of a system of lines in involution in respect to a general quadric should be another such system: and such is the fact: for, as I have shown in the *Comptes Rendus**, the necessary and sufficient condition of six lines being in involution is that they shall respectively intersect pairs of corresponding rays in two homographic pencils lying in two planes whose intersection contains the centres and two corresponding (coincident) rays of the two pencils—a condition which will not be affected by any polar transformation.

This leads to the remark that we may change the signification of the symbol (i, j) in the equation last indicated without destroying its validity as the condition of involution: namely, we may suppose two planes to be drawn through each line instead of two points being fixed upon it: and then if we understand by the determinant of two lines in space the determinant formed by the coefficients of the two pairs of equations which denote the lines, we may interpret (i, j) to mean the determinant of i, j and sum up the result obtained in the following proposition:

The determinants formed by six lines in involution, taken two and two together, are related in precisely the same manner as the squared distances from one another of six points in four-dimensional space.

The legitimacy of the second reading of (i, j) may be proved directly, as follows. For greater clearness let (i, j) when read with reference to pairs of planes through i and j, be called (I, J). Then

$$\begin{matrix} i_x & i_y & i_z & i_t \\ i_x' & i_y' & i_z' & i_t' \\ I_x & I_y & I_z & I_t \\ I_x' & I_y' & I_z' & I_t' \end{matrix}$$

will constitute an example of what in the *Johns Hopkins University Circular* for May, 1882†, I have called a *split matrix*, inasmuch as each of the first two lines multiplied term for term by each of the latter two gives

[* Vol. II. of this Reprint, p. 237.]

† Baltimore: John Murphy & Co.—It is interesting to notice (as there indicated) that the same theory of the split matrix here applied to mechanical involution has an important, although quite a different kind of bearing on the theory of algebraical involution. The two theories of involution have a considerable affinity to each other—groundforms and their coefficients in the equation of linear connection in the one theory, being regarded as the analogues of space-directions and the force-magnitudes acting along them in the other. (See *J. H. U. Circular*, June, 1882.) It was the sense of this connection which caused me to throw a retrospective glance on the theory of mechanical involution, abandoned by me since the remote date of the appearance of my papers on the subject in the *Comptes Rendus*. I ought to mention that I owe the idea of applying the split-matrix theory to the proof of the polar property of an involution-system, to a suggestion of Professor Cayley.

products whose sum is zero. Hence by virtue of the property of such a matrix, each complete minor of the upper pair will bear to the opposite complete minor in the lower pair the ratio of (i) to (I), where

$$(i)^2 = \begin{vmatrix} \Sigma i_x^2 & \Sigma i_x i_x' \\ \Sigma i_x i_x' & \Sigma i_x'^2 \end{vmatrix} \text{ and } (I)^2 = \begin{vmatrix} \Sigma I_x^2 & \Sigma I_x I_x' \\ \Sigma I_x I_x' & \Sigma I_x'^2 \end{vmatrix},$$

and of course the same conclusions apply *mutatis mutandis* when j, J take the place of i, I; from which it immediately follows that

$$(i, j) : (I, J) = (i)(j) : (I)(J).$$

Let now in the (i, j) determinant, which is equated to zero, each element in any θth column be multiplied by $\frac{I_\theta}{i_\theta}$, and then again each element in any θth row by the same; these multiplications will not affect the equality to zero of the determinant so modified, but the effect of the combined multiplications will be to change the element in the ith row and jth column, namely, (i, j), into $\frac{(I)(J)}{(i)(j)}(i, j)$, that is into (I, J). Thus it is proved that we may pass from the first reading of the (i, j) determinant to the second; and this in its turn serves to prove that if six lines are in involution their polars in respect to any quadric must also be in involution.

The theory of involution may of course be extended to a system of $\frac{n(n+1)}{2}$ lines in n-dimensional space.

65.

SUR LES PUISSANCES ET LES RACINES DE SUBSTITUTIONS LINÉAIRES.

[*Comptes Rendus*, XCIV. (1882), pp. 55—59.]

On sait ce que veut dire un déterminant de substitution. Ces déterminants ne diffèrent nullement, dans leur forme extérieure, des déterminants ordinaires, que l'on peut nommer *déterminants absolus*, mais les lois de combinaison ne sont pas les mêmes dans les deux cas. Ainsi, par exemple, l'inverse du déterminant absolu $\begin{vmatrix} a & \alpha \\ \beta & b \end{vmatrix}$ est

$$\begin{vmatrix} \dfrac{b}{\Delta} & -\dfrac{\beta}{\Delta} \\ -\dfrac{\alpha}{\Delta} & \dfrac{a}{\Delta} \end{vmatrix}$$

où

$$\Delta = ab - \alpha\beta,$$

tandis que pour ce même déterminant, envisagé comme déterminant de substitution, l'inverse est

$$\begin{vmatrix} \dfrac{b}{\Delta} & -\dfrac{\alpha}{\Delta} \\ -\dfrac{\beta}{\Delta} & \dfrac{a}{\Delta} \end{vmatrix},$$

et ainsi, en général, l'inverse d'un déterminant de substitution est ce que l'on peut nommer le *transversal* de l'inverse d'un déterminant absolu, c'est-à-dire ce que ce déterminant devient quand, en prenant la diagonale qui joint le premier au dernier terme comme axe, on fait décrire à l'inverse ordinaire une demi-révolution autour de cet axe. De même pour la multiplication de deux déterminants de substitutions A et B, chacun de l'ordre n; pour obtenir le produit de A par B, il faut multiplier ensemble le transversal de A par B, selon la règle ordinaire, ce qui donnera un déterminant C'; C, le transversal de C', sera le produit de la substitution A par la substitution B.

Ainsi, tandis que le carré d'un déterminant absolu quelconque est un déterminant symétrique, le carré d'un déterminant non symétrique de substitution reste asymétrique.

Soit un déterminant quelconque donné, et ajoutons le terme $-\lambda$ à chaque terme diagonal; on obtient ainsi une fonction de λ; je nomme les racines de cette fonction racines *lambdaïques* du déterminant donné, et j'obtiens facilement les deux théorèmes suivants:

(1) *Les racines* lambdaïques *de l'inverse d'un déterminant sont les réciproques des racines* lambdaïques *du déterminant lui-même.*

(2) *i étant un nombre entier et positif quelconque, les $i^{\text{èmes}}$ puissances des racines* lambdaïques *d'un déterminant* de substitution *sont identiques avec les racines* lambdaïques *de la puissance $i^{\text{ème}}$ du déterminant.*

En réunissant ces deux énoncés, on parvient à ce théorème plus général:

i étant une quantité commensurable quelconque, les $i^{\text{èmes}}$ puissances des racines lambdaïques *d'un déterminant de substitution sont identiques avec les racines* lambdaïques *de $i^{\text{ème}}$ puissance du déterminant.*

Si le déterminant est symétrique, on n'a pas besoin de le définir comme représentant une substitution, car, pour les déterminants symétriques (qu'ils soient envisagés comme absolus ou comme substitutifs), les lois d'opération deviennent identiques.

Avec l'aide du théorème sur les racines lambdaïques, je parviens facilement à la résolution de ce beau problème:

Extraire la racine $\mu^{\text{ième}}$, ou plus généralement trouver la puissance $i^{\text{ième}}$ d'une substitution donnée, i étant un nombre commensurable quelconque.

Voici la solution. Soit n l'ordre du déterminant de substitution donné.

Soient K un terme quelconque dans ce déterminant, K_θ le terme qui occupe, dans la puissance $\theta^{\text{ième}}$ du déterminant, la même position que K dans le déterminant lui-même. De plus, soient $K_0 = 1$ quand K est un terme dans la diagonale, et $K_0 = 0$ dans tout autre cas. Alors je dis que, pour une valeur commensurable quelconque de i, positive ou négative, en nommant la somme des quantités $\lambda_2, \lambda_3, \ldots, \lambda_n$, S_1, leur produit, S_{n-1}, et en général la somme de leurs combinaisons binaires, ternaires, etc., S_2, $S_3, \ldots$ on aura

$$K_i = \Sigma \frac{K_{n-1} - S_1 . K_{n-2} + S_2 . K_{n-3} - \ldots \pm S_{n-2} . K_1 \mp S_{n-1} . K_0}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \ldots (\lambda_1 - \lambda_n)} \lambda_1^i,$$

où $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ sont les racines lambdaïques du déterminant donné.

Si l'on fait $i = \frac{1}{\mu}$, où μ est un nombre entier, on voit que le nombre des $\mu^{\text{ièmes}}$ racines est μ^n et consistera en μ^{n-1} groupes de μ matrices pour chaque groupe, ou pour le même groupe on passe d'une matrice à une autre, en multipliant chacun des n^2 éléments qu'il contient par la même racine $\mu^{\text{ième}}$ de l'unité.

Il peut arriver que les racines *lambdaïques* du déterminant ne soient pas toutes inégales; alors la formule générale pour K_i subira une modification qu'on déduit facilement du théorème général, au moyen de l'introduction de différences infinitésimales entre les racines.

Il y a cependant un cas très particulier qu'on ne doit pas manquer de signaler: c'est le cas où le nombre de solutions devient infini pour une valeur finie de i, où, en effet, le problème à résoudre devient un véritable porisme; dans ce cas, des n^2 quantités qu'on cherche, $n^2 - n$, c'est-à-dire tous les termes qui ne sont pas en diagonale, restent absolument arbitraires. C'est le cas où le déterminant donné est de la forme la plus simple possible, c'est-à-dire où tous les termes qui ne se trouvent pas dans la diagonale du déterminant donné sont des zéros, et tous les termes qui sont dans la diagonale égaux entre eux. Pour plus de clarté, supposons que tous les termes qui ne disparaissent pas sont des *unités.*

(1) Pour que le problème soit résoluble, il faut que μ ne soit pas moindre que n.

(2) μ n'étant pas inférieur à n, la seule condition nécessaire et suffisante pour que la $\mu^{\text{ième}}$ puissance du déterminant Δ soit de la forme proposée est que les *racines lambdaïques* de Δ soient égales respectivement à μ *racines distinctes* (choisies à volonté) de l'unité.

Par exemple, si $n=2$, pour que la $\mu^{\text{ième}}$ puissance de la substitution $\begin{vmatrix} a & \alpha \\ \beta & b \end{vmatrix}$ soit de la forme $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, on n'a qu'à faire les racines de

$$\lambda^2 - (a+b)\lambda + ab - \alpha\beta = 0,$$

égale à
$$\cos\frac{2r\pi}{\mu} + i\sin\frac{2r\pi}{\mu}, \quad \cos\frac{2s\pi}{\mu} + i\sin\frac{2s\pi}{\mu},$$
respectivement.

Si l'on veut seulement que la $\mu^{\text{ième}}$ puissance de $\begin{vmatrix} a & \alpha \\ \beta & b \end{vmatrix}$ soit de la forme $\begin{vmatrix} A & 0 \\ 0 & A \end{vmatrix}$, A étant arbitraire, il suffira que les deux racines de λ soient dans le rapport de 1 à une $\mu^{\text{ième}}$ racine imaginaire quelconque de l'unité, de sorte qu'on peut mettre

$$\lambda_1 = k\left(\cos\frac{r\pi}{\mu} + i\sin\frac{r\pi}{\mu}\right),$$
$$\lambda_2 = k\left(\cos\frac{r\pi}{\mu} - i\sin\frac{r\pi}{\mu}\right),$$

ce qui donne pour la seule condition nécessaire et suffisante

$$(a+b)^2 = 4\left(\cos\frac{r\pi}{\mu}\right)^2 (ab - \alpha\beta).$$

C'est la solution bien connue du problème soulevé et résolu par le célèbre M. Babbage, dans son traité *Sur le Calcul des Fonctions: Trouver*

$$\phi(x) = \frac{ax+\alpha}{bx+\beta},$$

tel que $\phi^\mu(x) = x$. La même question a été bien plus récemment considérée de nouveau par M. Serret (voir son *Cours d'Algèbre supérieure*, t. II. pp. 356—362).

66.

SUR LES RACINES DES MATRICES UNITAIRES.

[*Comptes Rendus*, XCIV. (1882), pp. 396—399.]

UNE matrice dont les termes sont tous des zéros, sauf toutefois les termes de la diagonale principale, qui ont des unités, constitue ce que je nomme une *matrice unitaire.*

Je suppose une telle matrice (assujettie à la loi de multiplication donnée par la combinaison des substitutions linéaires) de l'ordre n. On peut demander quelle est la forme d'une autre matrice M du même ordre n, telle que la $i^{\text{ème}}$ puissance de M soit une matrice unitaire.

J'ai donnée une solution de cette question dans ma précédente Note*.

Cette solution n'exige que n conditions, qui doivent être remplies par n^2 éléments de M; mais, chose remarquable, ce n'est pas la solution la plus générale. Je vais à présent donner toutes les solutions dont la question est susceptible. Soient $\nu_1, \nu_2, \nu_3, \ldots, \nu_k$ des nombres entiers et positifs quelconques dont la somme est n, et $\rho_1, \rho_2, \ldots, \rho_k$, k quelconques des $i^{\text{èmes}}$ racines de l'unité. Soit M_λ la matrice M affectée de l'indice λ, c'est-à-dire modifiée par l'addition de $-\lambda$ à chacun des n termes de la diagonale.

Considérons les systèmes de matrices mineurs de M, de l'ordre

$$n-\nu_1+1,\ n-\nu_2+1, \ldots,\ n-\nu_k+1$$

respectivement; et prenons M tel que

$$\lambda-\rho_1,\ \lambda-\rho_2, \ldots,\ \lambda-\rho_k$$

soient facteurs de chaque mineur du premier, du second, ..., du $k^{\text{ième}}$ de ces systèmes respectivement; alors M sera une racine $i^{\text{ème}}$ de la matrice unitaire de l'ordre n.

Ainsi, pourvu que i soit égal ou supérieur à n, il y aura autant de *genres* de racines $i^{\text{èmes}}$ de cette matrice qu'il y a de partitions indéfinies de n.

[* p. 562, above.]

Le genre principal (*summum genus*) sera celui qui correspond à la partition de n en n unités, et le nombre de conditions exigées sera n.

Le genre le plus bas (*infimum genus*) sera celui où n est laissé sans décomposition, et le nombre de conditions pour ce cas sera n^2.

En général, à $n = \nu_1 + \nu_2 + \ldots + \nu_k$ on aura une solution pour laquelle le nombre de conditions exigées sera $\nu_1^2 + \nu_2^2 + \ldots + \nu_k^2$, de sorte qu'il restera $2\Sigma\nu_1\nu_2$ constantes arbitraires dans M.

Si i est moindre que n, quelques-uns des genres manqueront, mais il y en aura toujours quelques-uns qui resteront. Ainsi, par exemple, si $n = 3$ et $i = 2$, le *summum genus*, qui suppose l'existence de trois racines distinctes des racines quadratiques de l'unité, cesse nécessairement d'exister; mais on aura une valeur de M pour laquelle tous les mineurs premiers de M_λ contiendront le facteur $\lambda - 1$, et une autre valeur de M (du même genre) pour laquelle tous ces déterminants contiendront le facteur $\lambda + 1$.

En effet, la matrice trouvée par M. Cayley, dans son Mémoire sur les matrices (*Philosophical Transactions*, 1858),

$$\begin{vmatrix} \dfrac{\alpha}{\alpha+\beta+\gamma}, & \dfrac{-(\beta+\gamma)}{\alpha+\beta+\gamma}\dfrac{\nu}{\mu}, & \dfrac{-(\beta+\gamma)}{\alpha+\beta+\gamma}\dfrac{\nu}{\mu} \\ -\dfrac{(\gamma+\alpha)}{\alpha+\beta+\gamma}\dfrac{\mu}{\nu}, & \dfrac{\beta}{\alpha+\beta+\gamma}, & \dfrac{-(\gamma+\alpha)}{\alpha+\beta+\gamma}\dfrac{\lambda}{\mu} \\ -\dfrac{\alpha+\beta}{\alpha+\beta+\gamma}\dfrac{\mu}{\nu}, & \dfrac{-(\alpha+\beta)}{\alpha+\beta+\gamma}\dfrac{\nu}{\lambda}, & \dfrac{\gamma}{\alpha+\beta+\gamma} \end{vmatrix},$$

sera la matrice M, telle que chaque mineur de M_ρ contiendra $(\rho - 1)$; de même chaque mineur de $(-M)_\rho$ contiendra $\rho + 1$; on remarquera que 1 et -1 sont les racines carrées de l'unité, et l'on vérifiera aisément que M^2 ou, ce qui revient au même, $\Phi(-M)^2$ ont tous les deux la forme

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Le genre *infime* de solution sera

$$M = \begin{vmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{vmatrix}$$

où $$p^2 = 1, \quad q^2 = 1, \quad r^2 = 1.$$

Il y a une théorie analogue pour l'extraction des racines de la matrice *zéroïdale*, c'est-à-dire où tous les termes de la matrice sont des zéros, ce qui constitue encore un nouveau cas de porisme dans la théorie de l'extraction des racines des matrices.

Je n'entrerai pas dans les détails de cette question: il suffit de l'indiquer par le cas le plus frappant; je dis que, si M est une matrice de l'ordre n telle que le déterminant de M_ρ soit de la forme ρ^n (ce qui n'exige que la satisfaction de n conditions entre les n^2 termes de M), M^n sera une matrice zéroïdale. Ainsi, par exemple,

$$\begin{vmatrix} a & a\dfrac{\lambda}{\mu} \\ -a\dfrac{\mu}{\lambda} & -a \end{vmatrix}^2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}.$$

De même, comme solution particulière du cas de $n=3$, on trouve que si 1, ρ, ρ^2 sont les trois racines de l'unité,

$$\begin{vmatrix} (\rho-\rho^2)(c-b) & \dfrac{\mu}{\lambda}(a+\rho b+\rho^2 c) & -\dfrac{\nu}{\mu}(a+\rho^2 b+\rho c) \\ -\dfrac{\lambda}{\mu}(a+\rho^2 b+\rho c) & (\rho-\rho^2)(a-c) & \dfrac{\nu}{\mu}(a+\rho b+\rho^2 c) \\ \dfrac{\lambda}{\mu}(a+\rho b+\rho^2 c) & -\dfrac{\mu}{\lambda}(a+\rho^2 b+\rho c) & (\rho-\rho^2)(b-a) \end{vmatrix}^3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Je terminerai en ajoutant que j'ai déjà établi une théorie fonctionnelle générale des matrices, et que je ne regarde plus celles-ci comme des schemata d'éléments, mais comme des communautés, ou, si l'on veut, comme des quantités complexes.

Cette théorie n'est pas même bornée au cas de matrices simples. On peut faire subir à des lois générales d'Analyse les quantités complexes où chaque terme d'un complexe de l'ordre m est lui-même un complexe de l'ordre m', et chaque élément de ces nouveaux complexes encore un complexe de l'ordre m'', etc., de sorte qu'on a des complexes de rangs successifs qu'on peut prolonger indéfiniment.

67.

ON SUBINVARIANTS, THAT IS, SEMI-INVARIANTS TO BINARY QUANTICS OF AN UNLIMITED ORDER.

[*American Journal of Mathematics*, v. (1882), pp. 79—136.]

Er macht kein System, sondern es wird, es concrescirt in ihm, wie das Kind im Mutterleibe. (Schopenhauer) *Deutsche Rundschau*, July, 1882, p. 69.

§ 1. Proem.

Any rational integer function ϕ of the letters $a, b, c, \ldots$ indefinitely continued, which satisfies the partial differential equation

$$(a\delta_b + 2b\delta_c + 3c\delta_d \ldots)\,\phi = 0$$

may be termed a subinvariant in respect to the elements $a, b, c, \ldots$ or simply a subinvariant to or *quâ* those elements. It follows from this definition that any rational integer function of one or more subinvariants is itself one.

The same function of the letters $a, b, c, \ldots$ which, when regarded as the coefficient of the highest power of the first variable x in a covariant to the quantic $(a, b, c, \ldots \between x, y)^i$ or the polynomial $(a, b, c, \ldots \between x, 1)^i$ is termed a differenciant of the quantic or polynomial, when regarded as an individual of the infinite scale to which ϕ belongs, assumes the name of a subinvariant in respect to the letters $a, b, c, \ldots$.

Of course a differenciant derives its name from reference to the fact that when multiplied by a suitable power of a it may be regarded as a function of the differences of the roots of any one of the infinite series of polynomials, of some covariant of each of which it is the principal coefficient.

It follows also from the definition that if any composite function is a subinvariant, each of its factors must be so too. For if the function be $P^\alpha . Q^\beta . R^\gamma \ldots.$ writing $a\delta_b + 2b\delta_c + \ldots = E$, we must have

$$\alpha \frac{EP}{P} + \beta \frac{EQ}{Q} + \gamma \frac{ER}{R} + \ldots = 0,$$

which for denominators $P, Q, R, \ldots$ relatively prime to each other is obviously impossible unless $EP = 0$, $EQ = 0$, $ER = 0 \ldots$, that is, $P, Q, R \ldots$ are subinvariants.

Again, suppose U, V, Ω to be three subinvariants so related that the equation $XU + VU = \Omega$ is capable of being satisfied at all. I say that it must be capable of being satisfied by subinvariantive values of X, Y*.

For from the equation it follows that $EX . U + EY . V = 0$, of which the most general solution is

$$EX = K\left(\frac{V}{\Delta}\right), \quad EY = -K\left(\frac{U}{\Delta}\right).$$

Hence $$X = \left(\frac{V}{\Delta}\right) E^{-1} K + U_1, \quad Y = -\left(\frac{U}{\Delta}\right) E^{-1} K + V_1,$$

where U_1, V_1 are subinvariants. Substituting these values of X, Y in the original equation, there results $U_1 U + V_1 V = \Omega$, as was to be shown possible. The same or a similar manner of proof will serve to show that if for three functions U, V, W, $XU + YV + ZW = 0$, X, Y, Z, are, or may be replaced by subinvariants. I do not know for certain, but think that the proposition may be extended to any number of given functions $U, V, W, \ldots$.

It is scarcely necessary to add the fundamental theorem that if for the elements $a, b, c, \ldots$ be substituted the elements a, $a\lambda + b$, $a\lambda^2 + 2\lambda b + c, \ldots$ where λ is arbitrary, any subinvariant remains unchanged; the proof being that if such a change be made in the elements of any function F, ΔF (the change in F) is expressible by $(e^E - 1) F$, which, when F is a subinvariant, so that $EF = 0$, vanishes identically. Hence it is that subinvariants become differenciants†.

It may be worth while here to notice that if in place of the operator on ϕ in the above equation *any* numerical linear function of $a\delta_b$, $b\delta_c$, $c\delta_d \ldots$ be substituted‡, the value of ϕ which satisfies the transformed equation will be a subinvariant *quâ* the elements $a, b, c, \ldots$ divided respectively by appropriate

* For instance, in the above equation, U, V may be supposed to be two subinvariants of equal extent, exceeding by a unit that of Ω, their resultant in respect to their final letter. We know, by a principle demonstrated further on in the text, that Ω must be a subinvariant. The present theorem shows that X and Y also are (or may be replaced by) subinvariants.

† Or more simply for any number of letters $\alpha_1, \alpha_2, \ldots \alpha_i$, not fewer than the number of ratios between $a, b, c, \ldots$, if

$$a\Sigma\alpha_1 = ib, \; a\Sigma\alpha_1\alpha_2 = \frac{i(i-1)}{2} c, \; a\Sigma\alpha_1\alpha_2\alpha_3 = \frac{i(i-1)(i-2)}{2 . 3} d \ldots \text{ then } a\delta_b + 2b\delta_c + 3c\delta_d \ldots = \Sigma \frac{d}{d\alpha},$$

because $$\Sigma \frac{db}{d\alpha} = a, \; \Sigma \frac{dc}{d\alpha} = 2b, \; \Sigma \frac{dd}{d\alpha} = 3c \ldots.$$

Hence any subinvariant to the letters $a, b, c, \ldots$ is a function of the differences of $\alpha_1, \alpha_2, \ldots \alpha_i$.

‡ So, for example, $(a\delta_b + b\delta_c + c\delta_d \ldots)^{-1} 0$ is a subinvariant *quâ* the elements

$$a, \; b, \; \frac{c}{1 . 2}, \; \frac{d}{1 . 2 . 3} \ldots.$$

numbers; namely, if the linear function be $pa\delta_b + qb\delta_c + rc\delta_d$, these numbers will be $1, p, \frac{p.q}{1.2}, \frac{p.q.r}{1.2.3}$, as will be evident by making

$$\alpha = a, \quad p\beta = b, \quad \frac{p.q}{1.2}\gamma = c, \quad \frac{p.q.r}{1.2.3}\delta = d, \ldots$$

which being done the operator last above written may be changed into

$$\alpha\delta_\beta + 2\beta\delta_\gamma + 3\gamma\delta_\delta \ldots.$$

As a consequence of this it will readily be seen that if $\phi(a, b, c, d, \ldots)$ be a subinvariant to the elements $a, b, c, d \ldots$

$$\phi(0, b, c, d, \ldots), \quad \phi(0, 0, c, d, \ldots), \quad \phi(0, 0, 0, d, \ldots)$$

will respectively be subinvariants *quâ* the elements

$$b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \ldots,$$

$$c, \frac{d}{3}, \frac{e}{6}, \ldots,$$

$$d, \frac{e}{4}, \ldots,$$

and so on, the denominators following the law of figurate numbers.

This theorem, although foreign to the original and primary object of the present paper, as given in § 4, is of some considerable importance to the method of deduction. I mean the method (theoretically perfect but practically very difficult of application for quantics beyond the 4th order) according to which all the groundforms of a quantic, or which is the same thing, their ground-differenciants*, may be deduced by an exhaustive algebraical process in successive strata or categories from one another beginning with the known forms a, $ac - b^2$, $a^2d - 3abc + 2b^3, \ldots$ as the first category. See § 3.

It follows from the definition above given that a subinvariant may contain any given number of letters, and the number which it actually contains, less one (that is, the weight of the most advanced letter which appears in it), may be called its *extent*. Any subinvariant will then be a differenciant to a quantic whose order is not less than such extent.

Of course the definition of subinvariant may be extended to sets of letters $a, b, c \ldots; a', b', c' \ldots; a'', b'', c'' \ldots$. Any function ϕ of these sets of letters may be called a subinvariant, or when necessary, by way of distinction, a pluri-subinvariant, which satisfies the equality

$$(a\delta_b + 2b\delta_c + \ldots + a'\delta_{b'} + 2b'\delta_{c'} + \ldots + a''\delta_{b''} + 2b''\delta_{c''} \ldots)\phi = 0.$$

* I shall frequently use the term groundform to signify the leading coefficient of what is ordinarily so termed.

But for greater simplicity, except when a necessity arises for enlarging the horizon, I shall, in what follows, confine myself to the case of a single set of letters, that is, of uni-subinvariants*.

By an irreducible subinvariant is of course to be understood one which cannot be expressed as a rational integer function of any others. A differenciant to an irreducible quantic is of necessity a subinvariant, but not necessarily or even generally an irreducible subinvariant in the absolute sense in which the word is employed above; it will, however, be inexpressible as a rational integer function of any other subinvariants whose extent does not exceed the order of the quantic concerned, and may thus be said to be *relatively* irreducible. Thus, for example, the subinvariant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$$

is irreducible, relatively to the extent 3 or *quâ* the letters a, b, c, d, that is to say, cannot be expressed as a rational integer function of subinvariants whose elements are limited to a, b, c, d, but it is not an irreducible subinvariant in the absolute sense of the term, because it can be represented by a combination of the subinvariants

$$a,\ ac - b^2,\ ae - 4bd + 3c^2,\ (ac - b^2)\, e + 2bcd - ad^2 - c^3,$$

the letter e being eliminated by the process of taking the difference between the product of the 2nd and 3rd and that of the 1st and 4th of the preceding groundforms†.

Here I may take occasion to state a theorem of wide generality suggested by the above decomposition. It is well known that if ϕ be a subinvariant extending to the letter l as the highest letter which it contains, all the successive derivatives of ϕ in respect to l will also be subinvariants, as is evident from the fact that if $(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\,\phi$ is zero, the same must be true of $(\delta_l)(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\,\phi$, or what is the same thing, of

$$(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\,\delta_l\phi.$$

Suppose then that ϕ, ψ, ω, ... are any number of subinvariants limited to l as their highest letter, and regarded, each of them, as a homogeneous function of l and 1, then I say that any differenciant in respect to l of this system of quantics will be a subinvariant *quâ* the elements $a, b, c, \ldots k$. For we know that any differenciant of the system $\phi(x)$, $\psi(x)$, ... say

$$(\alpha,\ \beta,\ \gamma \ldots \lambda \between x,\ 1)^i;\ \ (\alpha',\ \beta',\ \gamma' \ldots \lambda' \between x,\ 1)^{i'}, \ldots$$

* Eventually I am inclined to substitute the word binariant for subinvariant, and to speak of simple, double, treble or multiple binariants. The functions similarly related to ternary forms will then be styled simple or multiple ternariants, and so in general.

† So it may be shown that the subinvariants of deg-orders 5 . 7, 5 . 1, 5 . 5 to the Quintic (which are perfectly determinate), may be regarded as the resultants in respect to g of the sextic groundforms 2 . 0 and 4 . 6, 2 . 0 and 4 . 0, 2 . 0 and 4 . 4 respectively, all four of which are linear in g. See Sextic Germ Table, § 2. [p. 578, below.]

remains unaltered when α, $\alpha+\beta x$, $\alpha+2\beta x+\gamma x^2 \dots \phi$, respectively, are substituted for $\alpha, \beta, \gamma \dots \lambda$, and at the same time

$$\alpha', \ \alpha'+\beta' x, \dots \psi, \text{ for } \alpha', \ \beta', \dots \lambda',$$

respectively, and so on; that is to say, any subinvariant of the equation above written may be regarded as a function of

$$\phi x, \ \phi' x, \ \phi'' x, \dots ; \ \psi x, \ \psi' x, \ \psi'' x, \dots ; \dots .$$

Hence in regard of the system of subinvariants any of its differenciants is a function of the members of the system, and the successive derivatives in respect to l of each member, all of which are subinvariants. Hence the differenciant in question may be regarded as a function exclusively of subinvariants, and is therefore a subinvariant of the letters $a, b, c, \dots k$. As a particular application of the theorem we see that the resultant in regard to their last letter of two subinvariants of like extent and the discriminant of any subinvariant in regard to its last letter are subinvariants. Thus, for example, if the discriminant of a cubic be exhibited as a quadratic function of d, namely, under the form $a^2d^2+(4b^3-6abc)\,d+(4ac^3-3b^2c^2)$, its discriminant, namely,

$$(2b^3-3abc)^2-a^2(4ac^3-3b^2c^2), \text{ that is, } 4\,(b^6-3ab^4c+3a^2b^2c^2-a^3c^3)$$

is as it ought to be a subinvariant, namely, it is $4\,(b^2-ac)^3$. So more generally, if we regard any number of pluri-subinvariants (all of the same extent in each set of letters) as a system of multi-partite polynomials in the extreme letter of each set, any differenciant of such system will be a subinvariant (of course with diminished extent in each set) in regard to the original letters. The simple instance already given will serve as a diagram to make the reason self-evident. The invariant in respect to d of the discriminant of the cubic is the same as in respect to x of

$$a^2(x+d)^2+(4b^3-6abc)(x+d)+(4ac^3-3b^2c^2),$$

that is, of $a^2x^2+2\,(a^2d-3abc+2b^3)\,x+(a^2d^2+4b^3d-6abcd+4ac^3-3b^2c^2)$,

hence being a function of the three coefficients, which are all of them subinvariants, it is itself a subinvariant*.

It has been shown above that the same form which regarded as a differenciant is irreducible, that is, is incapable of being decomposed into products of other differenciants of no higher extent than its own, when regarded as a subinvariant may be, and as a matter of fact, far oftener than not will be decomposable into products of subinvariants of higher extent. Thus the irreducible differenciants to any quantic naturally resolve themselves into two classes, those which are absolutely irreducible and those which are only relatively so; and it would seem that in any natural method of proof of Gordan's theorem these would, it is likely, have to be considered separately.

* The method of proof here employed, it will be seen, is the same in kind as that employed in the ordinary proof of Taylor's theorem.

There is comparatively little difficulty in proving that the first class are finite in number; the proof of the second class being likewise finite, must depend upon the fact that they are the resultants of a finite number of functions.

I use the word resultant in the above paragraph in an enlarged sense. If U, V, W, ... are any given polynomials in $x, y, \dots z, t, \dots u$, I call any quantity not containing $x, y, \dots z$ capable of being exhibited under the form of the syzygetic function $U_1U + V_1V + W_1W \dots$ a resultant of the given polynomials in respect to $x, y, \dots z$. For resultants thus defined, the following important proposition admits of easy proof, namely: Every such resultant is capable of being represented as a sum of products U_1U, V_1V, ... of which the orders in $x, y, \dots z$ are *limited* in extent, and consequently the most general representation of such resultant can contain only a finite number of arbitrary parameters. When the number of the eliminables $x, y, \dots$ is one less than the number of the given functions which contain them, we fall back upon the ordinary kind of resultant, having only one arbitrary parameter. When there is but one eliminable x, and any number of polynomials U, V, W, ... of orders $\alpha, \beta, \gamma, \dots$ in x, the order in x of each syzygetic product U_1U, V_1V, ... in a syzygetic function of U, V, W, ... which is competent to represent any resultant of the system, is (if I mistake not) at most one unit less than the sum of the two highest (or of the two as high as any) of the numbers $\alpha, \beta, \gamma \dots$.

The orders of the syzygetic multipliers being once determined, the number of indeterminate constants is known, and these will be subject to satisfy a known number of *linear* equations, namely, a number greater by unity than the order of the $U_1U + V_1V \dots$ polynomial, and thus the problem of finding the complete system of resultants of the original system of polynomials in one variable is brought to depend upon the problem of finding the complete system of resultants of a system of homogeneous *linear* functions of several variables, a problem of which the solution and the number of arbitrary parameters which at most can appear in it are perfectly well known and need not be here set forth.

The syzygetic products U_1U, V_1V, ... whose sum is competent to express every resultant of U, V ..., I have said, need none of them be taken of an order so high as the sum of the two greatest of the quantities $\alpha, \beta, \gamma \dots$. Thus for instance in the case of U, V, W, ... being linear functions, the syzygetic multipliers, as is well known, need only to be taken as constants; or again when $\alpha, \beta, \gamma, \dots$ form a descending series, the syzygetic products need only to be all of them made of the same order as the highest of the given functions. Take, to fix the ideas, three functions, U, V, W, all of them quadratics in x. The syzygetic multipliers may be taken all linear functions in x: there will thus arise six disposable constants subject to three con-

ditions, inasmuch as the coefficients of x^3, x^2, x, must vanish in the sum of the products: if two of the multipliers, say of U, V, were made quadratic functions, there would be eight disposable constants subject to four conditions, since an additional coefficient, namely, of x^4, would have to vanish in the sum of the products: there would therefore be one additional arbitrary parameter, namely, $8-4$ instead of $6-3$, but the form of the *resultant* would be not more general than on the preceding supposition, because if to U_1, V_1 (the most general values of the linear multipliers of U, V), λV, $-\lambda U$ respectively be added, there will then be four arbitrary parameters, and consequently the solution must be the same as on the second supposition, but the value of the resultant remains unaltered by the change made in U_1, V_1.

Or again if U, V, W were the two first quadratics and the second a linear function in x, their syzygetic multipliers might be taken two constants and a linear function respectively: by raising the orders of any two of these multipliers by a unit, an additional arbitrary constant would be gained, but the sum of the products resulting therefrom would not thereby gain in generality, as may be shown by the same method as in the preceding example.

It might probably not be difficult to give a universal rule for determining the lowest orders of the syzygetic multipliers required for expressing the resultant in its most general form, of functions of one or even of several variables, but this is an inquiry which it is necessary to postpone, as it might lead to too long a deviation from the immediate purpose in view, and there are some difficulties attending the subject more than present themselves at first sight.

It is enough to know, and that only for the case of a single eliminable, the existence of a limit to the orders of the multipliers, which it is quite easy to demonstrate. That being premised, it will follow as an easy consequence, that any combination *inter se* of subinvariants of any given extent and each containing the highest letter corresponding thereto can only give rise to a limited number of subinvariants of lower extent, and from that it is easy by repeated applications of the same *principle of the limit* to infer that only a finite number of *relatively* irreducible subinvariants of any given extent (that is, irreducible into combinations of subinvariants of the same or lower extent) can arise from the combinations of a finite number of subinvariants of any given higher extent; but it will appear in the sequel that the degree and consequently that the number of irreducible subinvariants of any given extent is subject to a limit; consequently if the number of relatively irreducible subinvariants of any given extent (or which is the same thing, if the covariants of a quantic of any given order) were unlimited in number, this could only be in consequence of there being no extent so large but

that subinvariants of that extent and containing the most advanced letter corresponding thereto, would be needed in order to exhibit the composition of the relatively irreducible, but in an absolute sense, reducible subinvariants referred to.

In § 4 I propose to show how to obtain the types (that is, deg-weights) of the absolutely irreducible subinvariants of the first few degrees. Besides the intrinsic interest of the inquiry, the result obtained without going beyond subinvariants of the 7th degree will serve to show conclusively that *it is not true* "that syzygants and groundforms of the same degree and order cannot appertain to the same binary quantic," but that when the order of the quantic is sufficiently elevated there *must* appertain to it, syzygants (*compound* ones) and groundforms of the same degree and order.

Let it be observed that the proposition here about to be disproved is not coextensive with the law of parsimony, but goes considerably beyond it—that is, implies much more than that law gives warrant for.

Let us for the moment call the number of linearly independent forms of the deg-order (j, ω) to a given quantic given by Cayley's rule, the denumerator to the type (j, ω), and the number of forms of such type that can be obtained by compounding together groundforms of lower types, the aggregator to the same type. Let us further suppose that the duad (j, ω) may be compounded of (j', ω'), (j'', ω'')*.

Suppose further that the aggregator to the type (j', ω') exceeds its denumerator, and also that there exists one or more, say Δ' linearly independent invariantive forms of the deg-order (ω'', j''), but that (*if possible*) the aggregator to the type (j, ω) is equal to or less than its denumerator, the difference being Δ. Obviously if such a case can occur, the law of parsimony (that is, the Newtonian rule of not assuming more causes to exist than are necessary to the explanation of a phenomenon or set of phenomena) will, on such a supposition, lead to the conclusion, not that there are Δ groundforms and *no* syzygies, but $\Delta + \Delta'$ groundforms and Δ' syzygies. Such a case does not present itself for quantics of the lower orders; it seems natural and logical therefore to seek for it in the case of a quantic of an infinite order, that is, in the case of subinvariants unlimited in *extent*. If it can be shown (as in § 4 it will be shown) that with an unlimited number of letters, an irreducible subinvariant and a compound syzygy of subinvariants coexist for a given degree and for the weight ω, it will follow from the nature of the process employed in what follows, that the same conclusion must hold when the *extent* of the subinvariants is limited, provided (at the very worst) that the limit is not less than ω, for it will be seen that no letter of higher weight than ω enters into the process which leads to the result under con-

* I mean that $j = j' + j''$, $\omega = \omega' + \omega''$.

sideration. It is in all human probability true that the proposition holds good in the form in which it was originally presented, namely, that *irreducible* syzygants and irreducible invariantive derivatives of the same type, to the same quantic cannot coexist; but whether the proposition so limited is sufficient to support the substitution of the process of tamisage performed upon the numerator of the representative generating fraction, in lieu of tamisage performed upon the development of that fraction in an infinite series, or how the method of substitutive tamisage, if at present inexact, may be modified *pari passu* with the needful modification in brute tamisage so as to recover its validity, is a matter which must be reserved for future consideration.

§ 2. Germs.

Before proceeding to the more immediate object of this paper I think it will be profitable to insert the following table of the multipliers of the highest letter or power of the highest letter f in the relatively irreducible subinvariants of the extent 5 (that is, the leading coefficients in the groundforms of the quintic), and a similar table for the groundforms of the sextic arranged according to the powers of g*. For many purposes these tables will be found as serviceable as the entire function of the letters or even as the entire covariant written out at length. Those relating to the quintic may be verified by comparison with the tables (as far as they extend) contained in the *Formes Binaires* of M. Faà de Bruno, but the order of arrangement of the terms in those tables is not what my method of representation points out as the most natural, and proceeds upon some principle not easy to divine. It is also necessary to state that there are very many errors and misprints in those tables. With regard to the particular choice of the groundforms of any deg-order I believe that in all cases but one the tables of M. de Bruno are in accordance with those employed by myself, and which are on the face of them the *simplest* that can be employed, with one exception, namely, in the expression for the covariant of deg-order 9.3 the multiplier of the power of f, or *germ* as it may well be styled, is $(ac-b^2)^3$, whereas in the extended tables of M. de Bruno the germ will be found to be some numerical linear function (its exact value I have forgotten) of

$$(ac-b^2)^3,\ a^2(ac-e^2)(ae-4bd+3c^2),\ \text{and}\ a^3\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

or which comes to the same thing, of the two former and

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd;$$

the covariant thus given of deg-order 9.3 is accordingly more complicated than it need have been.

* Any such multiplier I call the *germ* of the form to which it appertains.

It may be well to notice that whenever two consecutive terms in either table occur with the same germ but different powers of the last letter, the complete subinvariant of the antecedent is (to a numerical factor *près*) the differential derivative of the consequent in respect to that letter; thus, for example, the leading coefficient in the covariant to the quintic of the deg-order 7 . 5 will be found by simply differentiating the invariant of the degree 8 and dividing the result by the number 3.

In the table immediately following (c), (d), (e), (e'), Δ stand for

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ ae-4bd+3c^2,\ ace-ad^2+2bcd-c^3-ad^2$$

and
$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd$$

respectively. The quantities which appear in the outside vertical column are the germs; the double figures which fill the occupied spaces are the deg-orders. Thus, for example, 7 . 5 being opposite to the germ $(c)(d)$ and in the column headed by f^2, indicates that the covariant to the quintic of degree 7 and order 5 has for its differenciant a quantity of the form

$$(ac-b^2)(a^2d-3abc+2b^3)^2 + \text{a linear function of } f,$$

and so in general.

GERM TABLE TO THE QUINTIC.

	1	f	f^2	f^3	f^4	f^7
a	1 . 5					
(c)	2 . 6					
(d)	3 . 9	4 . 4				
(e)	2 . 2					
(e')	3 . 3					
a^2		3 . 5	4 . 0			
$a(c)$		4 . 6	5 . 1			
$a(d)$			6 . 4			
$(c)^2$		5 . 7	6 . 2			
$(c)(d)$			7 . 5	8 . 0		
$3a(e')-2(c)(e)$		5 . 3				
$a^2(c)$				7 . 1		
$a(c)^2$				8 . 2		
$(c)^3$				9 . 3		
$(c)^2(d)$					11 . 1	
$(c)^2$					12 . 0	
$(c)^3(e')$					13 . 1	
$a(c)^5$						18 . 0

In the annexed table $(c)(d)(e)(e')(f)(\Delta)$ retain their previous significations. The additional symbols (cf), (c^2f), (df), (cef) represent respectively the differenciants to the quintic of the deg-orders 4.6, 5.7, 4.4, 5.3, all of which are linear functions of f (see preceding table).

GERM TABLE TO THE SEXTIC.

	1	g	g^2	g^3	g^5	g^6
a	1.6	2.0				
(c)	2.8	3.2				
(d)	3.12	4.6				
(Δ)			6.0			
(e)	2.4					
(e')	3.6	4.0				
(f)	3.8					
$a(c)$			5.2			
$a(d)$			6.6			
$a(e)$		4.4				
$(c)(d)$				8.2		
$(d)(e)$			7.4			
$(c)(f)$	4.10	5.4				
$(d)^3$						15.0
$a(d)(e)$				9.4		
$(c)^2(f)$		6.6*				
$a(d)(f)$			7.2			
$(c)(c)^2(f)$				10.2		
$(c)(e)(f)$	5.8					
$a^2(c)(d)$					12.2	
a^5					10.0	
$(c)(c^2f)$				10.2		

§ 3. GROUNDFORMS.

Quantitative Deduction of their Categories.

I will now proceed to explain what I mean by the exhaustive or quantitative method of deducing the ground differenciants to a given quantic, referred to in the course of the preceding observations.

The well-known functions of alternately the second and third degrees $ac - b^2$, $a^2d - 3abc + 2b^3$, $ae - 4bd + 3c^2$, ... limited in extent to the order of the quantic under consideration, may be called the protomorphs or primaries.

Suppose then the groundforms to the cubic are to be deduced. The primaries or protomorphs, omitting a, are $ac - b^2$, $a^2d - 3abc + 2b^3$, and the residues (meaning thereby the remainders when these quantities are divided by a) are $-b^2$, $2b^3$. Hence $(a^2d - 3abc + 2b^3)^2 + 4(ac - b^2)^3$ will divide out by a (as it happens by a^2) and give the new groundform

$$a^2d^2 + 4ac^3 + 6abcd + 4b^3d - 3b^2c^2.$$

Between its residue $4b^3d - 3b^2c^2$, and the two former, it is obvious that no new relation can arise. Hence the four forms

$$a,\ ac - b^2,\ a^2d - 3abc + 2b^3,\ a^2d^2 + 4ac^3 - 6abcd + 4b^3d + 3b^2c^2$$

constitute the complete system of ground differenciants, and the corresponding co- and- invariants comprehend the complete system of such for the cubic.

Proceeding to the quartic, a new protomorph or base-form comes into view, namely, $ae - 4bd + 3c^2$, whose residue is $-4bd + 3c^2$ in addition to the antecedent ones $4b^3d - 3b^2c^2$, $2b^3$, $-b^2$, and since the second of these is the product of the first and last it follows that

$$-(a^2d^2 + \ldots) + (ac - b^2)(ae - 4bd + 3c^2)$$

must contain the factor a, and on performing the division there emerges the new groundform

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

so that $(a^2d^2 + \ldots)$ being equal to this multiplied by a less the product of two other groundforms, ceases itself to be one, and the groundforms now subsisting are the one last named in addition to the base-forms

$$a,\ ac - b^2,\ a^2d - 3abc + 2b^3,\ ae - 4bd + 3c^2,$$

which, since the new one is the only one of the *five* containing the letter e, can enter into no combination with them of which the residue is zero, and consequently the *deduction* is at an end and the five named constitute the complete system of groundforms.

Beyond this point the method of deduction has not hitherto been pushed, nor could it have been, without the use of the theorem concerning the subinvariantive character of the residues, in consequence of their enormous complexity when regarded as simple functions of the letters. In what follows the deduction is extended to the case of the quintic*.

Algebraical Deduction of the Groundforms of the Quintic†.

The complete system of groundforms to be deduced may be denoted by the deg-order or the deg-weight: viewed as subinvariants, the latter is the more natural mode of designation: if j and ω are the degree and weight, the order ϵ will be $5j - 2\omega$. For greater facility of reference to the known list of groundforms, it will be convenient to set out the order as well as the degree; the complete system of the designating j; $\epsilon\,.\,\omega$, of the twenty-three groundforms, that is, of the twenty-three relatively irreducible subinvariants of *extent* not exceeding five, will then be as follows: $1;5\,.\,0$, $2;2\,.\,4$, $2;6\,.\,2$,

* In Salmon's *Modern Algebra*, 3rd Ed., pp. 170—1, 195—6, the base-forms employed in the deduction of the quartic groundforms are not identical with those employed above, the third one being of the fourth instead of the second degree in the letters, and consequently not a groundform, whereby the deduction is rendered somewhat longer than that given in the text. The most eligible base-forms to employ in any case are alternately of the second and third degrees, whereas those given by Prof. Cayley, the author of this important method, are of degrees continually increasing by a unit.

† By algebraical, I mean in this connection, that which deals only with the ordinary algebraical processes of addition, multiplication and division, as contradistinguished from transcendental processes involving differential operation, or which is substantially the same thing, symbolical resolution.

The preceding deduction for the Cubic and the Quartic is by far the simplest mode of obtaining the complete systems of groundforms for these quantics, and proving their completeness, which, at an earlier period of the theory, was regarded as a problem of some little difficulty. See Faà de Bruno's *Formes Binaires*, Chapter 7, pp. 260—263, where the same results are obtained through the medium of "*Formes Associées*." I cannot but think that sooner or later this method, first discovered by the eagle-gaze of Cayley, will lead to the object which I presume he had in view when he originated it, namely, a proof of Gordan's theorem by ordinary algebra.

I think I see looming in the not far distance such a proof, depending ultimately upon the fact of a certain succession of increasing integer multiplets, subject to stated laws of limitation, not being capable of being indefinitely produced. To render sensible the sort of arithmetical theorem which I have in view, I subjoin a theorem *ejusdem generis* concerning singlets (simple integers), which, as far as I know, is new, and admits of easy proof.

A succession of integers of which no one is a multiple of one nor the sum of the multiples of two others cannot be continued ad infinitum.

To prove this we may begin with the case where one of the integers written down is a prime number, for which case the proof is immediate. Then it is easy from this to show that if the theorem is true for the case where one of the integers is a product of only i-primes, it must be true for the case where one of the integers is a product of only $(i+1)$ primes; for this case, by virtue of the supposition made, may easily be reduced to the case where one of the numbers is a relative prime to all the others, for which case the theorem is true, for the same reason as if the number in question were an absolute prime. Consequently the theorem is true universally.

By the quotient of a duad (in what follows) is to be understood the quotient of the second element by the first; by the sum of two duads, the duad whose elements are the sums of the

3;5.5, 3;9.3, 3;3.6, 4;0.10, 4;4.8, 4;6.7, 5;1.12, 5;3.11, 5;7.9, 6;2.14, 6;4.13, 7;1.17, 7;5.15, 8;0.20, 8;2.19, 9;3.21, 11;1.27, 12;0.30, 13;1.32, 18;0.45. The protomorphs or base-forms are the five first of these, namely,

1;5.0 is a, 2;6.2 is $ac-b^2$, 3;9.3 is $a^2d-3abc+2b^3$,

2;2.4 is $ae-4bd+3c^2$, 3;5.5 is $a^2f-5abe+2acd+8b^2d-6bc^2$.

Again, 3;3.6 is the determinant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

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corresponding elements of the two, and by a multiple of a duad the duad whose elements are the elements of that duad multiplied each by the same *integer*. The foregoing theorem may then be extended as follows:

A succession of duads, the quotients of all which but two are intermediate to the quotients of those two, and such that no duad is a multiple of any one or the sum of the multiples of any two or three of the others, cannot be indefinitely continued.

Again, one couple of quantities may be said to be *intermediate* to three others when the point representing the first is situated within the triangle whose apices represent the other three; a point being said to represent the two quantities which are equal to its two coordinates in respect to any two given axes. So a triplet of quantities, by aid of an analogous representation in space, may be said to be intermediate to four others when its representative point lies inside the pyramid whose apices represent those four.

It will readily be understood that these definitions may be translated into conditions of inequality between determinants, and thus translated may be extended so as to yield a definition of one pollad of $n-1$ elements being *intermediate* to n, or indeed to any number of other such pollads. Also the quotient-system of an n-ad will be understood to mean the system of $(n-1)$ quotients got by dividing the first element of the n-ad into the $n-1$ others. The following general theorem may then be enunciated:

A succession of n*-ads such that the quotient-systems of all but* n *of them are intermediate to the quotient-systems of those* n cannot be indefinitely continued, if every n-ad which is either a multiple of some one or a sum of multiples of 2, 3, ..., n or $n+1$ of the others, is excluded from the succession.

More generally, and with a less stringent negative condition, *a succession of* n*-ads such that the quotient-systems of all but ν given ones (ν being any number) are intermediate to the quotient-systems of those ν, cannot be indefinitely continued, if every* n*-ad which is a multiple or a sum of multiples of any or all of the* n*-ads of a group of $\nu+1$ others (whereof ν are the given ones) is excluded from the succession.*

The hypothetical ground of connection between this theorem and Gordan's algebraical one is as follows: It may be shown to be implied in the method of deduction, that if the number of groundforms to the quintic were infinite, then there must exist a certain infinite succession of products, *[some of the form $b^xQ^yR^zS^t$, the others of the form $b^\xi Q^\eta R^\zeta S^\tau T$, such that neither any product $b^xQ^yR^zS^t$ nor any product $b^\xi Q^\eta R^\zeta S^\tau$ could be (a power of one or) a product of powers of any number of the products not involving T. If then it could be shown that there exists a set of quadruplets of the kind x, y, z, t such that every other one of that kind and also every one of the kind ξ, η, ζ, τ is *intermediate* to that] set, the existence of such a succession would be impossible by virtue of the arithmetical theorem, and the possibility of the existence of an infinite number of groundforms would consequently be disproved. A similar kind of proof could conceivably, but with more difficulty, be extended to quantics of any order.

[* For the words placed in square brackets, see the correction, p. 621, below.]

This, not involving the letter f, has been previously deduced, and it has been shown that its integrating factor (that is, the power of a by which it must be multiplied to give a rational integer function of the base-forms) is a^3; it has, in fact, been shown (dropping the second integer and dealing only with deg-weights) that

$$(1\,.\,0)^3\,(3\,.\,6) = (1\,.\,0)^2\,(2\,.\,2)\,(2\,.\,4) - 4\,(2\,.\,2)^3 + (3\,.\,3)^2.$$

I shall denote the residue of any form ϕ by the symbol $\mathfrak{R}\phi$; each such residue is a function of the *five* letters b, c, d, e, f, being in fact a subinvariant in regard to the letters $b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \frac{f}{5}$, and therefore of the four groundforms proper to the diminished extent 4, that is, of the five following functions

$$b,\ \frac{bd}{3} - \frac{c^2}{4},\ \frac{b^2e}{2} - 3\frac{bcd}{6} + 2\frac{c^3}{8},\ \frac{bf}{5} - 4\frac{ce}{8} + 3\frac{d^2}{9},$$

$$\begin{vmatrix} b, & \frac{c}{2}, & \frac{d}{3} \\ \frac{c}{2}, & \frac{d}{3}, & \frac{e}{4} \\ \frac{d}{3}, & \frac{e}{4}, & \frac{f}{5} \end{vmatrix}$$

or (getting rid of the denominators) of

$$b,\ 4bd - 3c^2,\ 2b^2e - bcd + 2c^3,\ 6bf - 15ce + 10d^2,$$

$$\begin{vmatrix} 3b, & 3c, & 2d \\ 3c, & 4d, & 3e \\ 10d, & 15e, & 12f \end{vmatrix}$$

of which the deg-weights are $1\,.\,1$, $2\,.\,4$, $3\,.\,6$, $2\,.\,6$, $3\,.\,9$ respectively; the first of these is b, the others I shall call Q, T, R, S respectively. In all that follows I shall denote a numerical linear function of two or more quantities by enclosing them in brackets with commas interposed*; thus, for example, (ϕ, ψ, θ) will mean $\lambda\phi + \mu\psi + \nu\theta$, where λ, μ, ν are certain determinate (but unexpressed) numbers.

We know from the theory of the groundforms of extent 4 (that is, differenciants of a quartic) that the above five quantities are not algebraically independent, but are connected by an equation of the form

$$T^2 = (Q^3,\ b^3S,\ b^2QR).$$

We have also the following expressions for the residues of the groundforms denoted by their deg-orders, and their first deduct, namely,

$$\mathfrak{R}\,(2\,;6) = b^2,\ \mathfrak{R}\,(3\,;9) = b^3,\ \mathfrak{R}\,(2\,;2) = Q,\ \mathfrak{R}\,(3\,;5) = bQ,\ \mathfrak{R}\,(3\,;3) = T,$$

or, using deg-weights instead of deg-orders,

$$\mathfrak{R}\,(2\,.\,2) = b^2,\ \mathfrak{R}\,(3\,.\,3) = b^3,\ \mathfrak{R}\,(2\,.\,4) = Q,\ \mathfrak{R}\,(3\,.\,5) = bQ,\ \mathfrak{R}\,(3\,.\,6) = T.$$

* The brackets will sometimes for convenience be omitted.

Since $b^3 . Q = b^2 . bQ$, that is, $\mathfrak{R}(3, 9)\,\mathfrak{R}(2, 2) - \mathfrak{R}(2.6)\,\mathfrak{R}(3, 5) = 0$, it follows that $((3\,;9)(2\,;2),\ (2\,;6)(3\,;5))$ must contain a.

Also it is obvious that the effect of throwing out a from a differenciant to the quintic which contains it, is to diminish the degree by one unit, leaving the weight unaltered, and therefore diminishes the order by five units.

Hence $$\frac{1}{a}((3\,;9)(2\,;2),\ (2\,;6)(3\,;5)) = 4\,;6.$$

It will be more convenient here and hereafter to use exclusively deg-weights instead of deg-orders to denote the forms; the above equation thus expressed becomes

$$\frac{1}{a}((3.3)(2.4),\ (2.2)(3.5)) = 4.7.$$

Turning now to the deg-weights of the residues, it will be seen that 4.7 can only be composed of 1.1 and 3.6.

Hence $\mathfrak{R}(4.7) = bT$, which is not a product of residues; so 4.7 must be a new groundform. Again, (adhering to the use of deg-weights) we have

$$(\mathfrak{R}(3.5))^2 = b^2Q^2 = \mathfrak{R}(2.2)\,(\mathfrak{R}(2.4))^2.$$

Hence $$\frac{1}{a}((3.5)^2,\ (2.2).(2.4)^2) = 5.10.$$

The only mode of resolving 5.10 into sums of the duads 1.1, 2.4, 3.6, 2.6, 3.9, is by the addition of 2.4 and 3.6.

Hence $\mathfrak{R}(5.10)$ is a numerical multiple of QT, that is, of $\mathfrak{R}(2.4)$ and $\mathfrak{R}(3.6)$. Hence $((5.10), (2.4)(3.6))$ contains a; consequently 5.10 is not a groundform, but we shall have $\frac{1}{a}((5.10),\ (2.4)(3.6)) = 4.10$, and 4.10 can be resolved into $1.1 + 3.9$ and $2.4 + 3.6$. Hence $\mathfrak{R}(4.10) = (bS,\ QR)$ and 4.10* will be a new groundform.

So again $(3.3)(3.5) = b^3.bQ$, and $(2.2)^2(2.4) = (b^2)^2Q$. Hence

$$\frac{1}{a}\{(3.3)(3.5),\ (2.2)^2(2.4)\} = 5.8,$$

which can be resolved in only one way into a sum of the duads 1.1, 2.4, 3.6, 2.6, 3.9, namely, into $1.1 + 1.1 + 3.6$. Hence

$$\mathfrak{R}(5.8) = \mathfrak{R}(2.2)\,\mathfrak{R}(3.6),$$

and consequently 5.8 is not a groundform, but

$$\frac{1}{a}\{5.8,\ (2.2)(3.6)\} = [4.8],$$

which, in respect to the duads above mentioned, is resoluble (and only resoluble) into $2.4 + 2.4$ and $1.1 + 1.1 + 2.6$.

* 4.10 which is the same (using deg-orders) as 4.0 obviously cannot undergo further depression, and is consequently a groundform.

Hence $\Re(4.8) = (Q^2, b^2R)$, and since $Q = \Re(2.4)$ we have

$$\Re((4.8), (2.4)^2) = b^2R;$$

and since $([4.8], (2.4)^2)$ is of the deg-weight 4.8, we see that there is a form 4.8 such that $\Re(4.8) = b^2R$, which is consequently a groundform, since b^2R is not a rational integer function of any of the previous residues. Thus, then, from the base-forms 2.2, 3.3, 2.4, 3.5, besides the groundform not containing f, namely, 3.6, we have derived the three additional groundforms 4.7, 4.10, 4.8. Of these 4.7 and 4.8 belong to the same category as 3.6, being like it derived immediately from the base-forms. Whereas, in obtaining 4.10 it has been necessary to employ 3.6, so that it belongs to a more distant category. If we call the base-forms primaries, 3.6, 4.7, 4.8 will be secondaries, and 4.10 a tertiary. So again we shall find

$$\Re(3.3)\,\Re(3.6) = b^3.T, \text{ and } \Re(2.2)\,\Re(4.7) = b^2.bT.$$

Hence $\frac{1}{a}\{(3.3)(3.6),\ (2.2)(4.7)\} = 5.9$, and $\Re(5.9) = b^3.R$,

which cannot be compounded out of the preceding residues, so that (5.9) is another tertiary.

Again $\Re(4.7)\,\Re(2.4) = Q.bT$, and $\Re(3.5)\,\Re(3.6) = bQ.T$.

Hence $\frac{1}{a}((4.7)(2.4),\ (3.5)(3.6)) = 5.11$, and $\Re(5.11) = (b^2S,\ bQR)$,

for 5.11, in regard to the oft-quoted duads, is resoluble only into

$$1.1 + 1.1 + 3.6 \text{ and } 1.1 + 2.4 + 2.6.$$

Hence 5.11 is also a tertiary groundform.

Again

$$\Re(2.2)\,\Re(2.4)\,\Re(3.6) = b^2.Q.T, \text{ and } \Re 4.7\ \Re 3.5 = bT.bQ.$$

Hence $$\frac{1}{a}((2.2)(2.4)(3.6),\ (4.7),\ (3.5)) = [6.12],$$

and the duad 6.12 is resoluble into

$$3.9 + (1.1)^3{}^* (2.6) + (2.4) + (1.1)^2(3.6)^2 \text{ and } (2.4)^3,$$

corresponding to b^3S, b^2QR, Q^3, T^2. Now Q, T, b^2R are all residues, as already shown, and since b^2 and $(bS,\ QR)$ are residues $(b^3S,\ b^2R.Q)$, and therefore b^3S is a residue.

Hence a form denotable by 6.12 which shall be a linear function of $[6.12]$ and of the combinations of inferior groundforms, will have a residue zero, and consequently $[6.12]$ will not be a groundform, but the 6.12 last spoken of will be divisible by a, and the quotient will give a groundform 5.12, whose residue corresponding to the composition $3.6 + 2.6$ is RT. We shall thus have obtained for our tertiary or third batch of groundforms (descendants,

* It will be often found convenient to use $(p.q)^i$ to mean the sum of i duads $p.q$.

that is, in the second degree from the base-forms) the subinvariants denoted by 4 . 10, 5 . 9, 5 . 11, 5 . 12.

Again $\mathfrak{R}(3.3)\,\mathfrak{R}(4.10) = b^3(bS,\ QR)$;

$$\mathfrak{R}(2.2)\,\mathfrak{R}(5.11) = b^2(b^2S,\ bQR);\ \mathfrak{R}(2.4)\,\mathfrak{R}(5.9) = Q(b^3R).$$

Hence between these three equations the two arguments b^4S, b^3QR may be eliminated, and there results

$$\frac{1}{a}\{(3.3)(4.10),\ (2.2)(5.11),\ (2.4)(5.9)\} = 6.13,$$

and 6 . 13 will be resoluble only into

$$3.6 + 2.6 + 1.1, \text{ so that } \mathfrak{R}\,6.13 = bRT.$$

Again $\mathfrak{R}\,3.5\,\mathfrak{R}\,5.12 = bQ.RT$; $\mathfrak{R}\,3.6\,\mathfrak{R}\,5.11 = T(b^2S,\ bQR)$;

$$(\mathfrak{R}\,4.7).(\mathfrak{R}\,4.10) = bT(bS.\ QR),$$

on the right-hand side of which three equations $bQRT$, b^2ST are the only two arguments appearing, so that

$$(\mathfrak{R}\,3.5,\ \mathfrak{R}\,5.12,\ \mathfrak{R}\,3.6,\ \mathfrak{R}\,5.11,\ \mathfrak{R}\,4.7,\ \mathfrak{R}\,4.10)$$

may be made equal to zero. Hence we have a new deduct 7 . 17, and $\mathfrak{R}\,7.17$ will be found $= (Q^2S,\ b^2RS,\ bQR)$, and 7 . 17 will be a groundform, as is apparent at once from the fact that it is the same (using a deg-order instead of deg-weight) as 7 ; 1 which is obviously indecomposable into any inferior forms.

But it may be objected that conceivably there might exist a syzygy between (3 . 5) (5 . 12), (3 . 6) (5 . 11), (4 . 7) (4 . 10), so that the form 7 . 17 obtained by dividing a linear combination of the three products by a may really be a null quantity. But not to mention the unlikelihood that a syzygy should occur between so low a number as only three products of groundforms of elevated degrees, the existence of such a syzygy may be directly disproved as follows: (3 . 6) (6 . 11) will contain only the first power of f, and writing

$$5.12 = Lf^2 + 2Mf + N, \quad 4.10 = Pf^2 + 2Qf + R,$$

we shall have $4.7 = Lf + M, \qquad 3.5 = Pf + Q,$

so that if the supposed syzygy exists we must have $LQ - MP = 0$, but

$$L = -a^2,\ M = 5abc - 2acd + 8b^2d + 6bc^2,\ P = (a^2c - ab^2),\ Q = \ldots.$$

Hence since M does not contain a as a factor, MP cannot equal LQ, so that the conceivable syzygy does not exist, and the groundform 7 . 17 is correctly deduced*.

* I shall eventually supersede this proof of the non-existence of the syzygy under discussion by a method involving no algebraical computation. It is a remarkable feature in this deduction that although it is in its nature quantitative, no algebraical computations whatever need to nor will be employed in working it out and establishing its validity at each stage, thanks to the use made of the factors of integration, as will presently appear.

Again $\mathfrak{R}\, 5.9\, \mathfrak{R}\, 3.5 = b^3R.bQ$, $\mathfrak{R}\, 3.3\, \mathfrak{R}\, 5.11 = b^3(b^2S,\ bQR)$,

$$(\mathfrak{R}\, 2.2)^2.(\mathfrak{R}\, 4.10) = b^4(bS,\ QR),$$

between which equations b^5S, b^4QR can be eliminated; thus there will be a form [7.14] deduced from

$$\frac{1}{a}\big((5.9)(3.5),\ (3.3)(5.11),\ (2.2)^2(4.10)\big).$$

Also the sole components of $\mathfrak{R}\, 7.14$ will be easily seen to be

$$\big(3.6 \times (2.4)^2,\ 3.6 \times 2.6 \times (1.1)^2\big).$$

Hence $\mathfrak{R}\,[(7.14)] = (Q^2T,\ b^2RT)$, in which each of the two arguments is a residue*. Hence we may find a 7.14 which will be divisible by a and thus obtain a form 6.14, which (since 5.14 is necessarily non-existent) cannot be further depressed.

That this is not a *null* form will presently be demonstrated. It results that 6.14 is a new groundform, and we have now completed a new (quatertiary) group, that is, the third in order of descent from the primaries, namely, the group 5.12, 6.13, 7.17, 6.14.

Here, having reached the middle of this long deduction, it will be expedient to pause for a while and take stock of the relations so far established between the base-forms and their deducts.

I enclose, in what follows, the deg-weight numbers within square brackets, in order to indicate that the forms which they represent are not necessarily identical with the simplified forms represented by the same numbers, but are the immediate quotients which present themselves after dividing out by a or a power of a in the course of the deduction. We have thus

$$a^3[3.6] - a^2[2.2].[2.4] = (2.2)^3,\ (3.3)^2 \qquad (3)$$
$$a\,[4.7] = [2.2][3.5],\ [2.4][3.3] \qquad (1)$$
$$a^2[4.8] + a\,(?) = [3.3][3.5],\ [2.2]^2[2.4] \qquad (2)$$
$$a^2[4.10] + a\,(?) = [3.5]^2,\ [2.4]^2[2.2] \qquad (2)$$
$$a\,[5.9] = [4.7][2.2],\ [3.3][3.6] \qquad (4)$$
$$a\,[5.11] = [4.7][2.4],\ [3.5][3.6] \qquad (4)$$
$$a^2[5.12] + a\,(?) = [3.6][2.4][2.2],\ [4.7][3.5] \qquad (5)$$
$$a\,[7.17] = [5.12][3.5],\ [3.6][5.11],\ [4.10][4.7] \qquad (6)$$
$$a\,[6.13] = [4.10][3.3],\ [2.2][5.11],\ [2.4][5.9] \qquad (5)$$
$$a^2[6.14] = [5.9][3.5],\ [5.11][3.3],\ [2.2]^2[4.10] \qquad (6)$$

In the above table the quantities connected by one or more commas represent a linear function of themselves, and the sign of interrogation means

* For Q^2, b^2R, T are each of them residues.

"some known rational integral function of the base-forms." The numerals to the right (beginning with (3) and ending with (6)) indicate the power of (a) by which each corresponding deduct has to be multiplied in order to become an integral function of the base-forms, and which may be called its integrating factor. Thus for example the integrating factor of [5 . 9] is a^4, because the integrating factors of the two arguments in the linear function expressing a [5 . 9] are a, a^3 respectively; so again a^5 is the integrating factor of [5 . 12], because the integrating factors of the arguments of the linear function which expresses a^2 [5 . 12] + a (?) are a^3, a respectively. So again the arguments corresponding to a [7 . 17] having the integrating factors a^5, a^4, a^3 respectively, the integrating factor of [7 . 17] will be $1 + 5$ (the dominant of the numbers 3, 4, 5), that is, 6. This will be sufficient to show how the integrating factors are to be successively obtained, it being of course borne in mind that the integrating factor of a product of deducts is the product of the integrating factors of the deducts taken separately. With the aid of this table we may see *à priori* that the linear forms representing [7 . 17], [6 . 13], [6 . 14] cannot be identically *nulls*. In the preceding cases no proof is required because we know subinvariants can only be decomposed in one way into factors.

Thus, firstly, for [7 . 17], the integrating factors of the three arguments being a^5, a^4, a^3; for if a syzygy existed between them we should have $B_1 + aB_2 + a^2B_3 = 0$, where each B is a rational integer function of the base-forms not containing a as a factor.

Secondly, for [6 . 13], the separate integrating factors being a^2, a^4, a^4 respectively, did a syzygy exist, we should have $a^2B + B_1 + B_2 = 0$, and consequently [2 . 2] [5 . 11] would be in syzygy with [2 . 4] [5 . 9], which is impossible.

Thirdly, for [6 . 14], the separate integrating factors being a^4, a^4, a^2, the syzygy is impossible, for the same reason as in the preceding case.

I pass on now to the fifth group, that is, to the deducts four degrees of succession removed from the base-forms.

$\mathfrak{R}\,2\,.\,2\;\mathfrak{R}\,6\,.\,13 = b^2.\,bRT$, $\mathfrak{R}\,3\,.\,6\;\mathfrak{R}\,5\,.\,9 = T.b^3R$. Hence there is a deduct [7 . 15]. Its integrating factor will be a into the dominant of the integrating factors of 6 . 13, 5 . 9, which are a^4, a^5, that is, it will be a^6. Also in regard to the duads 1 . 1, 2 . 4, 2 . 6, 3 . 9, 3 . 6, the compositions of 7 . 15 are

$$(1\,.\,1)^3 + (2\,.\,6)^2,\ (1\,.\,1)^2 + (2\,.\,4) + (3\,.\,9),\ (1\,.\,1) + (2\,.\,4)^2 + (2\,.\,6),$$

or b^3R^2, b^2QS, bQ^2R, and the two latter being residues we may write $\mathfrak{R}\,7\,.\,15 = b^3R^2$. Its integrating factor is a into the dominant of the integrating factors of 6 . 13, 5 . 9 (which are a^5, a^4), and is therefore a^6; 7 . 15 is necessarily a groundform, for b^3R^2 is obviously indecomposable into simpler residues.

Again $\mathfrak{R}\,3\,.\,6\;\mathfrak{R}\,6\,.\,13 = T\,.\,bRT$, and $\mathfrak{R}\,5\,.12\;\mathfrak{R}\,4\,.\,7 = RT.bT$. Hence 8.19 is a deduct, and its decompositions in respect to the customary duads being

$3.6 \times 3.9 \times 2.4$, $3.6 \times (2.6)^2 \times 1.1$, we have $\mathfrak{R}\, 8.19 = (QST, bR^2T)$. Also 8.19 is a groundform, for the existence of such a form as 7.19 is impossible, inasmuch as 5 times 7 is less than the double of 19. Its integrating index will be the dominant of those of $(3.6)(6.13)$ and $(4.7)(5.12)$ [which are $3+5$ and $1+5$ respectively] increased by unity, that is, is 9. I use here and shall in future use the phrase "index of integration" to signify the index of the power of a which is the integrating factor.

Again, $\mathfrak{R}\, 4.7\, \mathfrak{R}\, 6.13 = bT.b.RT$, $\mathfrak{R}\, 5.12\, \mathfrak{R}\, 3.6\, \mathfrak{R}\, 2.2 = RT.T.b^2$. Hence there is a deduct [9.20].

The resolutions of the duad 9.20 in respect to 3.6, 3.9, 2.6, 2.4, 1.1 are

$$3.6+3.9+2.4+1.1,\ 3.6+(2.6)^2+(1.1)^2,\ 3.6+(2.4)^2+2.6,$$

corresponding to $bQST$, b^2R^2T, Q^2RT. Now Q^2, b^2R, RT are already known to be residues, and $\mathfrak{R}\, 2.2\, \mathfrak{R}\, 3.3\, \mathfrak{R}\, (4.0) = (bQST,\ Q^2RT)$. Hence b^2R^2T, Q^2RT, $bQST$ are all residues. Hence there exists a deduct 9.20 such that $\mathfrak{R}\, 9.20 = 0$, and consequently there is a deduct 8.20 which must be a groundform *, since 7.20 is *à priori* known to be impossible. Its resolutions (regarded as a duad) in respect to the customary duads are

$$(1.1)^2+(3.9)^2,\ (1.1)+(3.9)+(2.4)+(2.6),\ (2.4)^2+(2.6)^2,\ (1.1)^2+(2.6)^3,$$

so that $\mathfrak{R}\, 8.20 = b^2S^2$, $bQRS$, Q^2R^2, b^2R^3. The index of integration to $(4.7)(6.13)$ is $1+5=6$, and of $(5.12)(3.6)(2.2)$ is $5+3=8$. Hence the index of integration to 8.20 is $2+8$ or 10.

We have now obtained a new group of ground-deducts, fourth in descent from the primaries, namely, 7.15, 8.19, 8.20, whose integrating factors are a^6, a^9, a^{10} respectively.

Again, we have, firstly, the following group

$$\mathfrak{R}\, 2.2\, \mathfrak{R}\, 5.12\, \mathfrak{R}\, 6.13 = b^2.RT.bRT,\ (\mathfrak{R}\, 3.6)^2\, \mathfrak{R}\, 7.15 = T^2.b^3R^2.$$

Hence there is a deduct [12.27].

In writing out the decomposition table (*quâ* 1.2, 2.4, 2.6, 3.9, 3.6 of 12.27), no account need be taken of $(3.6)^2$, inasmuch as T^2 which it represents is a rational integral function of b, Q, R, S, consequently $(3.6)^3$ will not appear therein.

The table will thus be

$$3.6+(3.9)^2+(1.1)^3,\ 3.6+3.9+2.6+2.4+(1.1)^2,$$
$$3.6+3.9+(2.4)^3,\ 3.6+(2.6)^3+(1.1)^3.$$

* I have accidentally omitted here (and may possibly have done so in some other cases) the usual proof by means of the indices of integration, that the deduct is not a null.

Hence $\Re[12.27] = (b^3S^2T,\ b^2QRST,\ Q^3ST,\ b^3R^3T)$. But b^3R^2, b^2QS, RT have all been seen to be residues, hence b^3R^3T, b^2QRST are residues.

Also $(\Re 4.10)^2 = (b^2S^2,\ bQRS,\ Q^2R^2)$ is a residue, as is also bT. Hence $(b^3S^2T,\ b^2QS.RT,\ bQ^2R.RT)$ is a residue, and $bQ(bS,\ QR)$, $Q(b^2S,\ bQR)$ being each of them residues, b^2QS, bQ^2R are each of them separately residues. Hence b^3S^2T is a residue. Also $Q^2\,\Re 8.2 = (Q^3ST,\ bQ^2R^2T)$ is a residue, and bQ^2R^2T is a residue, because bQ^2R, RT are residues. Hence Q^3ST is a residue. Hence all the arguments in expression for $\Re[12.27]$, namely, b^3R^3T, b^2QRST, $b^3S^2T^2$, Q^3ST are residues; consequently a deduct 12.27 may be found such that $\Re 12.27 = 0$, and there will be a deduct 11.27 which cannot be still further reducible, because 10.27 is necessarily non-existent. Its index of integration will be two greater than the dominant of those of $(5.12)(6.13)$ and 7.15, which are $5+5$ and 6, that is, it is 12. Its residue $\Re 11.27$ will easily be seen to be

$$(b^3R^4,\ b^3RS^2,\ bQ^2R^3,\ bQ^2S^2,\ b^2QSR^2,\ Q^3RS).$$

Again, secondly,
$$\Re 5.9\ \Re 5.12 = RT.b^3R,$$
$$\Re 3.6\ \Re 7.15 = T.b^3R^2.$$

Hence there is a deduct 9.21 which cannot be further depressed, because 8.21 is necessarily non-existent, and it will readily be found that

$$\Re 9.21 = (b^3S^2,\ b^3R^3,\ b^2QRS.Q^3S),$$

and that the index of integration is $1+4+5$, that is, is 10.

Again, thirdly,
$$\Re 6.13\ \Re 7.17 = bRT(Q^2S,\ b^2RS,\ bQR^2)$$
$$\Re 5.11\ \Re 8.19 = (b^2S,\ bQR)(QST,\ bR^2T)$$
$$\Re 3.6\ \Re^2 5.12 = T.(RT)^2 = R^2T(Q^3,\ b^2QR,\ b^3S)$$
$$\Re 5.12\ \Re 4.8\ \Re 4.10 = RT.b^2R(bS,\ QR)$$
$$\Re 2.4\ \Re 3.6\ \Re^2 4.10 = Q.T.(bS,\ QR)^2$$
$$\Re 2.4\ \Re 3.6\ \Re 8.20 = Q.T(b^2S^2,\ bQRS,\ Q^2RQ^2,\ b^2R^3).$$

Hence it will be seen that the arguments on the right-hand side of the equation are the five following, namely, bQ^2RST, b^3R^2ST, b^2QR^3T, b^2QS^2T, Q^3R^2T, and no others. Hence the six products on the left may be linearly combined so as to give a result zero, and there will consequently be a deduct 12.30.

To prove that this is not a null, take the integrating factors of

$$(6.13)(7.17),\ (5.11)(8.19),\ (3.6)(5.12)^2,\ (5.12)(4.8)(4.10),$$
$$(2.4)(3.6)(4.10)^2,\ (2.4)(3.6)(8.20).$$

These will be found to be

$$5+6,\ 4+9,\ 3+5+5,\ 5+2+2,\ 3+2+2,\ 3+10, \text{ or } 11,\ 13,\ 13,\ 9,\ 7,\ 13.$$

Hence if there were any syzygy between these products it must be between the 2nd, 3rd and 6th, which have a common integrating factor a^{13}, but the

3rd and 6th products have a common factor 3.6; hence the three cannot be syzygetically connected, and consequently 12.30 is a *bona-fide* existing deduct, and being incapable of further depression, is necessarily a ground-form.

The index of integration will be a unit greater than the dominant of the indices last found, that is, it is 14.

Its residue will be found to be of the form

$$(b^3S^3,\ b^3R^3S,\ b^2QR^4,\ b^2QRS^2,\ bQ^2R^2S,\ Q^3R^3,\ Q^3S^2).$$

Again, fourthly,

$$\begin{aligned}
&\mathfrak{R}\ 6.13\ \mathfrak{R}\ 8.19 = bRT.(QST,\ bR^2T)\\
&\mathfrak{R}^2 5.12\ \mathfrak{R}\ 4.\ \ 8 = R^2T^2.b^2R\\
&\mathfrak{R}^2 5.12\ \mathfrak{R}^2 2.\ \ 4 = R^2T^2.Q^2\\
&\mathfrak{R}^2 3.\ \ 6\ \mathfrak{R}^2 4.10 = T^2.(bS,\ QR)^2\\
&\mathfrak{R} 2.4\ \mathfrak{R} 3.6\ \mathfrak{R} 5.12\ \mathfrak{R} 4.10 = QT.RT.(bS,\ QR).
\end{aligned}$$

In these five equations the arguments on the left-hand side are four in number, namely, $b^2R^3T^2$, $b^2S^2T^2$, $bQRST^2$, $Q^2R^2T^2$. Accordingly, a linear combination of the five quantities on the right-hand side will be zero, and there is a deduct 13.32 which cannot be further depressed (since 12.32 is necessarily non-existent), and may be easily seen to be an actual quantity and not a null, inasmuch as the indices of integration of the products of which the quantities to the left are the residues (the anti-residues as they may be termed), are

$5+9,\ 5+5+2,\ 5+5,\ 3+3+2,\ 3+5+2$, that is, 14, 12, 10, 8, 10,

of which only *a pair are equal*. Its index of integration is one unit more than the dominant of these numbers, that is, is 15.

Finally $\mathfrak{R}13.32 = (b^2RT,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T)$. The four last deducts 11.27, 9.21, 12.30, 13.32 form the batch fifth in descent from the primaries, and their indices of integration have been shown to be 12, 10, 14, 15.

We are now within sight of the goal of our wearisome pilgrimage. We may form eight equations leading to 18.45, the skew-invariant, as follows:

$$\mathfrak{R}4.10\ \mathfrak{R}7.17\ \mathfrak{R}3.6\ \mathfrak{R}5.12 = (bS,\ QR)(Q^2S,\ b^2RS,\ bQR^2).T.R.T \qquad (1)$$

$$\mathfrak{R}^2 4.10\ \mathfrak{R}3.6\ \mathfrak{R}8.19 = (bS,\ QR)^2.T.(QST,\ bR^2T) \qquad (2)$$

$$\mathfrak{R}^2 4.10\ \mathfrak{R}6.13\ \mathfrak{R}5.12 = (bS,\ QR)^2.bRT.RT \qquad (3)$$

$$\mathfrak{R}8.20\ \mathfrak{R}3.6\ \mathfrak{R}8.19 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)\ T(QST.bR^2T) \qquad (4)$$

$$\mathfrak{R}8.20\ \mathfrak{R}6.13\ \mathfrak{R}5.12 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)bRT.RT \qquad (5)$$

$$\mathfrak{R}11.27\ \mathfrak{R}3.6\ \mathfrak{R}5.12 = (b^3R^4,\ bQ^2R^3,\ bQ^2S^2,\ b^2QSR^2)\ T.RT \qquad (6)$$

$$\mathfrak{R}6.13\ \mathfrak{R}13.32 = bRT(b^2R^4T,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T) \qquad (7)$$

$$\mathfrak{R}9.21\ \mathfrak{R}^2 5.12 = (b^3S^2,\ b^3R^3,\ b^2QRS,\ Q^3S)R^2T^2. \qquad (8)$$

The arguments on the right-hand side of these equations will be seen to be the seven following: $T^2b^3R^5$, $T^2b^3R^2S^2$, $T^2b^2QR^3S$, $T^2b^2QS^3$, $T^2bQ^2R^4$, $T^2bQ^2RS^2$, $T^2Q^3R^2S$. Hence a linear function of the anti-residues to the eight products to the left can be made zero, and the sums of each set of duads being $19 \,.\, 45$, there emerges the deduct $18 \,.\, 45$ corresponding to the skew-invariant $18\,;0$.

That this is not a null may be shown in the usual manner as follows: The indices of integration of the several anti-residues are

$$2+6+3+5,\ 2+2+3+9,\ 2+2+5+5,\ 10+3+9,\ 10+5+5,\ 12+3+5,$$
$$5+15,\ 10+5, \text{ that is, } 16,\ 16,\ 14,\ 22,\ 20,\ 20,\ 20,\ 15.$$

The 5th, 6th and 7th indices constitute the only triad of equal indices, but the 5th, 6th and 7th anti-residues cannot be in syzygy, inasmuch as the two first of them have the factor $5 \,.\, 12$ in common. Hence the value of $18 \,.\, 45$ found as above will not be null.

Its index of integration will be one unit more than the dominant of the above numbers, that is, it is 23, and its residue will be of the form

$$(b^3R^6T,\ b^3R^3S^2T,\ b^3S^4T,\ b^2QR^4ST,\ b^2QRS^3T,\ bQ^2R^5T,\ bQ^2R^2S^2T,$$
$$Q^3R^3ST,\ Q^3S^3T).$$

We ought now to be able to show that there exists no other deduct of which the residue is not a rational integral function of the 22 residues which have been determined in order to prove that the system of groundforms obtained is complete. But this inquiry is one of considerable difficulty, and must be reserved for future consideration.

I will now bring together the several steps of the deduction (several of which, especially in the earlier stages, would admit of abridgement), separating the successive strata from one another and substituting the more familiar designation of deg-orders for the equivalent deg-weights. The single numbers on the left-hand side are the indices of integration to the corresponding deducts.

Table of Deduction for the Quintic.

(3) $a^3(3\,;3) + a^2(?) = (2\,;6)^3,\ (3\,;9)^2$

(1) $a(4\,;6) = (2\,;6)(3\,;5),\ (2\,;2)(3\,;9)$

(2) $a^2(4\,;4) + a(?) = (3\,;9)(3\,;5);\ (2\,.\,6)^2(2\,.\,2)$

(2) $a^2(4\,;0) + a(?) = (3\,;5)^2,\ (2\,;2)^2(2\,;6)$

(4) $a(5\,;3) = (4\,;6)(2\,.\,2),\ (3\,;5)(3\,;3)$

(5) $a^2(5\,;1) + a(?) = (3\,;3)(2\,;2)(2\,;6),\ (4\,.\,6)(3\,.\,5)$

(4) $a(5\,;7) = (4\,;6)(2\,;6),\ (3\,;9)(3\,;3)$

(5) $(a)(6;4)=(4;0)(3;9),\ (2;6)(5;3),\ (2;2)(5;7)$

(8) $a(7;1)=(3;5)(5;1),\ (3;3)(5;3),\ (4;6)(4;0)$

(6) $a^2(6;2)+a(?)=(5;7)(3;5),\ (3;9)(5;3),\ (2.6)^2(4.0)$

(8) $a(7;5)=(2;6)(6;4),\ (3;3)(5;7)$

(9) $a(8;2)=(3;3)(6;4),\ (5;1)(4;6)$

(10) $a^2(8;0)+a(?)=(4;6)(6;4),\ (5;1)(3;3)(2;6)$

(12) $a^2(11;1)+a(?)=(2;6)(5;1)(6;4),\ (3;3)^2(7;5)$

(10) $a(9;3)=(3;3)(7;5),\ (5;7)(5;1)$

(14) $a(12;0)=(6;4)(7;1),\ (5;3)(8;2),\ (3;3)(5;1)^2(5;1)(4.4)(4.0),$
$(2;2)(3;3)(8;0)$

(15) $a(13;1)=(6;4)(8;2),\ (5;1)^2(6;4),\ (5;1)^2(2;2)^2,\ (3;3)^2(4;0)^2,$
$(2;2)(3;3)(5;1)(4;0)$

(23) $18;0=(4;0)(7;1)(3;3)(5;1),\ (4;0)(3;3)(8;2),$
$(4;0)^2(6;4)(5;1),\ (8;0)(6;4)(5;1),\ (8;0)(3;3)(8;2),$
$(6;4)(13;1),\ (9;3),\ (5;1)^2$

In addition to the deducts which appear in the above table, the groundform 1 . 5 and the four protomorphs 2 ; 2 2 ; 6 3 ; 5 3 ; 9 have to be taken into account. Thus the twenty-three groundforms to the quintic will be seen to be distributed among seven batches or categories containing respectively 1, 4, 3, 4, 3, 3, 4, 1 individuals.

It was my intention to have simplified some of the steps of the deduction, and to have supplied the omissions, to show in one or two cases that the deducts as obtained are actual and not null forms*, but unfortunately the proof-sheets have been kept back, owing to the necessities of the printing-office, for some weeks, and in the meanwhile my attention has been drawn off to other parts of the subject, and I am unable to give sufficient time to call back to mind the intended ameliorations or rectifications of the text.

§ 4. Perpetuants.

On Absolutely Irreducible Binary Subinvariants.

Any rational integral value of $(\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d\ldots)^{-1}0$ is a binary subinvariant. If none of the numerical coefficients $\lambda,\ \mu,\ \nu\ldots$ are zero, the subinvariant is simple. If in the series of coefficients $\lambda,\ \mu,\ \nu,\ \pi,\ \rho\ldots$, any number i of breaks occur in consequence of i non-contiguous terms $\nu,\ \rho\ldots$ vanishing, it becomes a multiple subinvariant corresponding to a semi-invariant

* When the deduct is a zero instead of a possible new groundform, it indicates a syzygy between anterior groundforms.

of i distinct binary quantics. If, however, the subinvariant is to appertain to a system of quantics, all of unlimited order, it would be necessary for the breaks in the series to be each of them at an infinite distance from the initial term and from one another.

In what follows I shall confine my attention to simple binary subinvariants, and investigate the types, that is, the deg-weights (order ceases to be predicable) of those of them which are absolutely indecomposable, that is, incapable of being expressed as rational integral functions of others of lower types of any extent whatever.

It may be convenient to give a name to absolutely indecomposable subinvariants, and I propose, until an apter word presents itself, to call them perpetuants*. The present section then will be occupied with the successive determination of the types of all possible simple binary perpetuants up to a certain limit of degree.

We know, by Cayley's rule, that the number of linearly independent binariants of degree j and weight w is the difference between the number of partitions of w into j parts, and the number of partitions of $w-1$ into such parts, and therefore by Euler's law of reciprocity is the difference between the number of partitions of w into parts none exceeding j, and the number of partitions of $w-1$ into such parts; it is therefore the coefficient of x^w in

$$\left\{\frac{1}{(1-x)(1-x^2)\dots(1-x^j)}-\frac{x}{(1-x)(1-x^2)\dots(1-x^j)}\right\}$$

or the coefficient of x^w in $\dfrac{1}{(1-x^2)(1-x^3)\dots(1-x^j)}$, which I shall call the generating function for the degree j of the linearly independent subinvariants.

Thus for the degree 1 the generating function is simply 1, and there will be one subinvariant (a) of the degree 1 and weight zero.

For the degree 2 the generating function is $\dfrac{1}{1-x^2}$, which expanded gives the series $1+x^2+x^4+\dots$; there is consequently one semi-invariant of the degree 2 for every even weight 0, 2, 4, 6 ...; but the first of these will be merely the square of the one of degree 0 and weight 1; hence the generating function for the perpetuants of degree 2 is $\dfrac{1}{1-x^2}-1$ or $\dfrac{x^2}{1-x^2}$ giving rise to the deg-weights 2.2 2.4 2.6 ... corresponding to the well-known series of quadrinvariants or quadri-semi-invariants $ac-b^2$, $ac-4bd+3c^2$, ... Again,

* Perhaps *Revenants* would be more expressive to signify the forms (or ghosts of forms, if one pleases to say so) which never die out, but *continually return* as the leading coefficients of irreducible covariants. Such I need not say is not the case with conditionally irreducible integrals of the above partial differential equation (as for instance the discriminants to the cubic), which sooner or later die out and are seen no more as sources of irreducible covariants to quantics of a superior order.

for $j=3$ the generating function to the linearly independent binariants, or for brevity sake say the *total* generating function is $\dfrac{1}{(1-x^2)(1-x^3)}$.

To find the irreducible forms, or say the *limited* generating function, we must take away the cube of the one of degree 1 and weight zero, and the product of this one and each indecomposable one of the degree 2, and consequently the limited generating function will be

$$\frac{1}{(1-x^2)(1-x^3)}-\left(\frac{x^2}{1-x^2}+1\right) \text{ that is } \frac{x^3}{(1-x^2)(1-x^3)};$$

thus we obtain perpetuants of the deg-weights $3 \,.\, i$, where the least value of i is 3 and the number of such for $i=3, 4, 5, 6, 7, 8$; $9, 10, 11, 12, 13, 14$; $15, 16, 17, \ldots$ will be 1, 0, 1, 1, 1, 1; 2 1 2 2 2 2; 3, 2, 3, ...

Again, for $j=4$, the total generating function is $\dfrac{1}{(1-x^2)(1-x^3)(1-x^4)}$.

To determine the subtrahend consider the total partitions of 4 (the number itself not counting as a partition). These are 1^4, $1^2.2$, 1.3, 2^2. The three former will give rise to the partial subtrahends $1, \dfrac{x^2}{1-x^2}, \dfrac{x^3}{(1-x^2)(1-x^3)}$, but for 2^2, that is, $2\,.\,2$ the case is different.

Taking the development of $\dfrac{x^2}{1-x^2}$, that is, $x^2+x^4+x^6+x^8+\ldots$ the function corresponding to $2\,.\,2$ to be subtracted is not $\left(\dfrac{x^2}{1-x^2}\right)^2$, but the sum of the *homogeneous products* of the second order of the infinite succession $x^2, x^4, x^6, x^8, \ldots$, or calling s_1 the sum of the terms and s_2 the sum of their squares, is $\dfrac{s_1^2+s_2}{2}$, that is, is

$$\frac{1}{2}\left\{\left(\frac{x^2}{1-x^2}\right)^2+\frac{x^4}{1-x^4}\right\} \text{ or } \frac{x^4(1+x^2)+x^4(1-x^2)}{(1-x^2)(1-x^4)},$$

that is,

$$\frac{x^4}{(1-x^2)(1-x^4)}.$$

Hence the *limited* generating function for the degree 4 is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}-\left(\frac{x^3}{(1-x^2)(1-x^3)}+\frac{x^2}{1-x^2}+1\right)-\frac{x^4}{(1-x^2)(1-x^4)},$$

which is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}\{1-(1-x^4)-x^4(1-x^3)\},$$

that is

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Let us pause a moment in the deduction to draw an inference from this result. The lowest power of x in the development of the limited generating function for the degree 4 being x^7, we see that an absolutely indecomposable binariant of the 4th degree cannot be of lower weight than 7. Consider any semi-invariant of degree 4 to a quantic of order i. Its weight must be less

than $2i$. Hence if it is indecomposable, 7 must be less than $2i$, or i must be at least 4. Thus we see that there can be no absolutely indecomposable binariant of the 4th degree appertaining to a cubic. This shows *à priori* that the discriminant to the cubic, regarded as a subinvariant, is decomposable, as we know is the case*.

So in general if we know that no perpetuant of the degree j is of lower weight than k, we may be assured that no invariant or semi-invariant to a quantic of the degree j can be absolutely indecomposable if the order of the quantic is less than $\dfrac{2k}{j}$.

Agreeing to call the weight of any subinvariant divided by its degree its relative weight, we may put this result into words, by saying no quantic can possess an absolutely indecomposable invariant or semi-invariant of a given degree unless its order is at least twice as great as the minimum relative weight of a perpetuant of that degree. We may see further that the quartic can have no indecomposable invariant or semi-invariant of the degree 4, for its weight would be 8, but x^8 does not appear in the development of

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Pass we on now to the case of the 5th degree.

The indefinite partitions of 5 (leaving 5 itself out of the number) are 4.1, 3.2, $3.1.1$, $2.2.1$, 2.1^3, 1^5 which obviously give rise to the subtrahends

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)},\quad \frac{x^3}{(1-x^2)(1-x^3)}\cdot\frac{x^2}{1-x^2},\quad \frac{x^3}{(1-x^2)(1-x^3)},$$
$$\frac{x^4}{(1-x^2)(1-x^4)},\quad \frac{x^2}{1-x^2},\quad 1.$$

But from the mode in which the deduction has been carried on, it will be obvious on reflection that the sum of all these except the second which corresponds to a partition not ending with a unit will be equal to the total generating function for the case of the degree 4. So that the total subtrahend is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}+\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)}.$$

Hence the limited generating function for the degree 5 is

$$\frac{x^5}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}-\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)},$$

that is, is $\dfrac{x^5\{1-(1+x^2)(1-x^5)\}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$, which is $\dfrac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$, where for brevity I use in general (q) to denote $1-x^q$.

* It may easily be collected from the course of the ensuing investigation that *every* binary discriminant is decomposable into subinvariants of lower degrees than its own.

Here, for the first time, a new feature presents itself, namely, the presence of a negative coefficient in the numerator, and consequently of a series of such in the development in an infinite series of the generating function.

Each negative term $-kx^t$ in the development will obviously indicate the existence of k general syzygies of the degree 5 and weight t, or as we might call them, *privative* groundforms. The number of such terms will be finite, and they will be most readily obtained by writing the *l. g. f.* (limited generating function) under the form

$$\frac{-x^7(1-x^3)(1-x^5)+x^{15}}{(2)(3)(4)(5)}, \text{ that is } \frac{-x^7}{(2)(4)}+\frac{x^{15}}{(2)(3)(4)(5)}.$$

To find them it will be observed that the number of ways of composing 0, 2, 4, 6, 8, 10, 12, 14, 16 with the elements 2 and 4 are respectively 1, 1, 2, 2, 3, 3, 4, 4, 5, and that 1, 1, 2, 3, 5 are the number of ways of composing 0, 2, 4, 6, 8, with the elements 2, 3, 4, 5. Hence there will exist the negative terms

$$-x^7,\ -x^9,\ -2x^{11},\ -2x^{13},\ -2x^{15},\ -2x^{17},\ -2x^{19},\ -x^{21}\,*,$$

the sum of which is

$$-\frac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}.$$

Adding this with its sign changed to $\dfrac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$ there results

$$\frac{x^{18}+x^{20}-x^{21}-x^{23}+x^{24}+x^{25}+2x^{26}-x^{29}-2x^{30}-x^{31}-x^{32}+x^{33}+x^{35}}{(2)(3)(4)(5)},$$

which may be thrown under the form

$$x^{18}\left\{\frac{(3)+x^2(2)(3)+x^4+x^6(8)+x^8(3)(4)+x^8(4)(5)}{(2)(3)(4)(5)}\right\}.$$

It is therefore omni-positive in its development, which shows that no negative terms have been omitted, but that the 13 syzygies of odd weights ranging from 7 to 21 typically represented by $-\dfrac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}$ (say $-R_5$) constitute their entire aggregate. We see also that the minimum weight of a perpetuant of the 5th degree is 18, so that the double of the minimum relative weight is $\dfrac{36}{5}$, and accordingly there can exist no absolutely indecomposable binary subinvariants of the 5th degree, until we come to Quantics of the 8th order or upwards.

Proceeding to the degree 6, the total subtrahend from the *t. g. f.* (total generating function) for that degree would be *ut suprâ* the *t. g. f.* for the

* The numbers 1 1 2 2 2 . 2 1 are got by subtracting from the figures 1 1 2 2 3 3 4 4 5 the figures 1 1 2 3 5

degree one below (here 5), less expressions depending on the partitions of 6 not concluding with a unit, were it not for the presence of the negative terms represented by $-R_5$; the quantity to be subtracted corresponding to the partition 5.1, being now not the *l. g. f.* for degree 5, $\frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$, but this quantity rendered omni-positive in its development by the addition of R_5.

Hence the total subtrahend will be $\frac{1}{(2)(3)(4)(5)}+R_5+$ the quantities depending on the partitions 2.4 2.2.2 3.3.

To 2.4 will correspond the subtrahend $\frac{x^2}{(2)}\cdot\frac{x^7}{(2)(3)(4)}$.

To 3.3 will correspond $\frac{\phi x^2+(\phi x)^2}{2}$ where $\phi x=\frac{x^3}{(1-x^2)(1-x^3)}$, and to 2.2.2 by Crocchi's theorem*, will correspond the representative of the homogeneous products of the 3rd order of the terms in

$$\psi x=\frac{x^2}{1-x^2},\ \text{that is,}\ \frac{(\psi x)^3+3\psi x\psi x^2+2\phi x^3}{2.3}.$$

There might for a moment be felt a hesitation in applying the formula for homogeneous products to ϕx, in consequence of the coefficients in its development being no longer exclusively unities; but the force of this objection vanishes as soon as it is borne in mind that we may replace any term kx^t in the development of ϕx by k separate terms x^t, each of which corresponds to a distinct subinvariant.

Thus then to 3.3 will correspond the partial subtrahend

$$\frac{x^6}{2}\left\{\frac{1}{(1-x^2)^2(1-x^3)^2}+\frac{1}{(1-x^4)(1-x^6)}\right\}\ \text{or}\ x^6\frac{(1+x^2)(1+x^3)+(1-x^2)(1-x^3)}{2(2)(3)(4)(6)},$$

that is,
$$\frac{x^6+x^{11}}{(2)(3)(4)(6)},$$

and to 2.2.2 will correspond

$$x^6\frac{(1+x^2)(1+x^2+x^4)+3(1-x^6)+2(1-x^2)(1-x^4)}{6(2)(4)(6)},\ \text{or}\ \frac{x^6}{(2)(4)(6)}.$$

It may be remarked, in passing, that for any degree $2i$ the subtrahend corresponding to the partition consisting of i parts (each of the value 2), is $\frac{x^{2i}}{(2)(4)\dots(2i)}$, as may be shown, *à priori*, thus: using y in place of x^2 we have to find the sum of all the quantities ky^t where k is the number of ways of generating y^t as a product of i of the powers $1, y, y^2, y^3 \dots$, that is, k is the number of ways of composing t with i or less than i of the indefinite

* See for an instantaneous proof of this theorem, the *Johns Hopkins University Circular* for November 1882 [below, p. 653].

series of natural numbers, which by Euler's theorem, already cited, is the same as that of compounding t out of any number of parts none exceeding i. Hence the denominator of the subtrahend required will be

$$\frac{1}{(1-y)(1-y^2)\dots(1-y^i)}, \text{ that is, } \frac{1}{(2)(4)\dots(2i)}.$$

The numerator is obviously x^{2i}, and the complete value $\dfrac{x^{2i}}{(2)(4)\dots(2i)}$ as was to be found.

I may add, that this theorem (which is one concerning homogeneous product-sums expressed as functions of power-sums of the same elements), by an easy deduction from Crocchi's theorem, serves to show if the ith power-sum of a set of elements is $\dfrac{1}{1-c^i}$ (I substitute c for y) then the ith elementary symmetric function of the elements is

$$\frac{c^{\frac{i^2-i}{2}}}{(1-c)(1-c^2)\dots(1-c^i)}$$

and reversing the terms of this proposition we may say, that if

$$z^q - \frac{1}{1-c}z^{q-1} + \frac{c}{(1-c)(1-c^2)}z^{q-2} \dots \pm \frac{c^{\frac{n^2-n}{2}}}{(1-c)(1-c^2)\dots(1-c^n)}z^{q-12} + \dots = 0,$$

then the sum of the ith powers of z (q being not less than i) is $\dfrac{1}{1-c^i}$, to which may be added that the sum of the ith homogeneous products of z is

$$\frac{1}{(1-c)(1-c^2)\dots(1-c^i)},$$

as, for example, if $i=2$ the first of these sums, namely,

$$\frac{1}{(1-c)^2} - 2\frac{c}{(1-c)(1-c^2)} = \frac{1}{1-c^2}$$

and the other, namely,

$$\frac{1}{(1-c)^2} - \frac{c}{(1-c)(1-c^2)} = \frac{1}{(1-c)(1-c^2)}.$$

But this is a mere digression, a wild flower gathered on the wayside. Returning to the determination of the *l. g. f.** for the degree 6, we see that it will be

$$\frac{1}{(2)(3)(4)(5)(6)} - \frac{1}{(2)(3)(4)(5)} - \frac{x^9}{(2)(2)(3)(4)} - \frac{x^6+x^{11}}{(2)(3)(4)(6)} - \frac{x^6}{(2)(4)(6)} - R_5,$$

or $\dfrac{N}{(2)(3)(4)(5)(6)} - R_5$, where

* I repeat that *t. g. f.* stands for total generating function, and *l. g. f.* for limited generating function.

$$\begin{aligned} N &= x^6 - (1 + x^2 + x^4)(1 - x^5)x^9 - (x^6 - x^{16}) - x^6(1 - x^3)(1 - x^5) \\ &= x^6 + x^{14} + x^{16} + x^{18} + x^{16} + x^9 + x^{11} \\ &\quad - x^9 - x^{11} - x^{13} - x^6 - x^6 - x^{14} \\ &= - x^6 - x^{13} + 2x^{16} + x^{18}. \end{aligned}$$

Thus the *l. g. f.* for the degree 6 is

$$- R_5 + \frac{- x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)}.$$

$-R_5$ represents the fourteen compound syzygants of the degree 6; the fraction to which $-R_5$ is annexed, when developed, will give rise to only a *limited* number of terms with negative coefficients corresponding to the ground-syzygies; the remainder of the terms, infinite in number, will represent the infinite succession of groundforms. It may be well here to notice, as a universal fact, that in the development of the fraction $\frac{R(x)}{(2)(3)\ldots(n)}$ (where $R(x)$ is rational integral function of x) the number of negative terms or the number of positive terms will be finite according as $R(1)$ is positive or negative, and, as in the above fraction, $R(1) = 1$, it follows that there are only a finite number of negative terms, and consequently only a limited number of ground-syzygies, an important conclusion which will easily be seen to apply not only to the use of the degree 5 (in which syzygies first make their appearance) and 6, as here shown, but for all higher degrees, it being a universal law that the irreducible syzygies for subinvariants of any given degree, and therefore of any degree not exceeding a given limit, are finite in number.

The law that the development of $\frac{R(x)}{(1-x^2)(1-x^3)\ldots(1-x^n)}$, commencing from a certain point is omni-positive or omni-negative, according as $\phi 1$ is positive or negative when n exceeds 2, admits of easy proof. Of course the law could not be true when $n = 2$, as, for example, for $\frac{1-2x}{1-x^2}$ which remains *neutral*, that is, neither omni-positive nor omni-negative (which latter, if the law did apply, it ought eventually to become) throughout its entire extent.

Beginning with $\frac{Rx}{(1-x^2)(1-x^3)}$ the coefficient of x^i [where $i = 6t + \tau$ $(\tau < 6)$] will be not less than t, and not greater than $t+1$ in the development of

$$\frac{1}{(1-x^2)(1-x^3)}.$$

Hence in the development $\frac{-K + (K+\epsilon)x^\delta}{(1-x^2)(1-x^3)}$ the coefficient of x^i will be not less than $-K(t+1) + (K+\epsilon)\left(t - \frac{\delta}{6} - 1\right)$, and consequently for a sufficiently large value of i must be positive. *A fortiori* the same will be true for

$\frac{R(x)}{(1-x^2)(1-x^3)}$ when $K+\epsilon$ is the sum of the positive coefficients in Rx of powers of x none of whose indices are higher than δ, and K the sum of the negative coefficients of any powers of x; this proves the law for $\frac{R(x)}{(1-x^2)(1-x^3)}$ when $R(1)$ is supposed to be positive, and moreover the series will be omni-positive after a certain point in the strict sense of the following coefficients being neither negative nor zero.

Hence the law will be true for $\frac{Rx}{(1-x^2)(1-x^3)(1-x^4)}$, for we may divide $\frac{Rx}{(1-x^2)(1-x^3)}$, when expanded, into four series, whose indices $\equiv 0, 1, 2, 3$ respectively to modulus 4, and the negative terms in each of these being finite in number, it is clear that the effect of dividing any one of them by $1-x^4$ will be to give rise to a series omni-positive after a certain point, because each coefficient in the quotient of any one of the series divided by $\frac{1}{1-x^4}$ will at worst contain only the sum of a finite number of given negative coefficients, and a number of terms all greater than zero, whose sum, when that number is taken great enough, must exceed the arithmetical value of the former sum. Hence $\frac{R(x)}{(1-x^2)(1-x^3)(1-x^4)}$ will be the sum of four series, each omni-positive from a certain point, and will therefore be omni-positive from the most advanced of those points. In like manner

$$\frac{Rx}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

may be shown to be the sum of five series, each with an infinite omni-positive branch, and consequently will be itself of the same character, and so in general. Of course the same reasoning would show the truth of the law when $R(x^i)$ is negative, and that it may be extended to any denominator of the form $(1-x^i)(1-x^j)(1-x^k)\ldots$ provided any two of the indices $i, j, k\ldots$ are prime to one another. And of course a similar conclusion obtains (*mutatis mutandis*) when $R(x)$ is negative. The law might be proved more scientifically and more briefly as a consequence of the general algebraical representation of the denumerant of any equation in integers

$$[l_1x_1 + l_2x_2 + \ldots + l_ix_i = n]$$

as a sum of a non-periodical and of periodical parts, whereof the former is always of a higher dimension in n than any of the latter, except when all the l quantities have a common factor. See the annexed Excursus [p. 605, below].

I now proceed to find the lowest power of x in the fraction

$$\frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)},$$

say F, in which the coefficient is positive, in order to ascertain the minimum weight of an absolutely irreducible subinvariant of the 6th degree.

I think the easiest practical mode of proceeding to effect this is to use the table in my possession (having been previously calculated for me by Mr Franklin for another purpose) which gives the coefficients of the powers of x in $\frac{1}{(2)(3)(4)(5)(6)}$; those coefficients used as they stand, then advanced seven steps, then five steps further, then taken back two steps, and at the same time doubled, will give four series of numbers, the sum of the 1st and 2nd of which subtracted from the sum of the 3rd and 4th will give the successive coefficients from x^6 upwards in the development of F.

The four series are as underwritten:

(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,
							1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,
												1,	0,	1,	1,	2,	2,	4,	3,	6,
										2,	0,	2,	2,	4,	4,	8,	6,	12,	12,	18,

(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)
49,	60,	63,	78,	80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,
19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,	80,	97,	102,	120,	126;
6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,
18,	28,	26,	38,	40,	52,	54,	72,	72;	94,	98,	120,	126,	156,	160,	194,

(37)	(38)	(39)	(40)	(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)	(51)
270,	307,	322;	361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,
149,	154,	180,	189,	216,	227,	260,	270,	307,	322;	361,	378,	414,	441,	492,
80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,	270,	307,	322;
204,	240,	252;	298,	308,	360,	378,	432,	454,	520,	540,	614,	644;	722,	756,

(52)	(53)	(54)	(55)	(56)	(57)	(58)	(59)	(60)	(61)	(62)	(63)
854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,	1617,
515,	568,	594,	656,	682;	750,	783,	854,	891,	972,	1010,	1098,
361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,
848,	882,	984,	1030,	1136,	1188,	1312,	1364;	1500,	1566,	1708,	1782,

(64)	(65)	(66)	(67)	(68)	(69)	(70)	(71)	(72)	(73)	(74)
1734,	1802,	1932,	2002,	2142,	2223;	2369,	2457,	2618,	2709,	2881,
1144,	1236,	1287;	1391,	1443,	1555,	1617,	1734,	1802,	1932,	2002,
854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,
1944,	2020,	2196,	2288,	2472,	2574;	2782,	2886,	3100,	3234,	3468,

(75)	(76)	(77)	(78)	(79)
2985,	3164,	3276,	3472,	3588;
2142,	2223;	2369,	2457,	2618,
1617,	1734,	1802,	1932,	2002,
3604,	3864,	4004,	4284,	4446;

The first group of four numbers in which the 3rd and 4th terms combined exceed the 1st and 2nd will easily be seen to be, $\left.\begin{matrix}2369\\1617\\1236\\2782\end{matrix}\right\}$ which is 70 places from the first term, and for which the difference is 4018 less 3986 or 32. Starting from this point the series for F will be seen to be

$$32x^{76}-18x^{77}+81x^{78}+36x^{79}+188x^{80}+94x^{81}+211x^{82}+161x^{83}$$
$$+287x^{84}+242x^{85}+\dots;$$

so that there can be no practical doubt of the series being omni-positive from and after the 78th power of x*.

The relative weight of any one of the irreducible subinvariants corresponding to $32x^{76}$ is $\frac{76}{6}$, the double of which is $25\frac{1}{3}$. Hence there can be no irreducible semi-invariant of the 6th degree to a quantic below the 26th order, and, on account of the coefficient of x^{77} being negative, we see that a quantic of the 26th order can have no groundforms of the 6th degree in the coefficients except such as are invariants or quart-invariants.

As regards the syzygies irrespective of the compound ones represented by $-R_5$, we see that there will be primitive ones of all weights from 6 to 77 inclusive, with the exception of the weights 7 and 76, but that there will be no syzygies, whether reducible or irreducible, of the same weights as the irreducible subinvariants. Let us now pass on to the case of the 7th degree†.

The partitions of seven itself and those ending in unity excluded are 5.2 4.3 2.2.3.

Hence calling R_6 the sum of the negative terms in $\frac{-x^6-x^{13}+2x^{16}+x^{18}}{(2)(3)(4)(5)(6)}$, the *l. g. f.* for 7 will be

$$\frac{x^7}{(2)(3)(4)(5)(6)(7)}-\frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}\frac{x^2}{(2)}-\frac{x^7}{(2)(3)(4)}\frac{x^3}{(2)(3)}$$
$$-\frac{x^4}{(2)(4)}\frac{x^3}{(2)(3)}-R_5\frac{x^2}{1-x^2}-R_6.$$

If we call this $\frac{x^7+N}{(2)(3)(4)(5)(6)(7)}-R_5\frac{x^2}{1-x^2}-R_6$,

$$N=x^7(1-x^7)\{(1+x^2+x^4)(-x^9+x^{12}+x^{14})+x^{10}(1+x+x^2+x^3+x^4)$$
$$(1-x+x^2)+x^7(1-x^5)(1+x^2+x^4)\}=-(1-x^7)P,$$

where $P=\Sigma x^t-\Sigma x^\tau$, t having the values 12 14 16, 14 16 18; 10 11 12 13 14, 12 13 14 15 16; 7 9 11, and τ having the values 9 11 13; 11 12 13 14 15; 12 14 16.

* This conclusion will be strictly proved in the sequel with the aid of my general partition formulæ, in Section V.

[† For the 7th degree, cf. J. Hammond, *American Journal*, Vol. v. (1882), p. 225, under the heading: Disproof of Prof. Sylvester's Fundamental Postulate.]

Hence $$P = x^7 + x^{10} + x^{12} + 2x^{14} + 2x^{16} + x^{18},$$
and $$x^7 + N = -x^{10} - x^{12} - x^{14} - 2x^{16} - x^{18} + x^{17} + x^{19} + 2x^{21} + 2x^{23} + x^{25}.$$

The first term in the development of $\frac{x^7+N}{(2)(3)\dots(7)}$ is $-x^{12}$, indicating that the first irreducible syzygy is of the weight 12; it is not until a very high power of x is reached that a positive coefficient corresponding to a perpetuant makes its appearance.

The tables set out in a subsequent section exhibit *inter alia* the coefficients in the developments of $\frac{1}{(2)(3)\dots(7)}$ and $\frac{1}{(2)(3)\dots(6)}$, say F_7 and F_6 as far as the 174th power of x. Using instead of $\frac{x^7+N}{(2)\dots(7)}$ the equivalent value $x^7F_7 - PF_6$, if the coefficient of x^{q+7} in this is positive, the coefficient of x^q in F_7 must be greater than that of x^q in $(1 + x^3 + x^5 + 2x^7 + 2x^9 + x^{11})F_6$, and *à fortiori* greater than that of x^q in $8x^{11}F_6$, that is, greater than 8 times that of x^{q-11} in F_6. But a glance at the tables* for the developments of F_7, F_6 will show that this is never the case within the limits of q, furnished by the tables, that is, for any value of q not exceeding 174. It is certain, therefore, that the value of the lowest index of x^r, for which in $\frac{N}{(2)\dots(7)}$ the coefficient is positive, must considerably exceed 181, as indeed one might have anticipated from the series of similar exponents 2, 3, 7, 18, 76 corresponding to the cases previously considered, the ratio of increase in these numbers going on continually increasing†. To ascertain the value of the exponent in question there is left no resource but to endeavour to elicit it (as I shall presently proceed to do) from the general algebraical value of the coefficient. But before doing so it will be well to notice a very important inference that may be drawn from the form of the generating function, namely,

$$\frac{N}{(2)(3)(4)(5)(6)(7)} - \frac{R_5}{(2)} - R_6.$$

$\frac{R_5}{(2)}$ or $(1 + x^2 + x^4 + \dots)(x^7 + x^9 + 2x^{11} + 2x^{13} + 2x^{15} + 2x^{17} + 2x^{19} + x^{21})$ will represent the deg-weights of the compound syzygies corresponding to the multiplication of the syzygies of the deg-weights 5.7 5.9 5.11 5.13 5.15 5.17 5.19 5.21 5.23 by the groundforms of every even weight.

There will thus be seen to exist compound syzygies of every odd weight (no less than 13 in fact of weight 21 or any higher odd number). If then ω' be the lowest power of x in $\frac{N}{(2)(3)(4)(5)(6)(7)}$ with a positive coefficient and

* *Vide* the numerical tables at end of Section V of this Memoir.

† Subsequent calculations, however, have revealed to me that this ratio does not go on continually increasing.

with an odd exponent, there will coexist groundforms and syzygies of the same degree and weight appertaining to the quantic of an infinite order for every weight denoted by an odd number not less than ω'. From this it is easy to infer that there must exist syzygies and groundforms of the same deg-weight (and therefore of the same deg-order) for one or more quantics of an order not exceeding ω'; [and it may be added that ω' being a high number (not a number less than 23) there will be 13 syzygies of every odd weight equal to or greater than ω'].

For suppose that Q is a quantic of order i. In determining its ground-semi-invariants of the successive degrees the same process may be applied as in calculating the perpetuants, that is, the ground semi-invariants to a quantic of an unlimited order, except that in lieu of the complete development of the generating function $\frac{1}{(1-x^2)(1-x^3)\dots(1-x^j)}$ only such powers of x must be retained as are not higher than x^i. For the number of linearly independent subinvariants of the weight w and degree j will now be the difference between the number of ways of making up w with j parts none greater than i, less the number of ways of so making up $(w-1)$ which will be the difference between the number of ways of making up w and of making up $(w-1)$ with i parts none greater than j, which, if w does not exceed i, will be the same as if i were infinite. So far then as weights not superior in value to i are concerned, the total generating function for a quantic of the order i will be the same as for a quantic of an unlimited order, and consequently up to the weight i (inclusive) the generating functions for the ground subinvariants (to be obtained, be it remembered, by combining the total generating functions in the same manner, whatever the value of i may be) will be the same for a quantic of the ith as for the quantic of an unlimited order. Hence there must of necessity appertain irreducible covariants and compound syzygants of the same degree and order (namely, of the deg-order $7\,.\,5\omega'$) to a quantic of the order ω', and of course there is nothing to prevent such coexistence holding good for a quantic of an order very much lower than ω', the least value of which number say i, as far as I am able at present to see, can only be determined by putting each quantic of an order inferior to i successively upon its trial, a work of exceedingly great labour to undertake.

I use ω' to signify the lowest *odd* power of x in the development of the *g.f.* to perpetuants of the 7th degree affected with a positive coefficient, reserving ω to signify the lowest power (whether odd or even) so affected. Until further investigation we cannot say whether ω is equal to or less than ω', but we know that no absolutely irreducible subinvariant of the 7th degree can appertain to a quantic of an order lower than $\frac{2\omega}{7}$, a number whose exact

value we shall eventually succeed in ascertaining with the aid of a partition formula obtained by the method which will form the subject of the annexed "excursus."

Inasmuch as the theory is precisely the same for fractions in general as for those which correspond to denumerants (the name I give to the number of solutions in integers of one or more linear equations), I shall show how to find the general term in the development of any rational fraction, limiting myself however, for the present, to the theory of rational functions of a single variable, which covers the case with which alone we are here concerned, of denumerants of a single linear equation, or which is the same thing, the problem of exhibiting the number of modes of composing a general number n with given smaller numbers as an algebraico-exponential function of n.

When analysis is sufficiently advanced to admit of a perfectly methodical distribution of its subject-matter, the theorem for the expansion of rational functions, about to be given, will, it seems to me, take its place immediately after Newton's binomial theorem, as the second leading theorem of Algebra; my method of partitions (as stated and applied in Tortolini's *Ann.* Vol. VIII. 1856, and in the *Quarterly Mathematical Journal*, 1855, Vol. I. p. 141, to neither of which I have at present means of access*, but the latter of which is referred to by Prof. Cayley in the *Phil. Trans.* for 1880, footnote p. 47) virtually amounted to an enunciation of the theorem for the case of the reciprocal of a rational integral function all of whose roots are roots of unity, under such a form as almost of necessity to lead to the supposition of its remaining true (*mutatis mutandis*) in the general case; the actual averment of the generalization was, I believe, first made by Prof. Cayley†.

EXCURSUS.

On Rational Fractions and Partitions.

The method of finding the general term in the development of a rational fraction of a single variable in a series of ascending powers of the same may be regarded as a corollary to the following lemma, the proof of which is an instantaneous consequence of the fact that the coefficient of $\frac{1}{x}$, or to use Cauchy's word, the residue of $\frac{1}{(1-e^x)^i}$ developed in ascending powers of x

[* See Vol. II. of this Reprint, p. 90.]

† On second thoughts, and after more deliberate reflection, it occurs to me that I may have overstated in the text above the importance of the general theorem viewed as a theorem *an sich*; and that it is only from its special application to rational fractions whose infinity-roots are all of them roots of unity, that it derives its claim to be regarded as a cardinal theorem in Algebra.

when i is any positive integer is always -1: that this is so will be seen at once from the fact that the effect of changing i into $i+1$ in the above fraction is to increase it by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{i(1-e^x)^i}$, whose residue is obviously zero, so that the residue of $\frac{1}{(1-e^x)^i}$ will be unaffected by continually decreasing i by a unit until it becomes unity; and obviously therefore the residue in question is always -1.

The lemma may be stated as follows:

The constant term in any proper algebraical fraction developed in ascending powers of its variable is the same as the residue with its sign changed of the sum of the fractions obtained by substituting in the given fraction in lieu of the variable its exponential multiplied in succession by each of its values (zero excepted, if there be such) which makes the given fraction infinite.

Any value of a variable which makes a function infinite may conveniently be called an infinity root, and if it is not zero, a finite-infinity root. So too, a factor whose vanishing makes a function vanish may be termed an infinity factor.

Suppose Fx is a *proper* Algebraical fraction, then we may write

$$Fx = \Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - x)^\lambda} + \Sigma \frac{\gamma_\lambda}{x^\lambda},$$

where $\lambda = 1, 2, \ldots$; $\mu = 1, 2, \ldots j$ and of course any of the coefficients in either sum may be made zero, and then (using in general here and hereafter co_n to signify the coefficient of x^n in an ascending expansion of the function with which it is in regimen) we have

$$\mathrm{co}_{-1}\,\Sigma F(a_\nu e^x) \;\; [\text{where } \nu = 1, 2, \ldots j]$$

$$= \mathrm{co}_{-1}\,\Sigma\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\nu e^x)^\lambda} + \mathrm{co}_{-1}\,\Sigma\Sigma \frac{\gamma_\lambda}{a_\nu^\lambda e^{\lambda x}}$$

$$= \mathrm{co}_{-1}\,\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\mu e^x)^\lambda} = -\frac{c_{\lambda,\mu}}{a_\mu^\lambda} = -\,\mathrm{co}_0\,Fx$$

which proves the lemma.

Hence the coefficient of x^n in a rational function fx, which is the same as $\mathrm{co}_0 \frac{fx}{x^n}$ will be $-\,\mathrm{co}_{-1}\,\Sigma\,(r^{-n}e^{-nx}fre^x)$ or $\mathrm{co}_{-1}\,\Sigma\,\{r^{-n}e^{nx}f(re^{-x})\}$, [$r$ meaning each finite-infinity root of fx taken in turn], provided only that $\frac{fx}{x^n}$ is a proper algebraical function, that is, provided that n is greater than the degree of $f(x)$.

As for instance, if the degree of the fraction is zero, the theorem will not give the constant, but will give every coefficient of positive powers in the

ascending expansion of fx, and if it is negative, the theorem will give all but the coefficients of negative powers.

This theorem, as observed by Prof. Cayley, *Phil. Trans.*, 1856, p. 139, may be obtained "from the known theorem," that if fx be resolved into simple partial fractions, the sum of those which have any power of $a-x$ in their denominator will be the residue of

$$\frac{f(a+\zeta)}{x-a-\zeta} * .$$

Prof. Cayley quotes as "a theorem of Cauchy's and Jacobi's, that the coefficient of $\frac{1}{z}$ in $Fz =$ coefficient of $\frac{1}{t}$ in $\psi' t F \psi t$."

This is obviously not true in general, for we might take $Fz = \frac{1}{z}$ and $\psi t = a + t$ or t^2 and the alleged equality would not exist. It is, however, true whenever ψt is of the form $at + bt^2 +$ etc., as may be proved instantaneously by supposing Fz resolved into partial fractions, and making $z = \psi t$, so that $\int dz\, Fz = \int dt \psi' t F \psi t$, and observing that if the expansion of $\psi' t F \psi t$ contains $\frac{k}{t}$, that of $\int dz Fz$ must contain $\frac{k}{z}$, since otherwise when this integral is expressed as a function of t, it would not contain (as it is bound to do) the term $k \log t$. The theorem so limited is sufficient for the purpose in view, since on writing, in place of ζ, $-a(1-e^{-t})$ we see that the residue of $\frac{f(a+\zeta)}{x-a-\zeta}$ is the same as the residue of $\frac{f(ae^{-t})}{(1-ae^{-t}x)}$, and consequently the coefficient of x^n in so far as it depends on the infinity root a, will be the residue of $(a^{-n} e^{nt}) f(ae^{-t})$ as has been shown above to be the case. It may, possibly, be thought somewhat surprising that those familiar with the known theorem referred to and the general principle of transformation of residues should not have recognized, previous to the divulgation of my theorem, that the two things put together were competent to give a complete solution of the much ventilated problem of simple denumeration. But, perhaps, even supposing the mental conjunction of the two facts to have taken place, there would still have been needed an act of imagination (such as Kant justly remarks is at the bottom of every advance in geometry, where in reality the proof lies in the construction†) to have led to the choice of the particular transformation

* In his *Cours d'Algèbre*, Edition 1877, Vol. I. pp. 497—499, M. Serret obtains the same result under the form of the value (for $\zeta =$ zero) of

$$\frac{1}{\pi(m-1)}\left[\left(\frac{d}{d\zeta}\right)^{m-1} \frac{\zeta^m f(a+\zeta)}{x-a-\zeta}\right],$$

where m is the degree to which $(x-a)$ rises in the denominator of fx.

† Take as an example the theorem that the sum of the three angles of a triangle is equal to two right-angles: as soon as by a stroke of the imagination a line is conceived as drawn from one angle parallel to the opposite side, the truth of the proposition becomes virtually self-evident.

employed in this case, and to have entailed the consequences that are implied in it*.

In applying this theorem to finding the value of the denumerant to the equation $ax + by + \ldots + lt = n$, which I denote by $\dfrac{n}{a, b, \ldots l,}$ and is the same thing as the coefficient of x^n in the expansion of the rational fraction

$$\frac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}$$

or more generally to finding the value of the denumerant

$$\frac{n}{a_1, a_2, \ldots a_\alpha, b_1, b_2, \ldots b_\beta, \ldots l_1, l_2, \ldots l_\lambda,}$$

(where each letter has a fixed value independent of its subindex), that is, the coefficient of x^n in the development of $\dfrac{1}{(1-x^a)^\alpha (1-x^b)^\beta \ldots (1-x^l)^\lambda}$, say Fx, the first thing to be done is to determine and arrange in convenient groups the infinity roots of these functions. To effect this we have only to write down all the divisors of the set of numbers $a, b, \ldots l$, that is, all the integers which divide one or more of those numbers, say $\delta_1, \delta_2, \ldots \delta_\mu$. These divisors necessarily include the indices $a, b, \ldots l$ and *unity*, which latter we may suppose to be δ_1.

Giving then i every value from 1 to μ, the primitive δ_ith roots of unity will obviously be the infinity roots required, and we may separate the required function of n into μ distinct portions or waves, as I term them, where supposing $\nu_1, \nu_2, \ldots \nu_{\phi(\delta_i)}$ [$\phi(\delta_i)$ being the *totient* of δ_i, that is, the number of integers less than δ_i and prime to it] to be the primitive δ_ith roots of unity, the ith period or wave, say W_i, will be equal to the residue of

$$\Sigma r_q^{-n} e^{nt} F(r_q e^{-t}) \ [q = 1, 2, \ldots, (\phi\delta_i)].$$

Since every primitive root r_q is either equal to or is mated with its reciprocal, the above expression may be replaced by the somewhat more convenient one $\Sigma (r_q^n e^{nt}) F(r_q^{-1} e^t)$.

This again admits of a very important transformation, namely, we may write $\nu = n + \frac{1}{2}(\alpha a + \beta b + \ldots + \lambda l)$ and then

$$W_i = \text{co}_{-1} \Sigma \frac{r_q^{\nu} e^{\nu t}}{P\left(r_q^{\frac{a}{2}} e^{\frac{at}{2}} - r_q^{-\frac{a}{2}} e^{-\frac{at}{2}}\right)^\alpha}$$

* Thus, for example, the supposed investigator might have chosen to write $\sin t$ or $\log(1+t)$ in lieu of $1-e^t$ and the theorem thereby obtained would have been perfectly valid, but of little if any use, and the great bulk of transformations would certainly be of no use whatever; indeed, it is safe to say that the substitution practised, namely, that of $1-e^{\lambda t}$ [λ being taken at will] is the only one that would lead to a practical solution of the question.

(where P is used to signify that the product is to be taken of terms of like form to the one which is in regimen with it).

From this it follows that every wave W_i expressed as a function of ν, when ν is changed into $-\nu$, becomes $(-)^{\alpha+\beta+\dots+\lambda-1}W_i$, that is, retains its value absolutely or else merely changes its algebraic sign. To prove this it may be observed that whatever the index of the wave the above sum may be replaced by

$$\frac{1}{2}\,\mathrm{co}_{-1}\,\Sigma\left\{\frac{r_q{}^{\nu}e^{\nu t}}{P\left(r_q{}^{\frac{a}{2}}e^{\frac{at}{2}}-r_q{}^{-\frac{a}{2}}e^{-\frac{at}{2}}\right)^{\alpha}}+\frac{r_q{}^{-\nu}e^{\nu t}}{P\left(r_q{}^{-\frac{a}{2}}e^{\frac{at}{2}}-r_q{}^{\frac{a}{2}}e^{-\frac{at}{2}}\right)^{\alpha}}\right\}.$$

This is a consequence of r being either identical with $\frac{1}{r}$ as is the case for W_1 and W_2, or else being mated with it as belonging to the same group of primitive roots of unity.

Hence r_q may be changed into $r_q{}^{-1}$, and the expression to be residuated will undergo no change.

Again, if t is changed into $-t$, the residue changes its sign, and finally if r_q, t, and ν are simultaneously changed into $r_q{}^{-1}$, $-t$, $-\nu$ the expression to be residuated remains unaltered, except that it takes up a factor $(-)^{\Sigma\alpha}$. Consequently the effect of changing ν into $-\nu$, leaving everything else unaltered, will be to introduce the factor $(-)^{\Sigma\alpha-1}$; and this being true of every portion of the value of $\dfrac{n}{a\,\dots,\,b\,\dots,\,l\,\dots,}$ it follows that when that denumerant is expressed under the form $F\nu$, where $\nu=n+\frac{1}{2}\Sigma\alpha a$, $F(-\nu)=(-)^{-1+\Sigma\alpha}F(\nu)$.

There is consequently an enormous advantage gained, as well in the abbreviation of the calculations as in the conciseness of the result, by putting such a denumerant under the form of a function of the *augmented* argument ν instead of the original argument n; when so expressed I speak of the denumerant being in its canonical form.

In future, for greater simplicity, I shall disuse the indices α, β ... it being understood (unless the contrary is stated) that any of the indices a, b, c ... in the denominator of the denumerant $\dfrac{n}{a, b, c, \dots l,}$ or in its generating function $\dfrac{1}{(1-x^a)(1-x^b)\dots(1-x^l)}$ may be made equal to one another.

It is perhaps not unworthy of notice that the denumerant $\dfrac{n}{a, b, \dots, l,}$ may be expressed as the residue of a double sum without knowing the divisors of the indices. For it is obvious that we may express it as the sum of an infinite number of waves whose indices take in all values from unity up to infinity (since all those whose indices are non-divisors will be equal to zero)*,

* By a process, so to say, of *natural selection.*

and consequently as the residue of a sum of quantities obtained by substituting for r in the expression

$$\frac{r^\nu e^{\nu x}}{P\left(r^{\frac{a}{2}} e^{\frac{x}{2}} - r^{-\frac{a}{2}} e^{-\frac{x}{2}}\right)},$$

every primitive root of unity of every order up to the ωth inclusive, where ω is any number not less than the greatest of the quantities a, and therefore, if we please, equal to Σa, which saves the necessity of distinguishing the relative magnitudes of the several quantities a (ω it should be noticed must not be taken infinity, because that would render the sum to be residuated infinite). Thus then we see that the denumerant $\dfrac{n}{a, b, \ldots, l,}$ is the residue of

$$\Sigma \frac{e^{(t+2\pi ik)\nu}}{P\left\{e^{a\left(\frac{t}{2}+\pi ik\right)} - e^{-a\left(\frac{t}{2}+\pi ik\right)}\right\}},$$

where k represents every distinct quantity expressible by a proper fraction whose denominator is equal to or less than Σa*.

The result previously found concerning the relation of $F\nu$ to $F-\nu$ is in accordance with the observation due, I believe, to Jacobi, that if ϕn, ψn be the coefficients of x^n [n positive or negative] in the ascending and descending expansions of a proper rational fraction, then $\psi n = -\phi n$. For, in the particular fraction we are considering, it is obvious that calling the number of the factors (our former $\alpha + \beta + \ldots + \lambda$) i and $a + b + \ldots + l = s$, we shall have

$$\psi(-n-s) = (-)^i \phi n.$$

Therefore $\phi n = (-)^{i-1} \phi(-n-s)$ by Jacobi's observation.

If then $\nu = n + \frac{s}{2}$ and $\phi n = F\nu$ so that $\phi(-n-s) = F\left(-n-\frac{s}{2}\right) = F(-\nu)$ we shall have $F\nu = (-)^{i-1} F(-\nu)$, as already shown.

It is also a part of the same observation and shown in the same way that ϕn, used in the same sense as above, is zero for all values of negative n between *zero* and the *degree* of the fraction (*exclusive*); hence $F(\pm\nu)$ is zero for all values of ν from 0 to $\frac{s}{2} - 1$ inclusive if s be even, and from $\frac{1}{2}$ to $\frac{s}{2} - 1$ inclusive if s be odd†.

This fact alone is sufficient to give exactly the number of homogeneous equations required to determine (to a numerical factor près) the algebraico-

* The number of terms in this sum will be the sum of the totients of all the numbers up to the limit, an empirical expression for which (if my memory is not in fault) has been recently investigated by Mr Merrifield.

† In order not to break up the text, the footnote (which ought to come here) regarding the two statements above, as to the coefficient-functions of any proper fraction, is transferred to the last page of this Excursus [p. 621 below].

exponential form $F(\nu)$, that is, the effective* *trivial* zero values of $F(\nu)$ are exactly equal in number to the number of terms which that form contains, as I will proceed to show.

The number of the indices $a, b, c, \ldots$ in which any divisor is contained may be termed its frequency in respect to those numbers, and it is a very simple arithmetical fact that if the totient of every divisor of a set of given numbers be multiplied by its frequency in respect to the set, the sum of the products so obtained will be equal to the sum of the given numbers. When the set reduces to a single term this theorem becomes the familiar one, that any number is equal to the sum of the totients of all its several divisors, and from this to the general case there is but a step, for we may suppose the set of numbers written out in a line, and under every one of them which contains a divisor j the totient of j to be written, and every value from 1 upwards as far as the highest number of the set to be given to j. The rectangle (partly filled with totients and partly vacant) so formed, read off in columns, will, by the preceding case, give the sum of the set of numbers, and read off in lines, the sum of the products of each divisor by its frequency.

Let us now inquire into the number of the terms contained in the several waves. W_1, which always exists, will be the coefficient of $\frac{1}{t}$ in $\frac{e^{\nu t}}{P\,(e^{\frac{at}{2}} - e^{-\frac{at}{2}})}$, and therefore (always supposing the number of indices a to be i) will be the coefficient of t^{i-1} in the product of $\left(1 + \nu t + \nu^2 \frac{t^2}{1\,.\,2} + \ldots\right)$ into the ascending development of $\frac{1}{P\left(\frac{e^{\frac{at}{2}} - e^{-\frac{at}{2}}}{t}\right)}$, and will therefore be a function of ν consisting of multiples of $\nu^{i-1}, \nu^{i-3}, \ldots$ until a multiple of ν or a constant is reached, and therefore containing $E\,\frac{i+1}{2}$ terms, the first of which it may be well to notice (using $a_1, a_2 \ldots a_i$ in lieu of $a, b, \ldots l$ as the indices) will obviously always be $\frac{1}{\Pi\,(i-1)\, a_1 .\, a_2 . \ldots a_i}$†.

In like manner it will be obvious that for W_2 the degree of ν will be the frequency of 2 diminished by a unit, and the form of W_2 will be $(-)^n$ into a polynomial function of ν of that degree.

* I say *effective* because it will presently be seen that in a certain case one of the trivial zero values will be ineffective, that is, will only lead to an identity and not to an equation between the coefficients in question.

† The highest power of ν in any other wave (which is its frequency diminished by unity) will *in general be* less than $i-1$, and consequently the sign of the terms in the development of any rational fraction beyond a certain point must be unvarying, and the development from that

Again, any other wave W_i of frequency f_i will consist of a set of products of polynomial functions of ν of the degree $f_i - 1$ each multiplied by a sum of exponential quantities consisting of pairs of the form $c\Sigma(\rho^{\nu+\delta} + \rho^{\nu-\delta})$ or $c\Sigma(\rho^{\nu+\delta} - \rho^{\nu-\delta})$ according as $i - f_i$ is even or odd, where δ will be half the number of primitive ith roots of unity, say $\frac{\tau(i)}{2}$, where the numerator is the totient of i.

Hence the total number of constants to be determined in the algebraico-exponential function representing $\frac{n}{a_1, a_2, \ldots a_i}$ will be

$$E\frac{f_1+1}{2} + E\frac{f_2+1}{2} + \Sigma\frac{\phi\lambda f_\lambda}{2} \quad [\lambda = 3, 4, \ldots \infty].$$

(1) Suppose that f_1 and f_2 are not both even.

Then remembering that $\frac{f_1}{2} + \frac{f_2}{2} + \frac{f_3 \cdot \tau_3}{2} + \frac{f_4 \cdot \tau_4}{2} + \ldots = \frac{s}{2}$, the antecedent expression $= E\left(\frac{s}{2} + 1\right)$, for when f_1, f_2 are both odd, the two first terms on the left-hand side of this equation exceed the corresponding ones in the equation above it by $\frac{1}{2}, \frac{1}{2}$ respectively, and $E\left(\frac{s}{2} + 1\right)$ will exceed $\frac{s}{2}$ by unity (because $f_1 - f_2$ the number of the odd elements in the sum of all of them being even, s is even). And if f_1, f_2 are one odd and the other even, the right as well as the left-hand side of each equation will be increased $\frac{1}{2}$, for s will be now odd.

(2) Suppose that f_1, f_2 are both even, then

$$E\frac{f_1+1}{2} + E\frac{f_2+1}{2} + \frac{f_3\tau(3)}{2} + \ldots = \frac{f_1}{2} + \frac{f_2}{2} + \frac{f_3\tau(3)}{2} + \ldots = \frac{s}{2}.$$

Hence the number of constants to be determined is $1 + E\frac{s}{2}$, except when f_1, f_2 are both even, in which case it is $\frac{s}{2}$.

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point omni-positive or omni-negative, according as the numerator, on substituting unity for the variable, is positive or negative. The case of exception is when all the indices have a common numerant, say δ, for then the frequency of δ will be the same as of unity, and W_δ be of the same degree as W_1 in ν, so that the reason for uniformity of sign (at a sufficient distance from the origin) no longer subsists. This is the proof referred to at p. [600], in what precedes.

It is worth while imprinting on the memory the rule that the asymptotic value of

$$\frac{n}{a_1, a_2, \ldots a_i} \div n^{i-1} \text{ is } \frac{1}{\{1 . 2 . 3 \ldots (i-1)\}\, a_1 . a_2 \ldots a_i},$$

which ought, I imagine, to be susceptible of some simple proof or illustration by the method of nodes or cross-gratings, such as employed by Eisenstein to prove the law of reciprocity for quadratic residues, and by myself (*Johns Hopkins Circulars*, Nos. 13 and 14, pp. 179, 180, 209)* to demonstrate the impossibility of the existence of trebly periodic functions.

[* Below, pp. 635, 644.]

On *the first supposition* the trivial values of ν which make $F(\nu)$ zero are $0, 1, 2, \ldots \frac{s}{2} - 1$ when s is even, and $\frac{1}{2}, \frac{3}{2}, \ldots \left(\frac{s}{2} - 1\right)$ when s is odd, the number of such being $E\left(\frac{s}{2}\right)$ in either case, and there will be $E\left(\frac{s}{2}\right)$ homogeneous equations for finding the ratios of $E\left(\frac{s}{2}\right) + 1$ coefficients, which is exactly the right number.

On the second supposition, that is, when f_1, f_2 are both even, the number of the trivial values in question will be $\frac{s}{2}$, the same as the number of the coefficients, so that at first sight there would appear to be one superfluous equation—such, however, is not really the case—because the value 0 attributed to ν will lead not to a homogeneous equation between the coefficients but to the identity $0 = 0$. For evidently W_1, W_2 becoming odd functions of ν, will vanish when $\nu = 0$, and every other wave will also vanish; for when $\nu = 0$ it will consist exclusively of pairs of terms of the form $c(\rho^\delta - \rho^{-\delta})$ (because by hypothesis f_1 the number of the elements is even), and since ρ and $\frac{1}{\rho}$ may be interchanged, it follows that the sum of such pairs must be zero. Hence whatever the relation of the number of odd and the number of even elements to the modulus 2, there will be just as many homogeneous equations as are required for determining the ratios of the coefficients in the form which expresses the denumerant. The absolute values of the coefficients may be found by writing $F\left(\frac{s}{2}\right) =$ coefficient of x^0 in the generating function $= 1$, or by virtue of the observation made above, that the leading coefficient in W_1 for the elements $a_1, a_2, \ldots a_i$ is $\frac{1}{\pi(i-1)\, a_1, a_2, \ldots a_i}$.

When the denumerant is regarded as a function of n and not of ν, it is obvious *à priori* that being a particular integral of an equation in finite differences of the order s, its coefficients must be determinable in relative magnitude by the knowledge of $(s-1)$ values of the variable for which it vanishes, and this is almost but not quite sufficient in itself to establish the preceding result regarding the canonical form.

I will illustrate this method presently by one or two easy examples, but previously it will, I think, be desirable to give greater precision and uniformity to the nomenclature of simple denumerants.

If any such be denoted by $\frac{n}{a, b, \ldots l}$, (I have sometimes here or elsewhere referred to n as the numerator or denumerator or partible number, and to $a, b, \ldots l$, variously as the denominators or as the indices or as the elements of the denumerant), in future I shall call n the componend, and $a, b, \ldots l$ the components of the denumerant.

A denumerant with a single component as $\dfrac{n}{a,}$ which I call an elementary denumerant, deserves special attention, for it will presently be seen that every given simple denumerant is expressible as a sum of powers of its componend multiplied respectively by linear functions of elementary denumerants whose several components are the divisors of the components of the given one.

The elementary denumerant $\dfrac{n}{a,}$ being the number of solutions in positive integers of the equation $ax = n$, is obviously 1 or 0 according as n does or does not contain a. But we may also regard $\dfrac{n}{a,}$ as an analytical function and define it as the mean of the a values of ρ^n where ρ is any root of the equation $\rho^a - 1 = 0$, and so construed it will preserve a meaning even when n is taken a negative integer, and will mean 1 or 0, provided that n be an integer of either kind, according as it does or does not contain a without a remainder. It is in this extended sense that $\dfrac{n}{a,}$ or $\dfrac{\nu}{a,}$ will be employed in what follows.

Supposing r to be a primitive ith root of unity, W_i will consist of a sum of powers of ν each multiplied by the sum of quantities of the form $cr^{n+\delta}$ (where for the moment for greater clearness of elucidation I purposely retain n instead of using its augmentative ν). On giving n all values from $-\delta$ to $-\delta + i - 1$ inclusive, this sum will take i successive values to be determined from the equation containing the primitive roots, say $\epsilon_0, \epsilon_1, \ldots \epsilon_{i-1}$, so that its general value will be expressible under the form

$$\epsilon_0 \frac{n+\delta}{i,} + \epsilon_1 \frac{n+\delta-i}{i,} + \ldots + \epsilon_{i-1} \frac{n+\delta-i+1}{i,}.$$

We may then replace n by $\nu - \dfrac{s}{2}$, and on so doing and further replacing (where requisite) any numerator by its residue in respect to i, shall obtain a sum of the form

$$\eta_0 \frac{\nu}{i,} + \eta_1 \frac{\nu-1}{i,} + \ldots + \eta_{i-1} \frac{\nu-i+1}{i,} \text{ when } s \text{ is even,}$$

and of the form

$$\eta_0 \frac{\nu-\frac{1}{2}}{i,} + \eta_1 \frac{\nu-\frac{3}{2}}{i,} + \ldots + \eta_{i-1} \frac{\nu-i+\frac{1}{2}}{i,} \text{ if } s \text{ is odd.}$$

On this being done, remembering the extension given to the sense of an elementary denumerant and the theorem that the analytical value $F\nu$ of a denumerant is equal to $\pm F(-\nu)$, we see that in either case the above sums will be reducible to a sum of pairs of terms of the form $\eta\left(\dfrac{\nu+k}{i,} \pm \dfrac{\nu'-k}{i,}\right)$

[the same + or − sign subsisting throughout the whole series for any specified power of ν] but subject to the exception that when i is even, two of the pairs will be replaced by single terms, multiples of $\dfrac{\nu \pm \frac{i}{2}}{i,}$ and of $\dfrac{\nu}{i,}$ respectively, which become zero when the negative sign is the one to be employed*.

Thus taking $i = 2$, W_2 takes the form $(-)^n R\nu$, that is, $\dfrac{n}{2,} - \dfrac{n-1}{2,}$. W_1 it is scarcely necessary to repeat will contain no elementary denumerants, being purely an algebraical function of the resolvent. W_2 is such a function multiplied by $(-1)^n$. This multiplier is expressible under the form $\left(\dfrac{n}{2,} - \dfrac{n \pm 1}{2,}\right)$ which is always a function of n that remains unchanged when n is changed into $-n$. But when the two denumerants are expressed as functions of ν the case is different; if s (the sum of the components) is an even number, the above pair of terms becomes $(-)^{\frac{s}{2}}\left(\dfrac{\nu}{2,} - \dfrac{\nu \pm 1}{2,}\right)$ which is unaltered by the change of ν into $-\nu$, but when s is odd it becomes $(-)^{\frac{s-1}{2}}\left(\dfrac{\nu - \frac{1}{2}}{2,} - \dfrac{\nu + \frac{1}{2}}{2,}\right)$ which changes its sign when ν is changed into $-\nu$.

Before quitting the subject of nomenclature I may just observe that it will be convenient to call denumerants, when their resolvents are the natural numbers commencing with unity, *natural denumerants*, and when the natural numbers commencing with 2, curtate natural, or for greater brevity simply *curtate* denumerants, the highest number reached in either case being termed the order; D_i and Δ_i may then be used to denote natural and curtate denumerants of the order i†.

I now return to the application of the method of indeterminate coefficients to finding the value of denumerants whose components are given. This method is not practically applicable when the sum of the components is considerable, because that sum measures the number of linear equations to be solved. In the following section I shall work out in full, by the regular process, the case where the components are 2, 3, 4, 5, 6, 7, of which the result

* The sign is positive or negative according as the number of the components less the power of ν in question is odd or even, and it is easy also to see that the sum of all the coefficients of the elementary denumerants in the multiplier of each power of ν will be always zero.

† It is curtate denumerants which are almost exclusively required in the applications to the theory of invariants. If necessary to bring into evidence the componend we may use the more explicit notation $\overset{n}{D}_i$, $\overset{n}{\Delta}_i$ to signify natural and curtate denumerants of the order i with the componend n. Thus we may write $\overset{n}{D}_i - \overset{n-1}{D}_i = \overset{n}{\Delta}_i$ and $\overset{n}{D}_i - \overset{n}{D}_{i-1} = \overset{n-1}{\Delta}_i$.

It may be as well to notice that for curtate, as well as for natural denumerants, the divisors of the components are the natural numbers from unity to the order of the denumerant inclusive, so that the number of the waves for either of these sort of denumerants is equal to the order.

is more especially required for the purposes of the preceding section, and which has not previously been calculated. The other algebraical formulae for denumerants in their canonical form I shall give without exhibiting the work; the accuracy of most of them can be ascertained by comparison with Prof. Cayley's values of the same, exhibited as functions of the unaugmented componend in the *Phil. Trans.* for 1856 and 1858.

Let us suppose 1, 2, 3 to be the components,

we may write $\dfrac{n}{1,\,2,\,3,} = A\nu^2 + B + (-)^\nu C + \Sigma(\rho^{\nu+1} + \rho^{\nu-1})D,$

where $\rho^2 + \rho + 1 = 0$, or more simply, $A\nu^2 + B + (-)^\nu C - D\Sigma\rho^\nu = 0$.

Hence making $\nu = 0, 1, 2$ we have

$$B + C - 2D = 0$$
$$A + B - C + D = 0$$
$$4A + B + C + D = 0,$$

so that $2C + 3A = 0 \quad 3D + 4A = 0 \quad B + \left(\frac{8}{3} - \frac{3}{2}\right)A = 0,$

or $A = 6\sigma \quad B = -7\sigma \quad C = -9\sigma \quad D = -8\sigma;$

and to find σ, making $\nu = 3$, we obtain

$$(54 - 7 + 9 + 16)\sigma = 1 \quad \text{or } \sigma = \frac{1}{72}.$$

Hence $$\frac{n}{1,\,2,\,3,} = \frac{\nu^2}{12} - \frac{7}{72} - \frac{1}{8}\left(\frac{\nu}{2,} - \frac{\nu-1}{2,}\right) + \frac{1}{9}\left(2\frac{\nu}{3,} - \frac{\nu+1}{3,} - \frac{\nu-1}{3,}\right)$$

monomial denumerants being used to replace the exponential quantities $(-1)^\nu$; $\Sigma\rho^\nu$.

The leading coefficient $\frac{1}{12}$ it will be observed $= \frac{1}{(1\,.\,2)(1\,.\,2\,.\,3)}$, as it ought to be by the general rule.

The maximum negative value of $\frac{n}{1,\,2,\,3,} - \frac{\nu^2}{12}$ is $\frac{7}{72} + \frac{1}{8} - \frac{1}{9}$ or $\frac{1}{9}$, and its maximum positive value $\frac{2}{9} + \frac{1}{8} - \frac{7}{72}$ or $\frac{1}{4}$. Hence the value of $\frac{n}{1,\,2,\,3,}$ is always the nearest integer to $\frac{(n+3)^2}{12}$.

But by Euler's theorem of reciprocity $\frac{n}{1,\,2,\,3,}$ is the number of ways of resolving n into three or less than three parts, and consequently $\frac{n-3}{1,\,2,\,3,}$ is the number of ways of resolving n into exactly three parts, this therefore is always the nearest integer to $\frac{n^2}{12}$, as first observed I believe by the late lamented Prof. De Morgan.

Take as another case the components 1, 2, 3, 4 which give $\nu = n + 5$. We may write

$$\frac{n}{1,\,2,\,3,\,4,} = A\nu^3 + B\nu + (-)^\nu C\nu + D\Sigma(\rho^{\nu+1} - \rho^{\nu-1}) + E\Sigma(i^{\nu+1} - i^{\nu-1})$$

where $\rho^2 + \rho + 1 = 0$, $i^2 + 1 = 0$. Hence giving ν the successive values 1, 2, 3, 4, (omitting $\nu = 0$, which would lead to $0 = 0$) we obtain

$$A + B - C - 3D - 4E = 0$$
$$8A + 2B + 2C + 3D = 0$$
$$27A + 3B - 3C + 4E = 0$$
$$64A + 4B + 4C - 3D = 0.$$

Hence $\quad 72A + 6B + 6C = 0$, and $36A + 6B - 2C = 0$,

consequently $\quad 2C + 9A = 0 \quad 2B + 15A = 0 \quad -3D + 16A = 0$

or $\quad A = 6\sigma \quad B = -45\sigma \quad C = -27\sigma \quad D = 32\sigma \quad E = -27\sigma.$

Finally making $\nu = 5 \quad \sigma(750 - 225 + 135 + 96 + 108) = 1$, or $\sigma = \dfrac{1}{864}$,

and
$$\frac{n}{1,\,2,\,3,\,4,} = \frac{1}{144}\nu^3 - \frac{5}{96}\nu - \frac{1}{32}\left(\frac{\nu}{2,} - \frac{\nu-1}{2,}\right) + \frac{1}{9}\left(\frac{\nu-1}{3,} - \frac{\nu+1}{3,}\right) + \frac{1}{8}\left(\frac{\nu-1}{4,} - \frac{\nu-3}{4,}\right).$$

The principal coefficient is $\dfrac{1}{144}$ or $\dfrac{1}{\Pi 3\,.\,1\,.\,2\,.\,3\,.\,4}$, as it ought to be, according to the general rule, and this serves as a verification of the correctness of the whole work.

It will be found convenient to append here, instead of reserving for the following section, the analytical expression for the first wave of a general denumerant, which stands out markedly from the rest, inasmuch as it can be expressed once for all as an algebraical function of the componend and components without any regard being had to the arithmetical form of the latter.

Let $C(\tau_1\tau_2 \ldots \tau_j)$, $H(\tau_1\tau_2 \ldots \tau_j)$ or more briefly $C_j\tau$ $H_j\tau$ be understood to denote the perfectly well-known functions of $\tau_1, \tau_2, \ldots \tau_j$ which represent the elementary symmetric function and the sum of the homogeneous products of the jth order of those quantities of which τ_q represents the sum of the qth powers, so that, for example, $C_2\tau$, $H_2\tau$ will serve to denote $\dfrac{\tau_1^2 - \tau_2}{2}$, $\dfrac{\tau_1^2 + \tau_2}{2}$ respectively, upon which supposition we may write

$$e^{\tau_1 t + \tau_2\frac{t^2}{2} + \tau_3\frac{t^3}{3} + \ldots} = 1 + \tau_1 t + \frac{\tau_1^2 + \tau_2}{2}t^2 + \ldots + H_q\tau t^q + \ldots$$

Also let it be observed preliminarily that as a direct inference from Maclaurin's theorem, if ϕ represent any function of x but does not contain ν,

$$\text{co}_j\, e^{\nu x + \phi} = \text{co}_j\, e^\phi + \text{co}_{j-1}\, e^\phi \nu + \text{co}_{j-2}\, e^\phi \frac{\nu^2}{1\,.\,2} + \ldots$$

Furthermore for greater brevity let us agree to express the W_1 for j components $a_1, a_2, \ldots, a_j$ under the form $W_{1,j}$, and write it equal to $\frac{V_j}{\pi_j}$ where π_j indicates the product of the j components.

We may then write

$$V_j = \pi_j \, \text{co}_{-1} \frac{e^{\nu x}}{P\left(e^{a\frac{x}{2}} - e^{-a\frac{x}{2}}\right)}.$$

Now from the known expression for $\log \sin \theta$, we may write

$$\log\left(e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}\right) = \log\theta + \beta_1\theta^2 - \beta_2\theta^4 + \ldots \pm \beta_q\theta^{2q} + \ldots$$

where $$\beta_q = \frac{1}{\Pi 2q} \cdot \frac{B_{2q-1}}{2q}.$$

Hence $$V_j = \text{co}_{j-1}\, e^{\nu x - 2\tau_1\frac{x^2}{2} + 2\tau_2\frac{x^4}{4} - 2\tau_3\frac{x^6}{6}\ldots}$$

where $2\tau_q = \frac{B_{2q-1}}{\Pi 2q}\sigma_{2q}$ and the latter factor indicates the sum of the $2q$th powers of the components.

Hence writing $x^2 = t$ we have $V_j = \text{co}_{j-1}\, e^{\nu x - \tau_1 t + \tau_2\frac{t^2}{2} - \tau_3\frac{t^3}{3}\cdots}$

and consequently making $T = -\tau_1 t + \tau_2\frac{t^2}{2} - \tau_3\frac{t^3}{3}\cdots$

$$V_j = \text{co}_{j-1}T + \text{co}_{j-2}T.\nu + \text{co}_{j-3}T.\frac{\nu^2}{1.2} + \text{co}_{j-4}T.\frac{\nu^3}{1.2.3}\cdots$$

$$= \frac{\nu^{j-1}}{\Pi(j-1)} - H_1\tau\frac{\nu^{j-3}}{\Pi(j-3)} + H_2\tau\frac{\nu^{j-3}}{\Pi(j-3)}\cdots$$

the series ending with ν or with a constant according as j is even or odd.

Thus
$$V_2 = \nu,$$
$$V_3 = \frac{\nu^2}{2} - H_1(\tau),$$
$$V_4 = \frac{\nu^3}{6} - H_1(\tau)\nu,$$
$$V_5 = \frac{\nu^4}{24} - H_1(\tau)\frac{\nu^2}{2} + H_2(\tau),$$
$$V_6 = \frac{\nu^5}{120} - H_1(\tau)\frac{\nu^3}{6} + H_2(\tau)\nu, \text{ and so on,}$$

each V being an integral with respect to ν of the one which precedes it.

Substituting for each τ its value in terms of the Bernouillian numbers B and the σ's, and giving the former their arithmetical values we shall obtain

$$V_2 = \nu,$$

$$V_3 = \frac{\nu^2}{2} - \frac{\sigma_2}{24},$$

$$V_4 = \frac{\nu^3}{6} - \frac{\sigma_2}{24}\nu,$$

$$V_5 = \frac{\nu^4}{24} - \frac{\sigma_2}{48}\nu^2 + \left(\frac{\sigma_2^{\,2}}{1152} + \frac{\sigma_4}{2880}\right),$$

$$V_6 = \frac{\nu^5}{120} - \frac{\sigma_2}{144}\nu^3 + \left(\frac{\sigma_2^{\,2}}{1152} + \frac{\sigma_4}{2880}\right)\nu,$$

$$V_7 = \frac{\nu^6}{720} - \frac{\sigma_2}{576}\nu^4 + \left(\frac{\sigma_2^{\,2}}{2304} + \frac{\sigma_4}{5160}\right)\nu^2 - \left(\frac{\sigma_2^{\,3}}{82944} + \frac{\sigma_2\sigma_4}{103680} + \frac{\sigma_6}{181440}\right),$$

$$V_8 = \int_\gamma^0 d\nu\, V, \text{ and so on.}$$

Such are the expressions for V best adapted for actual use, since it is desirable to express $W_{1,j}$, that is, $\frac{V_j}{a_1 \,.\, a_2 \ldots a_j}$ explicitly in terms of powers of ν; but there is another somewhat noteworthy form which can be given to the V with an even subindex as follows:

It is obvious that

$$V_{2k} = \text{co}_{-1}\frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x}) + \frac{1}{2}(e^{\nu x} + e^{-\nu x})}{P\left(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x}\right)} = \text{co}_{-1}\frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x})}{P\left(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x}\right)}$$

for the neglected part of the numerator will contribute nothing to the residue*.

We may now calculate the logarithm of the entire quantity to be residuated instead of merely the denominator, and take the residue of its exponential ν; on so doing it will be obvious on reflection that we shall obtain the product of ν into a quantity of the very same form as the constant term in V_{2k-1}, when instead of σ_{2q} in the value of τ_q we substitute $-(2n)^{2q} + \sigma_{2q}$. If then we write $2U_q = \frac{B_{2q-1}}{\Pi 2q}\{(2n)^{2q} - \sigma_{2q}\}$ it is easy to see that we shall have $V_{2k} = \nu C_{k-1}(U)$.

* For V_{2k} the *effective* numerator of the residuand is a *sine* form, and may be subjected to the same treatment as its fellows in the denominator. The case is different with V_{2k-1}, for which the effective numerator of the residuand is a *cosine* form. But we may write

$$V_{2k-1} = \frac{d}{d\nu}V_{2k} = C_{k-1}U + \nu\delta C_{k-1}U,$$

and if we turn to account the fact that in $C_{k-1}U$ along with $(2\nu)^2, (2\nu)^4, \ldots (2\nu)^q \ldots$ are associated $-s_2, -s_4, \ldots -s_{2q} \ldots$ and choose to write $-\nu^{2q}\frac{d}{ds_{2q}} = \Delta^q$, it will be found that the above expression may be transformed so as to give the symbolical equation (more curious perhaps than useful) $V_{2k-1} = \left(\frac{1+\Delta}{1-\Delta}\right)^2 C_{k-1}U$, whereas as previously found $V_{2k} = \nu C_{k-1}U$.

Thus, for example, suppose $2k = 6$, we may write V_6 under the form

$$\nu\left\{\frac{(4\nu^2 - s_2)^2}{1152} - \frac{(16\nu^4 - s_4)}{2880}\right\}$$

to verify which it will be observed that

$$\frac{16}{1152} - \frac{16}{2880} = \frac{1}{72} - \frac{1}{180} = \frac{1}{120} \text{ and } \frac{8}{1152} = \frac{1}{144},$$

so that

$$V_6 = \frac{\nu^5}{120} - \frac{s_2}{144}\nu^3 + \left(\frac{s_2^{\,2}}{1152} + \frac{s_4}{2880}\right)\nu, \text{ as previously found.}$$

Before having done with this outline it may be well to call attention to the circumstance that the distribution of the infinity-roots into groups determined by the divisors of the components is not in all cases the best mode of grouping to adopt.

Thus suppose that the components $(a_1, a_2, \ldots a_i)$ are all prime relatively to each other, it will in such case be most expeditious, after taking out the algebraical part W_1, to separate what remains into i portions, referring respectively to *all* the non-unity a_1th, a_2th, ... a_ith roots of unity*.

This view enables us to give a concise answer to a question of some interest, namely, as to what is the number of solutions of the inequality

$$a_1x_1 + a_2x_2 + \ldots + a_ix_i < \mu\,(a_1a_2 \ldots a_i),$$

say $\mu\pi_i$, where μ is any positive integer and the coefficients are relatively prime each to each.

Certainly this number is no other than the denumerant $\dfrac{\mu\pi_i}{1, a_1, a_2, \ldots a_i}$ which might be calculated by the general formula, but would give a result neither concise nor elegant; we may on the other hand regard it as a sum of denumerants, say $\Sigma\dfrac{\mu\pi_i - \delta}{a_1, a_2, \ldots a_i}$, where δ takes all values from 0 to $\mu\pi_i - 1$. Now each such denumerant will consist of a purely algebraical and a purely periodical part, and it is very easy to see according to the view just indicated that the sum of all the latter will be zero. Hence the number required will be

$$\Sigma^0{}_{\mu\pi_i - 1}\frac{V_1}{\pi_i}.$$

I may illustrate this by the very simplest imaginable case, where there are but two components p, q and the number required is that of the solutions in integers of the inequality $px + qy < pq$ where p and q are relative primes.

Calling $pq = n$, the rule laid down will give for the number sought

$$\Sigma_0^{n-1}\frac{\nu}{n}, \text{ that is, } \Sigma_0^{n-1}\frac{n + \dfrac{p+q}{2}}{n} = \frac{pq - p - q - 1}{2}.$$

* This is tantamount to blending into one all the waves corresponding to the non-unity divisors of each component.

This result admits of a somewhat *piquant* verification. The number of integers less than pq and containing neither p nor q is $(p-1)(q-1)$, and if every two of these which are supplementary to one another (I mean whose sum is pq) be made into a pair, it is an easily demonstrable, but by no means an unimportant fact, that one of the pair will be a compound and the other a non-compound of p and q. Hence the total number of non-compounds is $\frac{1}{2}(p-1)(q-1)$, and therefore the total number of solutions of $px+qy<pq$ will be the remainder when the above is subtracted from pq, that is, $\frac{1}{2}(pq+p+q-1)$ as previously determined.

I will embrace this opportunity of noticing a correction that should be made to the long footnote in Section 3 given in the preceding number of the *Journal*. In lieu of the words* in the last paragraph of that note following the word *products*, line 3 and preceding the word *set*, line 8, read as follows:

Of the form $b^x Q^y R^z S^t T^u$ such that no one of them could be (a power of one or) a product of powers of any of the others. If then it could be shown that there exists in the succession a set of quintuplets x, y, z, t, u, such that the quotient system of every other quintuplet in the succession is intermediate to the quotient system of that

It may also be as well to notice here that the method of expressing in terms of ordinary space the intermediateness of a quadruplet, a triplet or a couplet, to four, three or two other such respective multiplets, may be profitably simplified by the use of quadriplanar, trilinear and bi-punctual coordinates, in flat spaces of three, two and one dimension respectively; for we may then without having recourse to quotient-systems regard each element of the multiplet as a coordinate of its representative point, inasmuch as the affection concerned being one relative exclusively to the inwardness or outwardness of a point in regard to a closed environment, obviously remains unchanged by projection.

What follows is the footnote referred to at foot of page [610] *where it was meant to be inserted.*

Each of the two statements regarding the coefficient-functions becomes next to self-evident when the coefficient of x^n in the reciprocal of $(1-\alpha x)(1-\beta x)\ldots(1-\lambda x)$ is put under the form of a sum of terms similar to $\alpha^n \div \left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right)\ldots\left(1-\frac{\lambda}{\alpha}\right)$ interpreted (when necessary) as meaning the function of $(n;\ \alpha,\ \beta\ldots\lambda)$ indefinitely near to the value of what such sum becomes when any equal elements are made to undergo arbitrary infinitesimal variations. Jacobi's proof of the theorem, I rather think, is got by proving it directly for each of the simple partial fractions into which any given proper fraction may be supposed to have been resolved.

A third method is to form the equation between u_n, u_{n-1}, ... u_{n+j-1}, and between

$$v_{-n},\ v_{-n-1},\ \ldots\ v_{-n-j-1}$$

[* p. 581 above.]

[u_n being the general coefficient in the ascending and v_{-n} in the descending development of $1 \div R(x)$]; the two equations become identical on changing u and n into v and $-n$, and $j-1$ homogeneous equations which help to determine the constants will be the same in both, namely, those got by making $n=-1, -2, \ldots -(j-1)$, consequently the two particular integrals u_n, v_n can differ only by a factor independent of n; if we write then $u_n : v_n :: P : Q$ and call the first and last coefficients in the denominator A and L, and pay attention to the fact that u_n, v_n can only become infinite when A, L vanish, and also to the indifference of the relation of R regarded as a quantic in x and 1 to the two sorts of development, it is plain to see that $P : Q :: A^\mu : \pm L^\mu$, but the x-weight of u_n is n and of v_{-n} is $-n$; hence $\mu=0$ and $u_n : v_n$ is independent, not only of n but of the coefficients in R, and to determine its value we may make $R=x^{j-1}-x^j$, which gives at once $u_n=-v_n$. This being true for all values of n, it is obvious that the relation will continue to subsist, when instead of unity any polynomial function of x of lower degree than that of the denominator (see below) is taken for the numerator.

Moreover, if the degree of the numerator be $j-\delta$, u_q and v_q will be seen (from what goes before) to vanish for every value of q common to the series

$$-1,\ -2, \ldots -(j-1) : 0,\ -1, \ldots -(j-2) : \ldots : (j-\delta-1),\ (j-\delta-2), \ldots -(\delta-1),$$

namely, for the values $-1, -2, \ldots -(\delta-1)$ or in other words either coefficient-function of the index of any power of the variable which appears neither in the ascending nor the descending development of a rational fraction is equal to zero.

Unless the fraction is a proper one u_n and v_n (the coefficient-functions) will not be continuous functions of n throughout; hence arises the necessity of this limitation in dealing with the generalized equation $u_n=-v_n$. Thus, for example, for the improper fraction $\frac{1+2x^2}{(1-x)^2}$, u_0 and v_0 are 1 and 2, but for any positive or negative value of n other than 0, u_n and v_n will be $3n-1$ and $-(3n-1)$ respectively. It may be added that the theorem will continue to subsist even for an improper fraction, provided that on freeing its numerator from a power of the variable, it becomes a proper one, for then the coefficient-functions remain continuous throughout.

This last proof, although more laboured than the preceding ones, seems to me the best because it goes straight to the heart of the question and does not depend on any apparently accidental results of calculation, but (so to say) compares the two twin functions in their nascent state, in the very act of birth.

The relation of the two coefficient-functions to one another and to the two general terms in the actual expansions becomes more clear if we use ϕn, ψn to denote the two former, reserving u_n, v_n for the two latter. Then besides the equation $\phi n+\psi n=0$ which is absolute, we have the equations $u_n=\phi n$, $v_n=\psi n$, limited as follows. Call Δ the deficiency of the numerator of the generating proper fraction, that is, the number of units that it stops short of its maximum possible value: then the first of these two equations holds good for all values of n not less than $-\Delta$, the latter for all values of n not greater than -1; if Δ is not zero, that is, if the degree of the numerator is not the integer next below that of the denominator, these two ranges will overlap for the values $-1, -2, \ldots, -\Delta$ of n, and for those values $\phi n=u_n=0$, $\psi n=v_n=0$. In the use made of these theorems in the text, the numerator is a mere constant, so that Δ has its maximum value, namely it is one unit less than the sum of the components (that sum being the degree of the generating function to a denumerant).

The general theorem may be brought into more distinct relief as follows: A finite fraction may be conceived as containing any number of powers of x positive or negative in numerator and denominator, and its two developments may be supposed to touch or be separate or to intersect one another. In the *last* case two coefficient-functions ϕn, $-\phi n$ exist applicable to all terms outside but inapplicable to any term inside the overlap. In the second case such functions exist which (besides being applicable, as in the case of contact, to all terms belonging to either of the two developments) vanish for all values of n in the chasm which separates them.

68.

TABLES OF GENERATING FUNCTIONS, REDUCED AND REPRESENTATIVE, FOR CERTAIN TERNARY SYSTEMS OF BINARY FORMS.

[*American Journal of Mathematics,* v. (1882), pp. 241—250.]

THE annexed tables have been calculated under my directions by Messrs Durfee and Ely, out of the fund placed at my disposition by the British Association for the Advancement of Science in the year 1881. Subsequent investigation will be necessary in order to ascertain whether there exist or not extra tabular groundforms which escape the operation of tamisage.

G. F. it will be understood stands for the words Generating Function.

SYSTEM OF TWO QUADRATICS AND ONE QUARTIC.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-\beta^2)(1-d^2)(1-d^3)(1-b\beta)(1-bd)$
$(1-\beta d)(1-b^2 d)(1-\beta^2 d)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4
	β^0	1				
b^0	β^1		$\bar{1}$			
	β^2			1		
	β^0		$\bar{1}$			
b^1	β^1		1	2		
	β^2		1		$\bar{1}$	
	β^3			$\bar{1}$		

		d^0	d^1	d^2	d^3	d^4
	β^1			$\bar{1}$		
b^3	β^2				1	
	β^3					$\bar{1}$
	β^0			1		
b^2	β^1		1		$\bar{1}$	
	β^2			$\bar{2}$	$\bar{1}$	
	β^3				1	

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-\beta^2)(1-d^2)(1-d^3)(1-b\beta)(1-b^2d^2)$
$(1-\beta^2 d^2)(1-b^2 d)(1-\beta^2 d)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5
	β^0	1					
b^0	β^1						
	β^2						
	β^3				1		
	β^4						
	β^0						
	β^1		1	1			
b^1	β^2		1	1	1		
	β^3						
	β^4				$\bar{1}$		
	β^0						
b^2	β^1		1	1	1		
	β^2						
	β^3				$\bar{1}$	$\bar{1}$	$\bar{1}$
	β^4						

		d^0	d^1	d^2	d^3	d^4	d^5	d^6
	β^0							
b^4	β^1				$\bar{1}$			
	β^2							
	β^3							
	β^4							$\bar{1}$
	β^0				1			
	β^1							
b^3	β^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	β^3					$\bar{1}$	$\bar{1}$	
	β^4							

TABLE OF GROUNDFORMS.

Deg. in coeff's of quadratic	Deg. in coeff's of quadratic	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1				
	2	1	1	1	
	3				1
1	0				
	1	1	1	1	
	2		1	1	1
2	0	1	1	1	
	1		1	1	1
3	0				1

SYSTEM OF QUADRATIC, CUBIC, AND QUARTIC.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-c^4)(1-d^2)(1-d^3)(1-bc^2)(1-b^3c^2)$
$(1-bd)(1-b^2d)(1-c^2d)(1-c^2d^3)(1-c^4d)$
$(1-c^4d^3)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
	c^0	1										
	c^2		$\bar{1}$									
b^0	c^4			2	2	2	1					
	c^6			1	1		$\bar{1}$	$\bar{1}$				
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$				
	c^{10}								1			
	c^{12}									$\bar{1}$		
	c^0		$\bar{1}$									
	c^2		2	4	2	1						
	c^4		2	2		$\bar{1}$	$\bar{2}$	$\bar{1}$				
b^1	c^6			$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	1	1			
	c^8			$\bar{1}$	$\bar{1}$		1	1	1			
	c^{10}				1	1		$\bar{1}$	$\bar{2}$	$\bar{2}$		
	c^{12}								$\bar{1}$		1	
	c^{14}									1		
	c^0			1								
	c^2		2	1	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^4		1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^6			$\bar{1}$			1	1				
b^2	c^8			$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{1}$			1	1	
	c^{12}					1	2	2	1	2	1	
	c^{14}										$\bar{1}$	
	c^2		1	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4	1	1		1			$\bar{1}$				
	c^6		$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		
b^3	c^8		$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$		1	2	1	1	
	c^{10}			1	1	2	1	2	2	3	2	
	c^{12}					1			$\bar{1}$		$\bar{1}$	$\bar{1}$
	c^{14}							1	1	1	$\bar{1}$	

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
	c^4			1								
	c^6				$\bar{1}$							
b^6	c^8					2	2	2	1			
	c^{10}					1	1		$\bar{1}$	$\bar{1}$		
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{14}										1	
	c^{16}											$\bar{1}$
	c^2			$\bar{1}$								
	c^4		$\bar{1}$		1							
	c^6			2	2	1		$\bar{1}$	$\bar{1}$			
b^5	c^8				$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1		
	c^{10}				$\bar{1}$	$\bar{1}$	1	3	3	2		
	c^{12}					1	2	1		$\bar{2}$	$\bar{2}$	
	c^{14}							$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{2}$	
	c^{16}										1	
	c^2		1									
	c^4		$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{1}$			1	1				
	c^8					1	2	2	1	1		
b^4	c^{10}					$\bar{1}$	$\bar{1}$			1		
	c^{12}						1	1	1	1	$\bar{1}$	
	c^{14}						1	1	1	$\bar{1}$	$\bar{2}$	
	c^{16}									$\bar{1}$		

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-c^4)(1-d^2)(1-d^3)(1-bc^2)(1-b^3c^2)(1-b^2d^2)$
$(1-b^2d)(1-c^4d^2)(1-c^2d^3)(1-c^4d)(1-c^4d^3)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
	c^0	1											
	c^2												
	c^4			1	2	2	1						
b^0	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{12}												
	c^{14}										$\bar{1}$		
	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^6												
b^1	c^8			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{14}												
	c^{16}										1		
	c^2		2	3	3	1							
	c^4		1	3	4	2	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6				$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
b^2	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}				$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{12}					1	2	2	1				
	c^{14}						1	2	2	1	1	1	1
	c^0				1								
	c^2		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^4	1	1	2	1	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
b^3	c^8		$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{2}$		1			
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{1}$	1	4	4	2	
	c^{12}				1	2	3	2	2	3	4	3	1
	c^{14}						1	2	3	2			
	c^{16}								1	1	1	$\bar{1}$	$\bar{1}$

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
	c^4			1									
	c^6												
	c^8					1	2	2	1				
	c^{10}					1	3	2	1				
b^7	c^{12}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{14}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}												
	c^{18}												$\bar{1}$
	c^2			$\bar{1}$									
	c^4												
	c^6			1	2	2		$\bar{1}$	$\bar{1}$				
b^6	c^8				2	3	2	1					
	c^{10}					1	2	3	3	3	1		
	c^{12}												
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{16}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^4	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$					
	c^6					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8					1	2	3	2	1			
b^5	c^{10}			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{12}					1	2	3	2	1			
	c^{14}					1	2	1	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{1}$	
	c^{16}								$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	
	c^2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4				$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^6	$\bar{1}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
b^4	c^8		$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	1	2	1				
	c^{10}				$\bar{1}$		2	4	5	6	3	1	
	c^{12}				1	2	2	2	2	2	2	1	
	c^{14}					1	3	3	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$
	c^{16}						1	1	1		$\bar{1}$	$\bar{1}$	
	c^{18}									$\bar{1}$			

TABLE OF GROUNDFORMS.

Deg. in coeff's of Quadratic	Deg. in coeff's of Cubic.	Degree in coeff's of Quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	2	1	2	3	2	1	
	4		2	4	5	3	1
2	0	1	1	1			
	2		2	3	3	1	
	4		1				
3	0				1		
	2	1	1	1			
	4	1	1				
4	2		1	1			

SYSTEM OF ONE QUADRATIC AND TWO QUARTICS.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)(1-b\delta)(1-b^2\delta)$
$(1-bd)(1-b^2d)(1-\delta d)(1-\delta^2 d)(1-\delta d^2)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6			d^0	d^1	d^2	d^3	d^4	d^5	d^6
	δ^0	1								δ^2			1				
	δ^1									δ^3							
b^0	δ^2			1					b^5	δ^4					1		
	δ^3									δ^5							
	δ^4					1				δ^6							1
	δ^0		$\bar{1}$							δ^1			$\bar{1}$				
	δ^1	$\bar{1}$	1	1	1					δ^2		$\bar{1}$					
b^1	δ^2		1	1					b^4	δ^3				1		1	
	δ^3		1		1					δ^4					1	1	
	δ^4						$\bar{1}$			δ^5				1	1	1	$\bar{1}$
	δ^5					$\bar{1}$				δ^6						$\bar{1}$	
	δ^0			1						δ^1		1	$\bar{1}$				
	δ^1		2		$\bar{1}$	$\bar{1}$				δ^2		$\bar{1}$				$\bar{1}$	
b^2	δ^2	1		$\bar{1}$	$\bar{2}$				b^3	δ^3					$\bar{2}$	$\bar{1}$	
	δ^3		$\bar{1}$	$\bar{2}$						δ^4				$\bar{2}$	$\bar{1}$		1
	δ^4		$\bar{1}$				$\bar{1}$			δ^5			$\bar{1}$	$\bar{1}$		2	
	δ^5					$\bar{1}$	1			δ^6					1		

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)(1-b^2\delta^2)(1-b^2\delta)$
$(1-b^2d^2)(1-b^2d)(1-\delta d)(1-\delta^2 d)(1-d^2\delta)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6
	δ^0	1						
	δ^1							
b^0	δ^2			1				
	δ^3							
	δ^4					1		
	δ^1		1	1	1			
b^1	δ^2		1	1	1			
	δ^3		1	1	1			
	δ^1		1	1				
	δ^2		1	1				
b^2	δ^3				1	1		
	δ^4				1		$\overline{1}$	$\overline{1}$
	δ^5					$\overline{1}$		
	δ^6					$\overline{1}$		
	δ^0				1			
	δ^1		1	1		$\overline{1}$	$\overline{1}$	
b^3	δ^2		1	1	$\overline{1}$	$\overline{2}$	$\overline{1}$	
	δ^3	1		$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	
	δ^4		$\overline{1}$	$\overline{2}$	$\overline{2}$			
	δ^5		$\overline{1}$	$\overline{1}$	$\overline{1}$			

		d^1	d^2	d^3	d^4	d^5	d^6	d^7
	δ^3			1				
	δ^4							
b^7	δ^5					1		
	δ^6							
	δ^7							1
	δ^4				1	1	1	
b^6	δ^5				1	1	1	
	δ^6				1	1	1	
	δ^1			$\overline{1}$				
	δ^2			$\overline{1}$				
b^5	δ^3	$\overline{1}$	$\overline{1}$		1			
	δ^4			1	1			
	δ^5					1	1	
	δ^6					1	1	
	δ^2				$\overline{1}$	$\overline{1}$	$\overline{1}$	
	δ^3				$\overline{2}$	$\overline{2}$	$\overline{1}$	
b^4	δ^4		$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{1}$		1
	δ^5		$\overline{1}$	$\overline{2}$	$\overline{1}$	1	1	
	δ^6		$\overline{1}$	$\overline{1}$		1	1	
	δ^7				1			

TABLE OF GROUNDFORMS.

Deg. in coeff's of quadratic.	Deg. in coeff's of quartic.	Deg. in coeff's of quartic.			
		0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
1	0				
	1		1	1	1
	2		1	1	1
	3		1	1	
2	0	1	1	1	
	1	1	1	1	
	2	1	1		
3	0				1
	1		1	1	
	2		1		
	3	1			

SYSTEM OF THREE QUARTICS.

G. F. for invariants, reduced form.

Denominator: $(1-\partial^2)(1-\partial^3)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)$
$(1-\partial\delta)(1-\partial d)(1-\delta d)(1-\partial^2 d)(1-\partial d^2)$
$(1-\partial^2\delta)(1-\partial\delta^2)(1-\delta^2 d)(1-\delta d^2)$.

Numerator :

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^0	1								
	δ^1									
	δ^2			1						
∂^0	δ^3									
	δ^4					1				
	δ^1		1	1						
	δ^2		1	1						
	δ^3				1	1				
∂^1	δ^4				1		$\bar{1}$	$\bar{1}$		
	δ^5					$\bar{1}$				
	δ^6					$\bar{1}$				
	δ^0			1						
	δ^1		1	1						
	δ^2	1	1	1			$\bar{1}$	$\bar{1}$		
∂^2	δ^3				1	$\bar{1}$	$\bar{2}$	$\bar{1}$		
	δ^4				$\bar{1}$	$\bar{3}$	$\bar{2}$		1	
	δ^5			$\bar{1}$	$\bar{2}$	$\bar{2}$	1	1		
	δ^6			$\bar{1}$	$\bar{1}$		1	1		
	δ^7					1				
	δ^1				1	1				
	δ^2				1	$\bar{1}$	$\bar{2}$	$\bar{1}$		
∂^3	δ^3		1	1		$\bar{4}$	$\bar{3}$	$\bar{1}$		
	δ^4		1	$\bar{1}$	$\bar{4}$	$\bar{5}$		2	1	
	δ^5			$\bar{2}$	$\bar{3}$		3	2		
	δ^6			$\bar{1}$	$\bar{1}$	2	2	1		
	δ^7					1				

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^4				$\bar{1}$				
	δ^5								
	δ^6						$\bar{1}$		
∂^8	δ^7								
	δ^8								$\bar{1}$
	δ^2				1				
	δ^3				1				
	δ^4		1	1		$\bar{1}$			
∂^7	δ^5				$\bar{1}$	$\bar{1}$			
	δ^6						$\bar{1}$	$\bar{1}$	
	δ^7						$\bar{1}$	$\bar{1}$	
	δ^1				$\bar{1}$				
	δ^2		$\bar{1}$	$\bar{1}$		1	1		
	δ^3		$\bar{1}$	$\bar{1}$	2	2	1		
∂^6	δ^4	$\bar{1}$		2	3	1			
	δ^5		1	2	1	$\bar{1}$			
	δ^6		1	1			$\bar{1}$	$\bar{1}$	$\bar{1}$
	δ^7						$\bar{1}$	$\bar{1}$	
	δ^8						$\bar{1}$		
	δ^1				$\bar{1}$				
	δ^2		$\bar{1}$	$\bar{2}$	$\bar{2}$	1	1		
∂^5	δ^3		$\bar{2}$	$\bar{3}$		3	2		
	δ^4	$\bar{1}$	$\bar{2}$		5	4	1	$\bar{1}$	
	δ^5		1	3	$\bar{4}$		$\bar{1}$	$\bar{1}$	
	δ^6		1	2	1	$\bar{1}$			
	δ^7				$\bar{1}$	$\bar{1}$			

Numerator—*Continued:*

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^0					1				
	δ^1				1		$\bar{1}$	$\bar{1}$		
	δ^2				$\bar{1}$	$\bar{3}$	$\bar{2}$		1	
∂^4	δ^3		1	$\bar{1}$	$\bar{4}$	$\bar{5}$		2	1	
	δ^4	1		$\bar{3}$	$\bar{5}$		5	3		$\bar{1}$
	δ^5		$\bar{1}$	$\bar{2}$		5	4	1	$\bar{1}$	
	δ^6		$\bar{1}$		2	3	1			
	δ^7			1	1		$\bar{1}$			
	δ^8					$\bar{1}$				

Representative form same as reduced form.

Table of Groundforms.

Deg. in coeff's of quartic	**Deg. in coeff's of quartic**	**Deg. in coeff's of quartic.** 0	1	2	3
0	**0**			**1**	**1**
	1		**1**	**1**	
	2	**1**	**1**	**1**	
	3	**1**			
1	**0**		**1**	**1**	
	1	**1**	**1**	**1**	
	2	**1**	**1**	**1**	
2	**0**	**1**	**1**	**1**	
	1	**1**	**1**	**1**	
	2	**1**	**1**		
3	**0**	**1**			

69.

ON A CERTAIN INTEGRABLE CLASS OF DIFFERENTIAL AND FINITE DIFFERENCE EQUATIONS.

[*Johns Hopkins University Circulars*, I. (1882), p. 178.]

IN Mr Moulton's edition of Boole's *Finite Differences* will be found quoted from the author of this notice a certain class of equations of which the *general* integral can be found as for example

$$\begin{vmatrix} u_x & u_{x+1} \\ u_{x+1} & u_{x+2} \end{vmatrix} = A\alpha^x,$$

that is,

$$u_x u_{x+2} - u^2{}_{x+1} = A\alpha^x,$$

or again

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} \\ u_{x+1} & u_{x+2} & u_{x+3} \\ u_{x+2} & u_{x+3} & u_{x+4} \end{vmatrix} = A\alpha^x,$$

and so on for a persymmetrical determinant of any order (n) constructed on the same principle as the two foregoing ones; an equation of the nth degree and $2n$th order will thus arise.

In this communication to the Seminarium the writer pointed out that an integral (but without any arbitrary constants) may be found for an equation of the same form as that above indicated on the left hand side but with $(n+1)$ different exponentials instead of a single one on the right hand side as for example

$$u_x u_{x+2} - u^2{}_{x+1} = A\alpha^x + B\beta^x + C\gamma^x$$

can be integrated provided that there are really three and not merely two distinct terms as would happen if A or B or C were one of them to vanish. But any number of the exponentials may be made indefinitely near to each other and the integral still hold good; in this way other integrable forms of equations can be obtained. As for instance

$$u_x u_{x+2} - u^2{}_{x+1} = A\alpha^x + (B + Cx)\beta^x,$$
$$u_x u_{x+2} - u^2{}_{x+1} = (A + Bx + Cx^2)\alpha^x$$

are integrable.

The same conclusions in all respects apply both as regards the general and the special integral case when any term u_x is replaced by y and u_{x+i} by $\left(\frac{d}{dx}\right)^i y$. The form of the special integral whether for differential or difference equations is rather too long to produce in this abstract but will be given in full in a future number of the *American Journal of Mathematics* [above, p. 546].

70.

ON A QUESTION IN PARTITIONS.

[*Johns Hopkins University Circulars*, I. (1882), p. 179.]

CLOSELY connected with the theory of the contacts or special intersections of quadric figures in space of any number of dimensions, and also with the more general but allied theory of the different genera and species of the roots of unitary matrices, is the question of the number of series that can be formed commencing with zero and ending with a given number i subject to the condition that each intermediate term of any such series shall be not greater than the mean between its antecedent and consequent. By arranging each of the indefinite partitions of i according to an ascending order of magnitude, it was shown that there was a one to one correspondence between each such arrangement and each such series, and, consequently, that the number of the series is equal to the number of indefinite partitions of the given final term i.

71.

ON A GEOMETRICAL PROOF OF A THEOREM IN NUMBERS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 179, 180.]

THE theorem in question is the well-known one that if a, b are incommensurable and x, y integers $ax + by + c$ may be made positively and negatively indefinitely small. This is tantamount to showing that on the plane of a reticulation*, nodes may be found indefinitely near to and on each side of an irrational straight line, that is, a line not parallel to any line of nodes. The proof is based on the Lemma that no infinite parallelogram, each side of which is an irrational line containing a node, can be vacuous of nodes in its interior. If this were not true a succession of shifts of the figure in the direction of the line joining the two nodes would lead to the absurd conclusion that the whole reticulation consists of a single line of nodes.

(1) Suppose the irrational line L contains a node and that there is no other node at less than a finite distance from it on one side of it, say to the right. Let it be moved to the right parallel to itself until it passes through another node N', then there will be a vacuous parallelogram of the kind declared impossible by the Lemma. [To this it may be objected that when L has moved from the left to M through a distance δ, M might be supposed to be an asymptote to an infinite series of nodes to its right. But if this were the case a node P might be found at a less distance than δ from M, and a node, Q, nearer to M than P is; if this line of nodes PQ be followed up until we reach the first node T on the other side of M, the most elementary geometry seems to show that T in *any* case is nearer to M than P is and consequently there would be a node between L and M contrary to hypothesis.] Hence there must be a node indefinitely near to L on each side of it.

(2) Suppose the irrational line L not to contain a node. If the theorem to be proved is not true, L may as before be moved parallel to itself (through

* By a reticulation is to be understood a pair of systems of an infinite number of indefinite equidistant parallel lines in a plane whose intersections form the nodes.

a finite distance) until it pass through a single node and there would be a vacuous parallelogram of which one side contains nodes, which has already been shown to be impossible.

Dr Story and Dr Franklin took part in the discussion and the valuable critical observations of the latter, led to the consideration of the objection stated and disposed of in the passage within brackets above. Professor Cayley made a remark to the effect that the diamond point in a graver's tool however fine, drawn in a straight direction across the face of a double grating must either pass through none of the intersections of the two systems of parallel lines or through an infinite number of them. The principle established in the bracketed passage admits of being stated in the following terms: "It is impossible for a straight line in the plane of a reticulation to be asymptotic in regard to nodes on one side of it and not so in regard to the nodes on the other side"; this proposition and the Lemma being conceded, the existence of any indefinite *vacuous* strip bounded by irrational parallel lines is disproved by imagining it distended on both sides, still retaining its form (in case neither bounding line contains a node), or in the contrary case on one side only (that is, in the direction away from the nodal line) until the distended figure passes through two nodes. The asymptotic rule shows that this construction would be possible—the Lemma that it leads to an impossible result. From this it follows that every irrational line is asymptotic in respect to the nodes lying on *each* side of it which is the thing to be proved.

Let a line be termed mono-asymptotic when it is asymptotic in regard to any scheme of points lying on one side of it,—amphi-asymptotic when it is so for schemes of points lying on each side of it. The foregoing argument may then be summed up as follows. Any irrational right line in the plane of a reticulation, must be amphi-asymptotic as regards the nodes. For if not, a line parallel to it must (under pain of contradicting the Lemma) be conceded to exist, which shall be mono-asymptotic in respect to them, but the existence of such a line has been proved to be impossible*. Similarly, it may be shown for a solid network, that no indefinite open prism whose parallel edges are doubly irrational (that is, neither parallel to a nodal line nor to a plane of nodes) can be vacuous of nodes, and also that no plane can be mono-asymptotic—from which, by very similar reasoning to that previously used, may be deduced the law, that no prism of finite dimensions, *vacuous* of nodes, can be constructed about an irrational line as its axis and that consequently any such line may be regarded as a sort of asymptotic axis to a

* The form of proof is a somewhat unusual combination of an *Ex-absurdo* with a *Dilemma*. A *denial* of the amphi-asymptoticism of an irrational straight line either dashes itself against the impossibility of the existence of a vacuous parallelogram or against the equal impossibility of the existence of a mono-asymptotic line.

helical spiral of nodes. Hence it follows that if a, b, c (taken two and two) are incommensurable with each other, the quadratic function

$$\{b(z-\gamma)-c(y-\beta)\}^2+\{c(x-\alpha)-a(z-\gamma)\}^2+\{a(y-\beta)-b(x-\alpha)\}^2,$$

and as a particular case

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

may be made indefinitely small with integer values of x, y, z.

Nor is this all, for not only can a node be found indefinitely near to the doubly irrational line $x:y:z::a:b:c$, but such node may be successfully sought for within any infinitesimal sector of space contained within two planes drawn through that line, or in other words a node can be found indefinitely near to the irrational line and to any plane drawn through it, that is, to any plane

$$bc(m-n)x+ca(n-l)y+ab(l-m)z,$$

where l, m, n are any quantities whatever*, so that x, y, z integer numbers can be found which shall simultaneously cause

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

to be less than any positive quantity k^2 and

$$(b-c)x+(c-a)y+(a-b)z$$

to lie between 0 and h, or 0 and $-h$ where h is also any assigned quantity. And of course the proposition can be stated in more general terms by considering an irrational line which does not pass through a node.

The same geometrical property admits of being defined under another form, namely, through the assertion that if any two given planes be drawn through a completely irrational line in an infinite Nodal Block†, a node may be formed indefinitely near to each of them—and this statement translates itself into the arithmetical proposition following:

If no linear equation

$$\lambda(b\gamma-c\alpha)+\mu(c\alpha-a\gamma)+\nu(a\gamma-b\alpha)=0$$

exists for integer values of λ, μ, ν, the two expressions

$$ax+by+cz+d,$$

and

$$\alpha x+\beta y+\gamma z+\delta$$

may simultaneously be made less than any given quantity k, by integer values of x, y, z.

* A particular form of this is $(b-c)x+(c-a)y+(a-b)z$. In order to give the theorem its greatest generality it is only necessary to substitute for x, y, z, $x-\alpha$, $y-\beta$, $z-\gamma$ where α, β, γ, are any real quantities whatever.

† By a Nodal Block is to be understood three systems in space of an indefinite number of equidistant parallel planes whose intersections are the nodes.

Although it would be difficult to follow the theory of nodal schemes into regions transcending the sensible dimensions of space, there need be no hesitation in accepting the truth of the generalized arithmetical theorem corresponding to this bolder than Icarian flight, namely, that

Any number of linear functions of one more than that number of integer Variables such that the determinant to the Matrix formed by the coefficients of the Variables supplemented by a line of arbitrary integers is incapable of being made zero, can by a right assignment of the Variables be brought to lie each of them between any assigned (indefinitely narrow) limits.

This proposition admits of a partial removal of the condition imposed in the above statement.

For an irrational line even if singly irrational, that is, parallel to a *nodal* plane although not so to a line of nodes, will be asymptotic to a series of nodes *if it lies in a nodal plane*, the only difference in this case being that the nodal sheath will be plain instead of being helical. Hence the two functions

$$ax + by + cz + d, \quad a'x + b'y + c'z + d',$$

can be made simultaneously indefinitely small, even though integer numbers A, B, C, *can* be found such that the determinant $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ A & B & C \end{vmatrix}$ is zero, provided that a rational number D can also be found, which will cause *all* the complete minors of the Matrix $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ A & B & C & D \end{vmatrix}$ to vanish.

A particular case of this arises when d and d' are each zero. Consequently the two twin functions

$$ax + by + cz \text{ and } a'x + b'y + c'z,$$

may *in all cases* be made each of them simultaneously to vanish, or else to become indefinitely small for integer values of x, y, z. Thus then, we have an immediate and intuitive proof of Jacobi's celebrated proposition for proving the impossibility of the existence of trebly periodic functions*.

Those gifted with the powers of a Stringham, a Newcomb, or a Charles S. Peirce to feel their way about in supersensible space, may, in like manner, obtain if not an *intuitive*, at least an immediate, or non-mediated proof of the theorem that: *Any number of* homogeneous *functions of one more than that number of integer variables may be made either* to vanish simultaneously or else *to become simultaneously less than any assignable quantity.*

* See M. Hermite's admirable *Note sur le calcul différentiel et le calcul intégral*, Paris, 1862, pp. 5—8, where the proposition in question is established by means of the theory of ternary and binary quadratic forms.

Whilst in the course of writing out the above matter the following note, addressed to him, from Dr F. Franklin, was received by Professor Sylvester:

"Your proof may be put into the following form:

Theorem.—In any stripe bounded by irrational parallels there must be a node.

For if not, let N and N' be *any* two nodes. Repeat the stripe a finite number of times, namely, until the aggregate of the stripes shall have included N and N'. No stripe can contain two nodes ν, ν', for if it did, by producing $\nu\nu'$ we see that each of the stripes must contain at least one node, which is contrary to the hypothesis. Hence we have an open parallelogram containing two nodes N, N', and only a finite number of others, which is absurd; for since the parallelogram intercepts a distance greater than NN', it must intercept on *every* nodal line parallel to NN' at least one node. Thus the theorem is proved.

It may be noted that while a stripe of finite width bounded by *rational* lines contains either no nodes or a *singly* infinite number of them, a stripe bounded by *irrational* lines always contains a *doubly* infinite number of nodes; which, although easily explicable, might at first sight strike one as paradoxical, inasmuch as the probable number in a *given finite portion* is the same for one sort of stripe as for the other."

One word in conclusion. The modes above given of presenting the theory with reference to planes passing through a singly or doubly irrational line ought not to be allowed to draw away attention from the image afforded by a doubly irrational line surrounded by an asymptotic spiral sheath (a point-helix winding round a fish-bellied-torpedo-like bobbin or core) tapering off to an indefinitely fine point in both directions, nor from the extension of the theory of continued fractions to which that image points.

Taking for greater simplicity the case of such a line passing through a node at the origin, the question invites solution to *devise an Algorithm* for finding the integer values of x, y, z which shall give the successive minima (corresponding to nodes of nearest approach) of the function

$$(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2,$$

that being the problem next in the order of natural succession to the solved one of finding the successive absolute minima of $ay - bx$.

In the latter case, a and b are supposed to be incommensurable—in the former, no linear equation with rational coefficients is supposed to exist between a, b, c*.

* Which is tantamount to saying that the line $x : y : z :: a : b : c$ must be doubly irrational.

72.

ON THE GEOMETRICAL FORMS CALLED TREES.

[*Johns Hopkins University Circulars*, I. (1882), pp. 202, 203.]

[In connexion with the reference to his name in the above] Professor Sylvester stated that to M. Camille Jordan was due the credit of being the first to discover the existence of the centre or centre-pair of each kind described in the above note. In entire ignorance of M. Jordan's work he rediscovered for himself the centre or centre-pair of the first kind, and was the first to make use of the method immediately flowing therefrom to solve the problem of finding the forms and the number of tree-graphs* corresponding to a hydro-carbon or hypothetical hydro-boron series with a given number of carbon atoms. His results, which he communicated from time to time to Professor Tait, of Edinburgh, were however as regards the ascertainment of the number of such graphs, purely arithmetical, but giving all the different forms of the so-called trees or (more properly speaking) ramifications for different values of the number of atoms up to a certain arithmetical limit. The problem was subsequently taken up from this point by Professor Cayley, who obtained general generating-function formulæ for effecting the denumeration of the graphs. Mr Sylvester then proceeded to explain his method of arriving at the first kind of centre or centre-pair of any given tree or ramification.

To this end he supposes all the terminal branches of the tree removed. A tree with a less number of nodes is thus brought into evidence which is subjected (if possible) to like treatment and so a third tree with still fewer nodes is arrived at. As this process cannot be indefinitely continued (for if so a finite number could be continually diminished) we must at length come

* In accordance with the nomenclature employed above, the writer uses here occasionally the word *tree*, but considers his original word *ramification* more correct. A tree is a ramification with one point fixed as a root or origin, and no such fixed origin is supposed to exist in the graphs in question.

to a tree or ramification whose terminal branches cannot be removed without leaving nothing in the form of a tree remaining. So long as not less than three nodes remain, since they must not form a triangle, for that would be inconsistent with their appertaining to a ramification, the process of lopping off terminals cannot be brought to a close. Eventually, therefore, this process must lead to a system of branches all radiating out from a single point, or which being removed, only an isolated point remains, or else to a sort of double-headed mop or broom consisting of two such radiating systems stuck into the two ends of an axis. This is the case of bicentric or axial, the former of a monocentric ramification. Thus every ramification may be said to belong either to a central or an axial class. He concluded with suggesting that some general chemical or physical property or set of such properties might reasonably be supposed to exist serving to distinguish between these two classes or genera in the case of the well developed series of the hydrocarbons.

73.

ON THE 8-SQUARE IMAGINARIES.

[*Johns Hopkins University Circulars*, I. (1882), p. 203.]

[WITH reference to the above communication] Professor Sylvester referred to the general question of representing the product of sums of two, four or eight squares under the form of a like sum, and mentioned that Professor Cayley had been the first to demonstrate, by an exhaustive investigation, the impossibility of extending the law applicable to 2, 4 and 8 to the case of 16 squares. The new kind of so-called imaginaries referred to by Professor Cayley are, as far as Mr Sylvester is aware, the first example of the introduction into Analysis of locative symbols not subject to the strict law of association, and he considers the law regulating the connexion of the two products represented by a succession of three such symbols, most interesting, inasmuch as such products are either identical, or if not identical, of the same absolute value, but with contrary signs: most persons, before this example had been brought forward, would have felt inclined to doubt the possibility of locative symbols (*vulgo* imaginary quantities*) whose multiplication table should give results inconsistent with the common associative

* Using θ, h, t, u to denote thousands, hundreds, tens, units, the year of grace in which we live may be represented by $\theta+8h+8t+2u$, θ, h, t, u, being locative symbols which it would be absurd to style *imaginary quantities*; but they are as much entitled to that name as the i, j, k, or any like set of symbols—the only essential difference being that the one set of symbols is limited, the other unlimited in number—and accordingly the law of combination of the one set is given by a finite and of the other set by an infinite *multiplication table*. We might mark off the specific difference between the two cases, by defining the latter set as *unlimited*, the former as *recurrent* or *periodic* locatives or locators; the *locatives* indicate out of what *basket*, so to say, the *quantities* appearing in an analytical expression are to be selected—the multiplication table determines the basket into which their product is to be thrown. Under a purely analytical point of view this is all that is wanted—but in the application of quaternions to problems in nature, it becomes necessary to give special significance to the baskets or rubrics (which would do as well) to which the quantities belong and understand them to signify that certain geometrical processes of *setting* are to be performed.

The true analytical theory of quaternions has nothing to do with this setting part of the

law, being capable of forming the groundwork of any real accession to algebraical science—the results of Professor Cayley referred to above, seem to show that such doubts are open to question. Mr Sylvester mentioned as bearing upon the subject of so-called imaginary quantities, that in his recent researches in Multiple Algebra he had come upon a system of Nonions, the exact analogues of the Hamiltonian Quaternions and like them capable of being represented by square matrices. Mr Charles S. Peirce, it should be stated, had to the certain knowledge of Mr Sylvester arrived at the same result many years ago in connexion with his theory of the *logic of relatives;* but whether this result had been published by Mr Peirce, he was unable to say*.

business, and regards quaternions as matrices of the second order of a certain determinate form, and accordingly the whole analytical side of the theory of quaternions merges into a particular case of the general theory of *Multiple Algebra*.

As far as the present writer is aware, Professor Cayley in his memoir on Matrices, (*Phil. Trans.* 1858), was the first to recognize the parallelism between quaternions and matrices, but the idea and method of effecting their complete identification is due to the late Prof. Benjamin Peirce or to his son Mr C. S. Peirce.

* Mr C. S. Peirce gave a form of this Algebra in a paper "On a Notation for the Logic of Relatives," published in 1870. The class of Associative Algebras to which this belongs were termed *quadrates* by the late Professor Clifford.

74.

ON A GEOMETRICAL TREATMENT OF A THEOREM IN NUMBERS.

[*Johns Hopkins University Circulars*, I. (1882), p. 209.]

THE author made some remarks additional to those made on the same subject at the preceding meeting of the seminarium. In a plane reticulation four cases present themselves, namely, a line may be drawn through a line of nodes, or through a solitary node, or parallel to a line of nodes, or so as neither to pass through any node nor to be parallel to a line of nodes. In the third case the distance of the nodes of nearest approach is constant: in the second and fourth cases it approximates continually to zero. So in a solid reticulation eight cases present themselves, namely, four in addition to those last detailed: for without lying in a nodal plane, the line of flight may (α) pass through a single node, or (β) it may be parallel to a line of nodes, or (γ) it may be parallel to a nodal plane but not to a nodal line, or (δ) it may not pass through any node. In case (β) the distance of the nodes of nearest approach is constant; in case (γ) it approximates to a constant finite limit: in cases (α) and (δ) it approximates to zero.

There are thus four cases in all for which the distance from the nodes of nearest approach is a continually decreasing infinitesimal, namely: two for which the line of flight does not pass through any node, and two for which it does pass through a node—these latter two being those which serve to establish the theorem relating to the non-existence of trebly periodic functions.

The author further drew attention to the singular metamorphosis undergone by the geometrical setting forth of this theorem. It may be put under the form of asserting that a trilateral whose three sides are conditioned to be exact multiples of, and parallel to, three given straight lines lying in a plane may either be made to form a closed triangle or else such that the line closing the trilateral shall be less than any assigned quantity. On the other hand, the very same fact lends itself to, and is absolutely equivalent in substance to the statement that an arrow let fly from a node of a solid reticulation whether it speed along a nodal plane or be shot miscellaneously at the stars must (the law of gravity being supposed to be suspended) pass *indefinitely near* an infinite number of nodes in the course of its flight. The corresponding theorem for space of five dimensions serves to show that Quaternion Functions cannot have a higher than a quadruple periodicity.

75.

ON THE PROPERTIES OF A SPLIT MATRIX.

[*Johns Hopkins University Circulars*, I. (1882), pp. 210, 211.]

SUPPOSE a square matrix split into two sets of lines which need not be contiguous and may be called ranges, say ABC, $DEFG$. Let the sum of the products of the corresponding elements of any two lines be called their product. It is well known (see Salmon's *Higher Alg.*, 3rd Ed., p. 82) that if the product of each line in the first range by every line in the other is zero, the opposite complete minors of the two ranges will be in a constant ratio, say in the ratio $l:\lambda$. Call the content of the matrix Δ: then it follows, if S, Σ denote the sums of the squares of the complete minors in the two ranges respectively, that

$$\frac{\lambda}{l}S = \frac{l}{\lambda}\Sigma = \Delta.$$

But by a theorem of Cauchy concerning rectangular matrices S is equal to the determinant $(A, B, C)^2$, that is, to the determinant

$$\begin{vmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{vmatrix}$$

and similarly $$\Sigma = (D, E, F, G)^2$$

so that $$\lambda^2 : l^2 :: (D, E, F, G)^2 : (A, B, C)^2$$

and $$S\Sigma = \Delta^2.$$

Suppose now that the product of *every* two lines in the entire matrix is zero. Then into whatever two ranges the matrix be divided the ratio $\lambda^2 : l^2$ (since all but the diagonal terms in the matrices which express the ratio $l^2 : \lambda^2$ vanish) will be expressed by the ratio of one simple product to another: thus for example for the ranges $ABC : DEFG$

$$\lambda^2 : l^2 :: D^2 . E^2 . F^2 . G^2 : A^2 . B^2 . C^2; \text{ also } \Delta^2 = A^2 . B^2 . C^2 . D^2 . E^2 . F^2 . G^2.$$

If we now further suppose that the sum of the squares of the elements in each line is unity, that is, that

$$A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = G^2 = 1,$$

it will follow that every minor whatever divided by its opposite will be equal to Δ $\left(\text{for on the hypothesis made, } \frac{\lambda}{l} = \frac{\Delta}{S} = \Delta\right)$.

Also Δ will be plus or minus unity since $\Delta^2 = 1$. Thus it is seen that we may pass by a natural transition from the theory of a split to that of an orthogonal or self-reciprocal matrix—to show which was the principal motive to the present communication. It is by aid of the theorem of the *split matrix* that I prove a remarkable theorem in Multiple Algebra, namely, that if the product of two matrices of the same order is a complete null, the sum of the nullities of the two factors must be at least equal to the order of the matrix—the nullity of a matrix of the order ω being regarded as unity, when its determinant simply is zero, as 2 when each first minor simply is zero, as 3 when each second minor is zero ... as $(\omega - 1)$ when each quadratic minor is zero and as ω (or absolute) when every element is zero. This theorem again is included in the more general and precise one following—*If any number of matrices of the same order be multiplied together, the nullity of their product is not less than the nullity of any single factor and not greater than the sum of the nullities of all the several factors.*

In Professor Cayley's memoir on Matrices (*Phil. Trans.*, 1858) the very important proposition is stated that if

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{matrix}$$

be any matrix of substitution, say m (here taken by way of illustration of the order 4) the determinant

$$\begin{vmatrix} a-m & b & c & d \\ a' & b'-m & c' & d' \\ a'' & b'' & c''-m & d'' \\ a''' & b''' & c''' & d'''-m \end{vmatrix}$$

is identically zero; or in other words, its nullity is complete. By means of the above theorem it may be shown that the nullity of any i distinct algebraical factors of such matrix is equal to i, i having any value from unity up to the number which expresses the order of the matrix, inclusive.

76.

A WORD ON NONIONS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 241, 242; II. (1883), p. 46.]

In my lectures on Multiple Algebra I showed that if u, v are two matrices of the second order, and if the determinant of the matrix $(z + yv + xu)$ be written as

$$z^2 + 2bxz + 2cyz + dx^2 + 2exy + fy^2$$

then the necessary and sufficient conditions for the equation $vu + uv = 0$ are the following, namely,

$$b = 0, \quad c = 0, \quad e = 0.$$

If to these conditions we superadd $d = 1, f = 1$, and write $uv = w$, then

$$u^2 = -1, \quad v^2 = -1, \quad w^2 = -1, \quad uv = -vu = w, \quad vw = -wv = u, \quad wu = -uw = v;$$

and 1, u, v, w form a quaternion system. The conditions above stated will be satisfied if

$$\text{Det.}\ (z + yv + xu) = z^2 + y^2 + x^2,$$

which will obviously be the case if

$$v = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad u = \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix},$$

where $\theta = \surd(-1)$. For then

$$z + yv + xu = \begin{vmatrix} z & y + x\theta \\ -y + x\theta & z \end{vmatrix}.$$

Hence the matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix} \begin{vmatrix} -\theta & 0 \\ 0 & \theta \end{vmatrix}$$

construed as complex quantities are a linear transformation of the ordinary

quaternion system 1, i, j, k; that is to say, if we form the multiplication table

	λ	μ	ν	τ
λ	λ	μ	0	0
μ	0	0	λ	μ
ν	ν	τ	0	0
τ	0	0	ν	τ

$$\lambda + \tau = 1 \qquad -\mu + \nu = i$$

$$-\theta\lambda + \theta\tau = k \qquad \theta\mu + \theta\nu = j.$$

Since u, v contain between them 8 letters subject to the satisfaction of 5 conditions, the most general values of λ, μ, ν, τ ought to contain 3 arbitrary constants; but it is well-known that any particular (i, j, k) system may be superseded by a $\lambda\,(i', j', k')$ system, where i', j', k' are orthogonally related linear functions of i, j, k; and as this substitution introduces just 3 arbitrary constants, we may, by aid of it, pass from the system of matrices above given, to the most general form. The general expression for the matrices containing 3 arbitrary constants may also be found directly by the method given in my lectures, which will be reproduced in the memoir on Multiple Algebra in the *Mathematical Journal.* What goes before is by way of introduction to the *word* on Nonions which follows.

Just as the necessary and sufficient condition that u, v, two matrices of the second order, may satisfy the equations $vu = -uv$, $u^2 = 1$, $v^2 = 1$, is that the determinant to $z + yv + xu$ may be $z^2 + y^2 + x^2$, so I have proved that the necessary and sufficient condition, in order that we may have $vu = \rho uv$, $u^3 = 1$, $v^3 = 1$ (u, v being matrices of the third order, and ρ an imaginary cube root of unity) is that the determinant to $z + yu + xv$ may be $z^3 + y^3 + x^3$; but if we make

$$u = \begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix}, \qquad v = \begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix},$$

then
$$z + yu + xv = \begin{vmatrix} z & 0 & y + x \\ \rho y + \rho^2 x & z & 0 \\ 0 & \rho^2 y + \rho x & z \end{vmatrix}$$

of which the determinant is

$$z^3 + (y + x)(\rho y + \rho^2 x)(\rho^2 y + \rho x) = z^3 + y^3 + x^3.$$

Hence there will be a system of Nonions (precisely analogous to the known

system of quaternions) represented by the 9 matrices

$$\begin{array}{ccccc} & & 1 & & \\ & u & & v & \\ u^2 & & uv & & v^2 \\ & u^2v & & uv^2 & \\ & & u^2v^2 & & \end{array}$$

and just as in the preceding case the 8 terms ± 1, $\pm u$, $\pm v$, $\pm uv$ form a closed group, so here the 27 terms obtained by multiplying each of the above 9 by 1, ρ, ρ^2 will form a closed group. The values of the 9 matrices will easily be found to be

$$\begin{array}{ccccc} & & \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} & & \\ & \begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} & \\ \begin{vmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho \\ 1 & 0 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho \\ \rho & 0 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^2 \\ 1 & 0 & 0 \end{vmatrix} \\ & \begin{vmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} & & \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{vmatrix} & \\ & & \begin{vmatrix} 0 & 0 & \rho \\ \rho & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} & & \end{array}$$

These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870).

I will only stay to observe that as the condition of the Determinant to $z + uy + vx$ (which for general values of u, v is a general cubic with the coefficient of z^3 unity) assuming the form $z^3 + y^3 + x^3$, implies the satisfaction of 9 conditions, and as u, v between them contain 18 constants, the most general form of a system of Nonions must contain $18 - 9$, or 9 arbitrary constants; but how these can be obtained from the particular form of the system above given, remains open for further examination.

[*Note.* For the remark made above] "These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870)," read "Mr C. S. Peirce informs me that these forms can be derived from his *Logic of Relatives*, 1870." I know nothing whatever of the fact of my own personal knowledge*. I have not read the paper referred to, and am not

* I have also a great repugnance to being made to speak of Algebras in the plural; I would as lief acknowledge a plurality of Gods as of Algebras.

acquainted with its contents. The mistake originated in my having left instructions for Mr Peirce to be invited to supply in my final copy for the press, such reference as he might think called for. He will be doing a service to Algebra by showing in these columns how he derives my forms from his logic*. The application of Algebra to Logic is now an old tale—the application of Logic to Algebra marks a far more advanced stadium in the evolution of the human intellect; the same may be said as regards the application by Descartes of Analysis to Geometry, and the reverse application by Eisenstein, Dirichlet, Cauchy, Riemann, and others, of Geometry to Analysis—so that if Mr Peirce accomplishes the task proposed to him (his ability to do which I do not call into question), he will have raised himself as far above the level of the ordinary Algebraic logicians as Riemann's mathematical stand-point tops that of Descartes.

It is but justice to Boole's memory to recall the fact that, in one of his papers in the *Philosophical Transactions,* he has made a reverse use of logic to establish a certain theorem concerning inequalities, which is very far from obvious, and which I think he states it took him ten years to deduce from purely algebraical considerations, having previously seen it through logical spectacles—I mean, by the aids to vision afforded him by his logical calculus: this theorem I believe (or at least did so when it was present to my mind) must of necessity admit of a much more comprehensive form of statement.

* I had understood Mr Peirce to say that these forms were actually contained in his memoir.

77.

ON MECHANICAL INVOLUTION.

[*Johns Hopkins University Circulars*, I. (1882), pp. 242, 243.]

MANY years ago I gave in the *Comptes Rendus* of the Institute of France, one or more geometrical constructions of the problem of Mechanical Involution.

When forces can be introduced along six given lines in space whose statical sum is zero, a certain geometrical condition must be fulfilled by the 6 lines which are then said to be in involution. If two homographic pencils of rays in different planes have two corresponding rays coincident (but their centres apart), any six lines, each of which cuts two corresponding rays, will form an involution system. In the communication to the Society I showed that the analytical condition of involution might be expressed by means of equating to zero a certain compound determinant. I have found since that this determinant is given by Cayley in the *Cam. Phil. Soc. Tr.* 1861, part 2.

Let 1, 2, 3, 4, 5, 6 be the six lines and on each of them let two arbitrary points be taken; let the quadri-planar coordinates of the two arbitrary points on any of the lines, say j, be called j_x, j_y, j_z, j_t; j_x', j_y', j_z', j_t', respectively, the condition of involution referred to will be

$$\begin{vmatrix} & 1.2 & 1.3 & 1.4 & 1.5 & 1.6 \\ 2.1 & & 2.3 & 2.4 & 2.5 & 2.6 \\ 3.1 & 3.2 & & 3.4 & 3.5 & 3.6 \\ 4.1 & 4.2 & 4.3 & & 4.5 & 4.6 \\ 5.1 & 5.2 & 5.3 & 5.4 & & 5.6 \\ 6.1 & 6.2 & 6.3 & 6.4 & 6.5 & \end{vmatrix} = 0$$

where any binary combination $ij = ji$, and where either of them represents the determinant

$$\begin{vmatrix} i_x, & i_y, & i_z, & i_t \\ i_x', & i_y', & i_z', & i_t' \\ j_x, & j_y, & j_z, & j_t \\ j_x', & j_y', & j_z', & j_t' \end{vmatrix}$$

Six lines in involution represent indifferently lines along which forces or axes of couples can be introduced, whose statical sum is zero. Consequently such a system is the analogue in space at one and the same time to three force-lines converging to a point, or to three points in a line regarded as centres of moments, in a plane. But *in plano* the concurrence of right lines is the polar property to the collineation of points. Hence we ought to expect that the polar reciprocal in respect to any quadric of an involution system, should also be an involution system; and such is obviously the case by virtue of the fact that the correspondence of the rays in the two homographic pencils, referred to above, will not be affected when for each ray in either pencil is substituted its polar in respect to any quadric. (A direct proof will be found in the *American Mathematical Journal*, Vol. IV., part 4*.) I concluded with pointing out the analogy between the problem of Mechanical Involution and what I call Algebraical Involution, which takes place when x, y being each of them matrices of the order ω, a linear equation connects the ω^2 ground-forms represented by the distinct terms of the product

$$(1, x, x^2, \dots x^{\omega-1}\between 1, y, y^2, \dots y^{\omega-1}).$$

Mechanical involution in a plane, in 3-dimensional, in 4-dimensional space, etc., is the analogue of algebraical involution between two matrices of the order 2, 3, 4, etc.; the $\frac{1}{2}(\omega^2+\omega)$ directions in ω-dimensional space being the analogues of the ω^2 ground-forms of matrices of the order ω. Each of the two problems consists of two parts: to obtain the condition of involution being the one part, to assign the relative magnitudes, in the one case, of the forces which cause their statical sum to vanish, and in the other case of the coefficients which enter into the linear function, the other part of the problem. The form of the solution of this second part of the algebraical problem (subject only to a certain ambiguity) has been given in my lectures, and will appear in the Memoir on Multiple Algebra in the *American Journal of Mathematics*; but the former part of the algebraical problem, that is, the determination of the *condition* of Algebraical Involution, except for the case of matrices of the second order, I have not yet succeeded in solving.

[* Cf. p. 560, above.]

78.

ON CROCCHI'S THEOREM.

[*Johns Hopkins University Circulars*, II. (1883), p. 2.]

IN *Battaglini's Journal* for July, 1880, Signor Crocchi has given a theorem which may be stated in the following terms. If s_i, σ_i, h_i denote respectively the sum of the elementary combinations, of the powers, and of the homogeneous products each of the ith order of any number of elements, then h_i is the same function of σ_1, $\overline{\sigma}_2$, σ_3, $\overline{\sigma}_4$, ... that s_i is of σ_1, σ_2, σ_3, σ_4, ...

Signor Crocchi's proof is very elegant but a little circuitous. An instantaneous proof may be derived from the relation of reciprocity which connects s and h, namely, that if

$$h_i = f(s_1, s_2, \ldots s_i) \text{ then } s_i = f(h_1, h_2, \ldots h_i),$$

which is an immediate deduction from the well-known fact that

$$(1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots)(1 - h_1 y + h_2 y^2 - h_3 y^3 + \ldots) = 1.$$

For from this relation spring the equations

$$s_1 - h_1 = 0, \quad s_2 - h_1 s_1 + h_2 = 0, \quad s_3 - h_1 s_2 + h_2 s_1 - h_3 = 0 \ldots$$

which equations continue unaltered when the letters s and h are interchanged; for when such interchange takes place, the functions equated to zero of an even rank remain unaltered and those of an odd rank merely change their sign.

Returning to the immediate object in view, if a, b, c, ... are the elements subject to the s, h, σ symbols, we may write

$$\Sigma \log(1 + ay) = \log(1 + s_1 y + s_2 y^2 + s_3 y^3 + s_4 y^4 + \ldots)$$

or,
$$\Sigma \log(1 - ay) = -\log(1 + h_1 y + h_2 y^2 + h_3 y^3 + h_4 y^4 + \ldots).$$

The first equation by differentiation performed in each side gives

$$\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \sigma_4 y^3 + \ldots = \frac{s_1 + 2s_2 y + 3s_3 y^2 + 4s_4 y^3 + \ldots}{1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots},$$

and similarly the second equation gives

$$\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \sigma_4 y^3 + \ldots = \frac{h_1 + 2h_2 y + 3h_3 y^2 + 4h_4 y^3 + \ldots}{1 + h_1 y + h_2 y^2 + h_3 y^3 + \ldots},$$

that is, $(\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \ldots)(1 + s_1 y + s_2 y^2 + \ldots) = s_1 + 2s_2 y + 3s_3 y^2 + \ldots$

and $\qquad (\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \ldots)(1 + h_1 y + h_2 y^2 + \ldots) = h_1 + 2h_2 y + 3h_3 y^2 + \ldots.$

By comparison of coefficients of the powers of y, the first of these two equations affords the means of finding any σ in terms of the s quantities, and the second of these any σ in terms of the h quantities. But if we change s into h and $\sigma_2, \sigma_4, \ldots$ into $-\sigma_2, -\sigma_4, \ldots$ the first equation becomes the second. Hence if

$$s = f(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots)$$
$$h = f(\sigma_1, \bar{\sigma}_2, \sigma_3, \bar{\sigma}_4, \ldots). \quad \text{Q.E.D.}$$

It is not without interest to set out the reciprocity of the 6 relations which exist between s, σ, h. The synoptical scheme of such reciprocity may be exhibited symbolically as follows:

$$h/s = s/h, \quad h/\sigma = s/\pm\sigma, \quad \sigma/h = \pm\sigma/s.$$

As an illustration of the second of these symbolic equalities take

$$s_3 = \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \qquad s_4 = \frac{\sigma_1^4 - 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 - 6\sigma_4}{24};$$

the corresponding equations are

$$h_3 = \frac{\sigma_1^3 + 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \qquad h_4 = \frac{\sigma_1^4 + 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 + 6\sigma_4}{24},$$

and it is worthy of observation that the sum of the numerical coefficients is always (as in the above examples) zero for the function of the σ quantities which gives an s of any order, and unity for the function of the same which expresses any h*.

* This statement is proved instantaneously by taking one of the elements equal to unity and all the rest zero; and the latter part of it gives a new proof of Cauchy's theorem which he obtains by a consideration of all the possible cyclic representations of the substitutions of n elements. The theorem is that if n elements be divided in every possible way into λ set of l, μ set of m, ν set of n ... elements, then

$$\Sigma \frac{1}{\pi\lambda\pi\mu\pi\nu \ldots l^\lambda m^\mu n^\nu \ldots} = 1.$$

For we know by a theorem of Waring that

$$s_n = \Sigma \pm \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots} \sigma_l^\lambda \sigma_m^\mu \ldots.$$

Hence by Crocchi's theorem the sum of the coefficients in h_n expressed in σ's is equal to

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m \ldots}$$

but it is also equal to unity. Cauchy's theorem is therefore proved.

Frequent occasion presents itself (especially in the theory of numbers) for expressing any s in terms of σ's, but probably up to the time when Signor Crocchi wrote on the subject there had never been any occasion to express h in terms of the σ's: for had such occasion ever arisen it seems almost impossible that the relation between the two corresponding sets of formulæ could have escaped observation.

In some recent researches, however, of the writer of this note on the irreducible semi-invariants of a quantic of an unlimited order, it becomes indispensable to convert homogeneous products into sums of powers, and Crocchi's theorem comes into play. (See sec. 4 of Article on Subinvariants, *Am. Math. Journ.*, Vol. v., part 2 [p. 597, above].)

The relation $\sigma/h = \pm\, \sigma/s$ is interesting under the point of view that virtually it contains an example of a sort of *invariance* of form which may possibly contain within itself the germ of an important theory. It informs us that if, in the function of h's which expresses any σ, in lieu of each h the function of s quantities to which it is equal be substituted, the form of the σ function will remain unchanged, except that when the order of the σ is an even number, its algebraical sign is reversed. Thus, for example,

$$\sigma_3 = h_1^3 - 3h_1h_2 + 3h_3, \quad h_1 = s_1, \quad h_2 = s_1^2 - s_2, \quad h_3 = s_1^3 - 2s_1s_2 + s_3.$$

Consequently if we write $\phi = x^3 - 3xy + 3z$, and for y and z substitute $x^2 - y$, $x^3 - 2xy + z$, respectively, the value of ϕ remains unaltered. So in like manner if we write

$$\phi = x^4 - 4x^2y + 4xz + 2y^2 - 4t,$$

and substitute for y, z, t;

$$x^2 - y, \quad x^3 - 2xy + z, \quad x^4 - 3x^2y + 2xz + y^2 - t,$$

respectively, no change ensues in ϕ except that it undergoes a change of sign.

So in general the σ functions with even and those with odd subindices may be regarded as the analogues of symmetrical and skew-invariants, respectively.

Again in the formula for s the sign *plus* or *minus* depends on the oddness or evenness of $\lambda + \mu + \ldots$. Hence if in

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots}$$

only those values of $\lambda, \mu, \ldots$ are admitted which make $\lambda + \mu + \ldots$ always odd or always even, either sum so formed will be equal to $\frac{1}{2}$, because the difference of the two sums is zero and their sum unity.

This theorem can, of course, be deduced like the former one from the method of cycles applied now, not to the entire number of the substitutions, but to that half of them which correspond to the *alternate group* of each, of which the number of representative cycles (monomial ones included) is always odd or else always even, according as the number of elements is one or the other.

79.

ON CERTAIN SUCCESSIONS OF INTEGERS THAT CANNOT BE INDEFINITELY CONTINUED.

[*Johns Hopkins University Circulars*, II. (1883), pp. 2, 3.]

A SUCCESSION of decreasing integers we know cannot be indefinitely continued, but there are also successions of increasing integers subject to certain stated conditions, but otherwise arbitrary, which are similarly incapable of indefinite extension.

The following is a simple instance of the kind. Suppose integers to be written down one after the other, no one of which is a multiple of any other, nor the sum of a multiple of any other and of a multiple of a specified one. *Such a succession cannot be indefinitely continued.*

Let a be the specified integer.

(1) Suppose that all the other integers of the succession are prime to a.

Then if b be any other of the integers, the equation $ax + by = c$ is soluble in integers if c is greater than ab, as follows at once from the consideration that the numbers $c - b$, $c - 2b$, $c - 3b$ ... $c - ab$ must be all distinct residues to the modulus a, inasmuch as the difference of any two of them being of the form $(i - j)b$ where $i - j$ is less than and b prime to a, cannot be divisible by a.

But if the succession could be indefinitely produced, it must contain a number greater than ab. Hence the theorem is proved for the case where a is prime to every other integer in the succession.

(2) Suppose the theorem to be true for the case where the quotient of a divided by i prime numbers (not necessarily all distinct) is prime to all the other terms of the series: it must be true when the number of such prime numbers is $i + 1$. For let p be one of them and $a = pa'$, consider all the terms of the succession divisible by p apart from the rest.

Let pa', pb', pc' ... be those terms. By the law of the succession the equation $pa'x + pb'y = pc'$ cannot be satisfied for any values of b', c', and consequently $a'x + b'y = c'$ cannot be satisfied.

Hence by hypothesis since a' divided by i factors is prime to b', c' ... the succession of terms divisible by p must be finite in number, and this will be true for every factor p. Hence the succession b, c, ... will contain only a finite number of terms having any factor in common with a. Moreover the succession containing a and terms prime to a exclusively, must also be finite by the preceding case. Consequently the whole succession will be finite, and the theorem if assumed to be true for $i = 0$, or any positive integer, is true for $i + 1$.

But by the preceding case the proposition is true when $i = 0$. Hence it is true universally.

In the long footnote to the Article on Subinvariants in Vol. V., pp. 92, 93 of the *Am. Journal of Math.*, will be found given the mode of extending this theorem to the case of successions of complex integers or multiplets, when a proper restriction is laid upon the ratios to one another of the simple numbers which constitute the multiplets, and a possible connexion pointed out between the finiteness of such successions and that of the system of ground-forms to a binary quantic [p. 580, above].

80.

ON THE FUNDAMENTAL THEOREM IN THE NEW METHOD OF PARTITIONS.

[*Johns Hopkins University Circulars,* II. (1883), p. 22.]

THE new method of partitions which I gave to the world more than a quarter of a century ago is an application of a theorem which, I think it must be conceded, is, after Newton's Binomial Theorem, the most important organic theorem which exists in the whole range of the Old Algebra. What Newton's theorem effects for the development of *radical*—that theorem accomplishes for the development of *fractional* forms of algebraical functions.

One (but not the most perfect) form in which it can be presented is the following. If Fx be any proper algebraical fraction in x, whose infinity roots (that is, the values of x which make Fx infinite) are $a, b, \ldots l$, quantities all supposed to differ from zero, then the coefficient of x^n for any value of n will be the residue, that is, the coefficient of $\frac{1}{x}$ in

$$\Sigma\,(\lambda^{-n}e^{nx})\,F(\lambda e^{-x}) \quad [\lambda = a, b, \ldots, l].$$

By supposing Fx broken up into proper simple fractions of the form $\Sigma\,\frac{fx}{(a-x)^i}$ it is very easy to see that the theorem will be true in general if true for $\frac{fx}{(a-x)^i}$, and from this it is but a step to see that the theorem will be true in general if true for the simplest form of rational function, that is, $\frac{1}{(1-x)^i}$.

All then that remains to do is to show that the coefficient of x^n in this fraction is the same as the coefficient of $\frac{1}{x}$ in $\frac{e^{nx}}{(1-e^{-x})^i}$ which may be done as follows:

$$\frac{1}{(1-e^{-x})^i} = \left(1-\frac{\delta_x}{1}\right)\left(1-\frac{\delta_x}{2}\right)\left(1-\frac{\delta_x}{3}\right)\dots\left(1-\frac{\delta_x}{(i-1)}\right)\left(\frac{1}{1-e^{-x}}\right)$$

$$= (1 - A\delta_x + B\delta_x^2 - C\delta_x^3 \dots)\left(\frac{1}{x} + \dots\right)$$

$$= \left(\frac{1}{x} + \frac{A}{x^2} + \frac{1.2B}{x^3} + \frac{1.2.3C}{x^4} + \dots\right) + \text{positive powers of } x.$$

Therefore the coefficient of $\frac{1}{x}$ in

$$\frac{1 + nx + \frac{n^2}{1.2}x^2 + \frac{n^3}{1.2.3}x^3 + \dots}{(1-e^{-x})^i}$$

$$= (1 + An + Bn^2 + Cn^3 + \dots)$$

$$= \left(1+\frac{n}{1}\right)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\dots\left(1+\frac{n}{i-1}\right)$$

$$= \frac{(n+1)(n+2)\dots(n+i-1)}{1, 2, \dots (i-1)} = \text{coefficient of } x^n \text{ in } \frac{1}{(1-x)^i}. \quad \text{Q.E.D.}$$

This method of proof, however, is not the simplest or best; as soon as we mould the theorem into a form most easily admitting of being expressed in general terms that very form itself suggests a simpler (nay, so to say, an instantaneous) proof, and moreover relaxes an unnecessarily stringent condition in the previous statement of the theorem.

Of course by a finite infinity root of a function no one can fail to understand a value of the variable differing from zero which makes the function infinite. This then is the true statement of the theorem in general terms.

In any proper-fractional function developed in ascending powers of a variable, the constant term is equal to the Residue (with its sign changed) of a sum of functions obtained by substituting in the given function in place of the variable the product of each, in succession, of its finite infinity roots into the exponential of the variable.

That is to say, if we take the proper-fraction

$$Fx = \frac{\phi x}{x^i (x-a)^j (x-b)^k \dots (x-l)^\omega},$$

the constant term (with its sign changed) in this fraction developed in ascending powers of x is the same as the Residue of $\Sigma F(\lambda e^x)\,[\lambda = a, b, \dots l]$.

To prove this it is only necessary to suppose the fraction Fx separated into simple partial fractions with constant numerators and the theorem becomes self-evident*.

* It must, however, previously be shown that the residue of $\frac{1}{(1-e^x)^i}$, where i is a positive

It follows, therefore, writing n in place of i that the coefficient of x^n in ascending-power series for the fraction

$$Gx = \left(\frac{\phi x}{(x-a)^j \dots (x-l)^\omega}\right)$$

will be the Residue with its sign changed of $\Sigma\,(a^{-n}e^{-nx})\,G\,(ae^x)$, or which is the same thing is the Residue of $\Sigma a^{-n}e^{nx}\,G\,(ae^{-x})$, which theorem we now see is true not merely for the case where G is a proper-fraction, that is, is a function of x whose degree is a negative integer, but remains true when the degree of G is any number inferior to n, for when that condition is satisfied $\frac{G}{x^n}$ is a proper fraction, which is all that is required in order for the parent theorem to apply.

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integer, is the same as that of $\frac{1}{1-e^x}$, that is, is -1; this becomes obvious from the consideration that the change of i into $i+1$ alters the quantity to be residuated by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{(1-e^x)^i}$ divided by i, of which the residue is necessarily zero—that being true for the differential derivative of any series of powers of a variable.

81.

NOTE ON THE PAPER OF MR DURFEE'S.

[*Johns Hopkins University Circulars*, II. (1883), pp. 23, 24; 42, 43.]

MR DURFEE'S very elegant and interesting theorem above given may, by help of Euler's law of reciprocity, be expressed in the following terms.

Let fx and ϕx represent respectively:

$$\frac{1}{1-x}+\frac{x^4}{(1-x)(1-x^2)(1-x^3)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}+\dots+\frac{x^{2i^2+2i}}{(1-x)\dots(1-x^{2i+1})}+\dots$$

and $$\frac{x^2}{(1-x)(1-x^2)}+\frac{x^8}{(1-x)(1-x^2)(1-x^3)(1-x^4)}+\dots+\frac{x^{2i^2}}{(1-x)\dots(1-x^{2i})}+\dots,$$

then the number of self-conjugate partitions of $2m+1$ and of $2m$ are the coefficients of x^m in the ascending expansions of fx, ϕx, respectively.

Thus, suppose $2m+1=13$, the coefficient of x^6 in fx developed, that is, $\frac{6}{1,}+\frac{2}{1,\ 2,}$ or 3 is the number of self-conjugate partitions of 13.

These will be found to be 7 1 1 1 1 1 1, 4 4 3 2, 5 3 3 1 1. To find the conjugate to any partition $a, b, c \dots, l$, the most expeditious method is to find n_i, the number of the elements in the partition not less than i: $n_1, n_2, \dots, n_l$ (l being supposed to be the largest value of any element) will then be its opposite.

Thus, for example, for the partition 5 3 3 1 1, $n_1=5$, $n_2=3$, $n_3=3$, $n_4=1$, $n_5=1$, and $n_1\ n_2\ n_3\ n_4\ n_5$ reproduces 5 3 3 1 1.

If $2m = 12$ we have to find the value of $\frac{4}{1,\ 2,}$ which is again 3, and the 3 self-conjugate or self-opposite partitions of 12 will be seen to be 4 4 2 2; 5 3 2 1 1; 6 2 1 1 1 1.

In M. Faà de Bruno's tables of symmetric functions, which are only complete for the case of equations of not higher than the 11th degree, the number of self-conjugate partitions which appear among the headings and sidings of the tables is either 1 or 2, and it was therefore reasonable to try the effect of making arrangement of the partitions such as to bring the self-conjugate or pair of self-conjugate partitions into the centre of the line or column; but as soon as that degree is passed such a kind of principle (the rule founded upon which M. de Bruno does not state) becomes *prima facie* inapplicable at all events without undergoing modifications of which at present we know nothing.

Thus M. de Bruno's tables end just where his proposed principle of arrangement becomes inapplicable, stopping short at the case of the 12th degree, which has since been tabulated by Mr Durfee in the *American Journal of Mathematics.*

The term "opposite" or "conjugate" is used by Mr Durfee in the sense in which I am in the habit of employing it to signify the relation between what M. Faà de Bruno calls *combinaisons associées.* I think it right to recall attention to the fact that the credit of calling into being this kind of conjugate relation, is due to Dr Ferrers (the present Master of Gonville and Caius College, Cambridge), who some 30 years ago or more was the first to apply it to obtain an intuitive proof of Euler's great law of reciprocity, the very same as that which I have here employed to transform Mr Durfee's theorem. Euler demonstrated his law by help of his favourite instrument of generating functions.

By instituting in the case of combinations of *unrepeated* elements quite another and more exquisite kind of conjugate relation applicable to all such with the exception of those which belong to the infinite succession 1, 2, 2 3, 3 4, 3 4 5, 4 5 6, 4 5 6 7, 5 6 7 8, Mr Franklin, of this University, succeeded in finding an instantaneous demonstration of another well-known but very much more recondite theorem in partitions, also due to Euler, expressible by the statement that the indefinite product

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots$$

has for its development

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26}\ldots,$$

where the indices are the complete series of direct and retrograde pentagonal numbers.

By a singular oversight in my note in the last *Circular*, I omitted to state that Mr Durfee's rule is tantamount to affirming that the number of self-conjugate partitions or (which is the same thing) of symmetrical partition graphs for n, is the coefficient of x^n in the series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})} + \ldots$$

and since this series is identical with the infinite product

$$(1+x)(1+x^3)(1+x^5)\ldots$$

the number of self-conjugate partitions is the number of ways of distributing n into unrepeated odd-integers, a result which can be obtained directly by regarding any symmetrical partition graph as made up of a set of successively diminishing equilateral elbows or say carpenters' rules, each of which necessarily contains an odd number of points: the number of such elbows for any given graph will be the same as the number of points in the side of Mr Durfee's square nucleus, and consequently we have an intuitive proof of the theorem that the infinite product

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

is equal to the infinite series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i + \ldots$$

because the coefficient of $a^i x^n$ is the same in both expressions. By a similar method I obtain an intuitive and almost instantaneous solution of the problem to expand in infinite series the infinite products which express a Theta Function and its *reciprocal*, and many other questions of a similar nature.

It was the anticipation of the parallelism between the expressions for the number of special partitions in the unrepeated-numbers and the repeated-numbers theories which led me to find *à priori* the partition-into-odd-integers expression for the number of self-conjugate partitions, and thus started me on the track of the graphical method of transforming infinite products into infinite series: the light of analogy may sometimes "lead astray" but it is more often "light from heaven."

82.

ON DR F. FRANKLIN'S PROOF OF EULER'S THEOREM CONCERNING THE FORM OF THE INFINITE PRODUCT $(1-x)(1-x^2)(1-x^3)\ldots$.

[*Johns Hopkins University Circulars,* II. (1883), p. 42.]

REVOLVING in my mind Mr Franklin's remarkable proof of Euler's theorem concerning the above infinite product inserted in the *Comptes Rendus* of the Institute of France for 1880, I have found it useful to employ a certain terminology to enable myself to seize some of the points which it contains with a firmer grasp and to clothe it in what seems to me a more purely discursive, as distinguished from what, by analogy to geometrical processes, I am wont to call a diagrammatic form of reasoning; thinking that others may find advantage in what has been useful to myself, I avail myself of the pages of the *Circular* to give it publicity.

Let us agree to understand by a *distribution* of n any combination of *unrepeated* integers in descending order, whose sum is equal to n. The number of such component integers may be termed the *order* of the distribution.

If the initial components of such distributions be $m-1, m-2, \ldots, (m-i)$ [where i may be equal to but cannot exceed the order] not followed by an element $m-i-1$, I call i (the number of terms in such initial sequence) the *consecutant* and the final (that is, the least) component, the *concluant* of the distribution.

Lemma. Any distribution of a given integer, which does not form a single sequence whereof the concluant is either equal to or greater by a unit than the consecutant, may be converted by one or the other (but not by either) of two reversible processes (say of loading or unloading) into another distribution in which the order is diminished or increased by a single unit.

By loading is to be understood the process of taking away the concluant (say ω) and increasing the ω first terms of the initial sequence each by a

unit; and by unloading, that of taking away a unit from each of the components in the initial sequence and adding on an element equal to the consecutant as the new concluant.

1st. Suppose that the distribution does not form a single sequence.

If the concluant is equal to or less than the consecutant it is obvious that loading will be possible but not unloading, because the latter would give rise to a new concluant equal to or greater than the original one.

On the other hand, if the concluant is greater than the consecutant, unloading will be possible but not loading, because there will be too few terms in the initial sequence to exhaust (by the addition of one unit to each) the number of units in the concluant.

2nd. Suppose that the entire distribution forms a single sequence.

If the concluant is *less* than the consecutant loading will still be possible, because the number of terms in the sequence after taking away the concluant will still be not greater than the concluant.

Again, if the concluant is more than a unit greater than the consecutant, unloading will still be possible because the new concluant will be less than the original one even after it has lost a unit by the process of unloading.

Hence the Lemma is proved.

And as a Post-lemma, it may be stated that when the distribution forms a single sequence such that the concluant is equal to or only one unit greater than the consecutant, neither loading nor unloading will be possible. The loading on the first supposition is defeated by the fact that the diminished sequence will be one too few in number to absorb the units which make up the concluant—and the unloading on the second supposition is defeated by the fact that the new concluant will be equal to (that is, will be a repetition of) the old one when by the act of unloading it is diminished by a unit.

From the lemma and post-lemma combined, it follows as an inference that all the distributions of any number n may be taken in pairs (consisting of one of an even and one of an odd order), unless it should be the case that one of such distributions is a term in the series

$$1,\ 2,\ 3.2,\ 4.3,\ 5.4.3,\ 6.5.4,\ \ldots,\quad (2i-1).(2i-2)\ldots i,\quad 2i.(2i-1)\ldots(i+1),\ \ldots$$

which represent distributions of the several integers

$$1,\ 2,\ 5,\ 7,\ 12,\ 15,\ \ldots\ \frac{3i^2-i}{2},\ \frac{3i^2+i}{2},\ \ldots$$

to which the process either of loading or unloading (contraction or expansion by a unit) is inapplicable.

Hence if we denote by n_o, n_e, the number of distributions of n, into an odd and even number of unrepeated parts, we must have $n_o - n_e = 0$, except when $n = \frac{3i^2 \mp i}{2}$, in which case $n_e - n_o = (-)^i 1$.

Consequently we have

$$(1-x)(1-x^2)(1-x^3)\ldots,$$

that is,
$$1 + \ldots + (n_e - n_o)x^n + \ldots = \sum_{i=+\infty}^{i=-\infty} (-)^i x^{\frac{3i^2+i}{2}},$$

which is Euler's theorem.

To make the demonstration absolutely objection-proof it ought to be shown that if X is convertible into Y by loading or unloading, Y will be convertible into X by the reverse process—but this is almost self-obvious; for if X has become Y by loading, the new consecutant cannot be greater than the old one and will therefore not be greater than the new concluant, but equal to or less than it, and therefore the process of *unloading* is the one applicable to Y, and if X has become Y by unloading, the new consecutant cannot be less than the old one and will therefore be greater than the new concluant, and therefore the process of *loading* is the one applicable to Y; this completes the proof, and leaves I think nothing further to be desired.

In Mr Durfee's question, treated of in the last number of the *Circulars*, the object of research is the number of self-conjugate partitions (with repeated or unrepeated components) of a given integer n; in Mr Franklin's, the object sought for is the number (1 or 0) of (so to say celibate or) unconjugate distributions of an integer: the Ferrers-law of conjugation is of universal application to all partitions—the Franklin-law only to partitions with unrepeated components.

There is, however, a singular parallelism between the two theories; let us agree to call the self-conjugate in the one, and the non-conjugate partitions in the other, in each case alike *special* partitions—and denote the number of distributions of n into an odd number and into an even number of *unrestricted* parts by $(n)_o$ and $(n)_e$ respectively. Then just as the difference between n_o and n_e is the number of special partitions in the one, so it may be shown that the difference between $(n)_o$ and $(n)_e$ (which is well-known to be the same as the total number of partitions of n into unrepeated odd parts) is the number of special partitions in the cognate theory.

83.

ON THE USE OF CROSS-GRATINGS TO OBTAIN CERTAIN DEVELOPMENTS CONNECTED WITH THE THEORY OF ELLIPTIC FUNCTIONS.

[*Johns Hopkins University Circulars,* II. (1883), pp. 43, 44.]

It will be convenient to regard the components of any partition as arranged in a natural, say a descending order of magnitude: a partition graph means a series of points, say the knots in a web or the intersections of a cross-grating, lying in lines parallel to two fixed lines: the number of points, or lines parallel to one of the boundaries chosen at will, will represent the successive components of the partition and the number of the lines themselves will be the number of parts in the partition.

The lines in question may for greater distinctness be termed *magnitude* lines and the crossing ones, *part* lines. The graph may be termed regular when the magnitude lines never increase as they recede from the rectilinear boundary to which they are parallel. This, we see intuitively, cannot happen without the same condition being true of lines parallel to the *part* boundary: so that we may say that a regular partition graph is one in which the lines and columns of points neither of them ever increase in length as they recede from their respective boundaries. If such a graph corresponds to a partition *without* repetitions, the lines drawn in the magnitude direction must continually contract (that is, contain fewer and fewer points) as they recede from their maximum boundary.

The correlation referred to in the preceding paragraph is tantamount to saying that if there be two systems of partitions in one of which a given number is set out in every possible way as a sum of i parts none exceeding j in magnitude, and another in which the same number is set out in every possible way as a sum of j parts none exceeding i in number, such partitions arranged in natural order will have a one-to-one correspondence, being representable by the same regular graphs with the names of the magnitude and part boundaries interchanged, so that there will be the same number of partitions in the two systems.

A partition is self-conjugate when its representative graph, after an interchange of the names of the part- and magnitude-lines, gives the same reading.

Such a graph, therefore, must be symmetrical.

Suppose the partible number to be n.

Then its graph may be resolved into i angles fitting into one another, consisting of continually decreasing odd numbers; and the number of such graphs will be the number of ways of composing n with unrepeated odd numbers: but it may also be analyzed into a square containing i^2 points and two similar and equal appendages each containing $\frac{n-i^2}{2}$ points; and consequently their number will be the number of ways in which $\frac{n-i^2}{2}$ may be made up with the numbers $1, 2, \ldots i$, or what is the same thing $n-i^2$ with the numbers $2, 4, \ldots 2i$; it is consequently the coefficient of n in the development of

$$\frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})};$$

but it is also by virtue of the preceding remark the coefficient of $x^n a^i$ in the continued product

$$(1+ax)(1+ax^3)(1+ax^5)\ldots.$$

Hence this continued product

$$=1+\frac{x}{1-x^2}a+\frac{x^4}{(1-x^2)(1-x^4)}a^2+\frac{x^9}{(1-x^2)(1-x^4)(1-x^6)}a^3$$
$$+\ldots+\frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i+\ldots.$$

The expansion of the reciprocal of

$$(1-ax)(1-ax^3)(1-ax^5)\ldots$$

may be obtained in a similar manner; the coefficient of $x^n a^j$ in this product is the number of ways in which n can be composed with j free odd numbers. If we construct a graph with j angles or elbows fitting into one another, such that the number of nodes in each such angle from the outermost inward corresponds with any such partition in descending order, the graph so constructed will be still symmetrical but no longer regular; a line of nodes instead of being necessarily equal or less in number than an antecedent one may jut one degree beyond it, but if the j points in the diagonal be removed (as in the example following, the points

$$\begin{array}{cccccc}
1 & \bullet & \bullet & \bullet & \bullet & \\
\bullet & 2 & \bullet & \bullet & \bullet & \\
\bullet & \bullet & 3 & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & 4 & \bullet & \\
\bullet & \bullet & \bullet & \bullet & & \\
 & & \bullet & & &
\end{array}$$

whose places are supplied by the numbers 1, 2, 3, 4) then the figure that is left is decomposable into two regular graphs with one boundary line horizontal, or vertical, and the other oblique. Hence the fraction above given expanded in powers of a becomes

$$1+\frac{x}{1-x^2}a+\ldots+\frac{x^i}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i+\ldots$$

the only difference from the preceding case being that i points now instead of i^2 are taken away from the graph.

I might give numerous other exemplifications of the power and grasp of this method, but the following two may suffice for the present. I propose first to prove the equation between the reciprocal of

$$(1-ax)(1-ax^2)(1-ax^3)\ldots$$

and the infinite series

$$1+\frac{x}{1-x}\cdot\frac{a}{1-ax}+\frac{x^4}{(1-x)(1-x^2)}\cdot\frac{a^2}{(1-ax)(1-ax^2)}+\ldots$$
$$+\frac{x^{i^2}}{(1-x)(1-x^2)\ldots(1-x^i)}\cdot\frac{a^i}{(1-ax)(1-ax^2)\ldots(1-ax^i)}+\ldots.$$

The coefficient of $x^n a^j$ in the continued product is the number of regular graphs that can be formed with n nodes, containing j lines of them with no limitations to the number of the columns. We may suppose, therefore, the number of columns to be made successively 1, 2, 3, ... Consider the case where there are i columns forming a square; the graph being regular the lines and columns will intersect in i^2 nodes, and there will be left $n-i^2$ nodes to be made up of $j-i$ quantities none greater than i (namely, the horizontal filaments of nodes in the columns underlying the square), and of other quantities not greater than i but otherwise unlimited (namely, the vertical filaments of nodes in the hollowed out indefinite parallelogram abutting alongside of the square): that number we well know is the coefficient of $x^n a^{j-i}$ in

$$\frac{1}{(1-ax)(1-ax^2)\ldots(1-ax^i)}\cdot\frac{1}{(1-x)(1-x^2)\ldots(1-x^i)}x^{i^2}.$$

Hence for every value of j the coefficient of $x^n a^j$ in the infinite product is the coefficient of $x^n a^j$ in the infinite series, and consequently the two forms when developed must be identical.

Not to weary my readers I hurry on to the development in an infinite series of the product of the two infinite products

$$(1+ax)(1+ax^3)(1+ax^5)\ldots \text{ and } (1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\ldots.$$

Here it will be expedient to explain what I mean by a parallel bipartition of n; it is simply a couple of sets of numbers written on opposite sides of a line of demarcation, so that the number of the numbers on the left always

exceeds that on the right by a given difference δ, which may be any number from zero upwards, and such that the sum of all the elements collectively is equal to n.

When this difference is zero, such a bipartition may be called equi-parallel, in other cases parallel with a difference δ.

It is then clear that the coefficient of $x^n a^j$ or $x^n a^{-j}$ in the above product is nothing else but the number of parallel bipartitions of n to the difference j limited to contain only odd numbers which must not appear in the same arrangement more than once on the same side of the line of demarcation.

In particular the coefficient of x^n in the term not containing a will be the number of equi-parallel bipartitions of n restricted to odd numbers not repeated on the same side of the separating line.

Form a graph as follows: Supposing one of the bipartitions to consist of θ parts on each side, say $a, b, c, \ldots l$; $\alpha, \beta, \gamma, \ldots \lambda$; the parts being on each side taken in descending order, construct angles or elbows in which the horizontal sides contain

$$\frac{a+1}{2}, \frac{b+1}{2}, \ldots \frac{l+1}{2},$$

and the vertical sides

$$\frac{\alpha+1}{2}, \frac{\beta+1}{2}, \ldots \frac{\lambda+1}{2}$$

points, then these will contain

$$\frac{a+\alpha}{2}, \frac{b+\beta}{2}, \ldots \frac{l+\lambda}{2}$$

points respectively; on fitting them into one another we shall obtain a regular graph with θ lines or columns made up of $\frac{n}{2}$ points, and conversely every regular graph of $\frac{n}{2}$ points may be resolved into angles with sides p, p'; q, q'; r, r' ... corresponding to an equi-parallel unrepeated odd-number bipartition

$$2p-1,\ 2q-1,\ 2r-1,\ \ldots;\ 2p'-1,\ 2q'-1,\ 2r'-1,\ \ldots.$$

Hence the coefficient of x^n in the term not containing a in the development is the number of regular graphs that can be formed with $\frac{n}{2}$ points; and therefore the term not containing a is

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\ldots}.$$

Now consider the term containing a^j to which corresponds a parallel bipartition with j more elements to the left than to the right of the separating

line: arrange the sets on each side of the line in descending order, strike off the j highest on the left-hand side and construct a graph G with the sets which remain, as in the last case; then subtract 1, 3, 5, $(2j-1)$ respectively from the j elements [struck off] to the left, and place, taken in ascending order, half the numbers of points remaining, over the top line of the graph G; there will result a regular graph G'; and by an obvious reverse process every such graph can be made to correspond with a bipartition of unrepeated odd numbers having j more numbers to the left than to the right. Hence the number of the parallel bipartitions to the difference j will be the number of indefinite partitions of

$$\tfrac{1}{2}\{n-(1+3+\ldots+2j-1)\} \text{ or } \frac{n-j^2}{2},$$

that is, the coefficient of x^n in

$$\frac{x^{j^2}}{(1-x^2)(1-x^4)(1-x^6)\ldots}.$$

Hence the given bi-product when developed must be identical with

$$\frac{1}{(1-x^2)(1-x^4)\ldots}\{1+x(a+a^{-1})+x^4(a^2+a^{-2})+x^9(a^3+a^{-3})+\ldots\}.$$

In the preceding volume of the *Circular* I showed how the self-same method of points (but very differently applied) serves to establish and leads to wide generalizations of the theorem of Jacobi, upon which depends the proof of the impossibility of the existence of 3-period functions.

In a future number of the *Circular*, or else in the *American Journal of Mathematics*, I propose to show how to obtain intuitively by a graphical construction the expression for the product of the two infinite products

$$\frac{1-a^k}{1-a}\cdot\frac{1-a^{3k}}{1-a^3}\cdot\frac{1-a^{5k}}{1-a^5}\ldots \text{ and } \frac{1-a^{-k}}{1-a^{-1}}\cdot\frac{1-a^{-3k}}{1-a^{-3}}\cdot\frac{1-a^{-5k}}{1-a^{-5}}\ldots.$$

The *true inwardness* of this powerful analytical method depends in the first place on the idea of *correspondence,* assisted in the second place (in some but not in all instances) by the idea of graphical representation of partition numbers restrained to assume a natural order of succession.

Mr Ferrers' method, which has lain so long dormant and sterile in mathematical soil, has after an interval of 30 years begun to germinate, and seems to be about to burst forth into luxuriant growth and efflorescence.

It is Mr Durfee's graphical determination of the number of self-conjugate partitions of n, inserted in a preceding *Circular*, that has let in the light and air and supplied the fertilizing influence needful to bring this about.

84.

ON THE NUMBER OF FRACTIONS IN THEIR LOWEST TERMS WHOSE NUMERATORS AND DENOMINATORS ARE LIMITED NOT TO EXCEED A CERTAIN NUMBER.

[*Johns Hopkins University Circulars*, II. (1883), pp. 44, 45.]

THE fractions for greater simplicity may be supposed to be proper fractions, except that it is expedient to count in $\frac{1}{1}$ as one of them. To any given limit or argument as it may be called, n, corresponds a finite system of fractions in their lowest terms, which may be arranged in order of magnitude; when so arranged the system will be found to possess some remarkable properties, first apparently noticed by Mr Farey, an English mathematician, in 1816, subsequently made the subject of a proof by Cauchy in the same year (reproduced in his *Exercices de Mathématiques*, t. I. 1826), and again demonstrated and extended by Mr J. W. L. Glaisher in an interesting paper in the *London and Edinburgh Philosophical Magazine* for 1879, the same journal in which the subject was first broached.

I am under the impression that I have seen somewhere the names of one, if not two, English mathematicians who have endeavoured to obtain an empirical law for the number of fractions corresponding to any given limit, but all my endeavours to come upon the traces of those investigations, if such exist, have hitherto proved fruitless. Had anything been done in this direction prior to 1879, there is little doubt that reference would have been made to it by Mr Glaisher, who goes carefully in his paper of that date into the bibliography of the subject*.

This number for the limit or argument j is obviously no other than the sum of the *totients* of all the numbers from unity up to j. I shall use Tx

* Mr Glaisher, however, takes no notice of M. Halphen's important extension of Farey's theory, published in the *Proceedings of the Mathematical Society of France*, and followed by another on the same subject by M. Lucas, nor of Herzer's table in 1864, nor Hrabak's, 1876.

to denote the sum of all the totients of all the integers not exceeding x, where x is any positive quantity whatever, and show how to make Tx the subject of a functional equation, from which limiting functions to its value may be deduced. To this end consider the two sets of terms $1, 2, 3, \ldots i$ and $i+1, i+2, \ldots j$, where $j=2i$ or $2i+1$ indifferently.

The number of times that an integer r is contained in any given set of quantities, or rather the number of quantities in the set which contain r, I call the *frequency* of r in respect to the set.

Looking to the two sets here in question it is easily seen that the frequency of any integer *quâ* the second set must either be equal to its frequency *quâ* the first set or exceed it by a single unit. The equi-frequent and unequi-frequent integers are determinable by the following theorem which I call the theorem of harmonic division.

Let j_μ in general denote the integer part of j/μ if it is a fraction, or the whole of it if it is an integer.

Write down the successive ranges

$$j,\ j-1,\ j-2,\ \ldots\ j_2+1\,;\ j_2,\ j_2-1,\ \ldots\ j_3+1\,;$$
$$j_3,\ j_3-1,\ \ldots\ j_4+1\,;\ j_4,\ j_4-1,\ \ldots\ j_5+1;\ \ldots$$

where it will be understood that if $j_k=j_{k+1}$, the range $j_k \ldots$ becomes abortive.

Any number which appears in the first, third, fifth ... range is equi-frequent and any number which appears in the second, fourth ... range is unequi-frequent in respect to the two given series

$$1,\ 2,\ \ldots\ i;\ i+1,\ i+2,\ \ldots\ j.$$

This theorem will be found to be demonstrable without the slightest difficulty.

The second theorem required is one of which a demonstration almost instantaneous and conclusive is given in the Excursus on Rational Fractions and Partitions (*Am. Jour. of Math.*, Vol. v., No. 2*), namely, that the sum of the products formed by multiplying the frequency of any integer in respect to a given set of quantities by its totient is equal to the sum of the quantities contained in the set.

This proposition shows that if fr, $f'r$ be the frequencies of r in respect to the two last-named sets and τr its totient

$$\sum_{r=1}^{r=\infty}(f'r-fr)\,\tau r=(1+2+\ldots+j)-2\,(1+2+\ldots+i),$$

and the theorem of harmonic division shows that the left-hand side of this equation is equal to

$$\sum_{\lambda=j_2+1}^{\lambda=j}\tau\lambda+\sum_{\lambda=j_4+1}^{\lambda=j_3}\tau\lambda+\sum_{\lambda=j_6+1}^{\lambda=j_5}\tau\lambda+\ldots$$

[* Above, p. 611.]

because $f'r - fr = 1$ for the odd-ordered, and $= 0$ for the even-ordered harmonic ranges.

The separate sums above written are obviously the same respectively as

$$Tj - T\frac{j}{2},\ T\frac{j}{3} - T\frac{j}{4},\ T\frac{j}{5} - T\frac{j}{6},\ \ldots.$$

Hence, if we write

$$\vartheta j = Tj - T\frac{j}{2} + T\frac{j}{3} - T\frac{j}{4} + T\frac{j}{5} \ldots \textit{ ad inf.}$$

when j is an even integer

$$\vartheta j = (1 + 2 + \ldots + 2i) - 2(1 + 2 + \ldots + i) = i^2 = \frac{j^2}{4},$$

and when j is an odd integer

$$\vartheta j = \{1 + 2 + \ldots + (2i + 1)\} - 2(1 + 2 + \ldots + i) = (i + 1)^2 = \frac{(j + 1)^2}{4}.$$

If now we write for any positive quantity x,

$$\vartheta x = Tx - T\frac{x}{2} + T\frac{x}{3} - T\frac{x}{4} + \ldots,$$

it may be shown by aid of the above results that for all values of x not less than unity,

$$\vartheta x = \text{ or } > \frac{x^2}{4} - \frac{x}{2},\quad \vartheta x = \text{ or } < \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4},$$

and from these two inequalities limiting values to $T(x)$ may be deduced by a process of successive approximation in principle the same as that employed by me in the *Am. Jour. of Math.*, Vol. v., No. 2, pp. 124, 125*, in connexion with Tchebycheff's theory, but differing from it considerably in the mode of application and in the character of the results to which it leads.

The subject has been for so very short a time studied by me that I feel it desirable to reserve this part of its development for a future communication, but I am in a position to state that it is possible to find superior and inferior limits to Tx, say Lx and Λx, such that Lx shall be of the form $Mx^2 + Nx +$ a rational integral function of $\log x$ and Λx of a similar form, $M'x^2 + N'x +$ another rational integral function of x, where M, M' differ by a quantity less than any quantity that may be assigned from one another and from a number λ found from the equation

$$\lambda\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} \ldots\right) = \frac{1}{4},$$

that is, equal to $3/\pi^2$.

Accordingly the ultimate, or so to say asymptotic, value of $\frac{Tx}{x^2}$ is $3/\pi^2$ where Tx is the number of pairs of integers not exceeding x, which are relatively prime to each other; consequently since the total number of such

[* Above, p. 530.]

pairs is ultimately in a ratio of equality to $\frac{x^2+x}{2}$, it will follow, if the above assertion is correct, that the chance of two arbitrary independent integers, being relatively prime to one another, is $\frac{6}{\pi^2}$; the odds in favour of two such numbers being relatively prime, are thus very nearly expressed by the ratio of 60792 to 39268; that is to say, are pretty nearly as 76 to 49 or a trifle better than 3 to 2.

In what precedes I have used the simplest means or formula sufficient for obtaining a functional equation to the sum-totient Tx, but the theorem of harmonic division admits of a very wide generalization, and accordingly the functional equation admits of an indefinite number of distinct presentations.

Thus instead of j belonging to the series $2i$, $2i+1$ it may be considered as belonging to the series ki, $ki+1$, ..., $ki+(k-1)$, and the theorem of harmonic division then is as follows: calling the range commencing with j_λ and ending with $j_{\lambda+1}+1$, range number λ, where λ may be understood to be any integer from 0 upwards (0 itself included) if the residue of λ in respect to k is μ, and if $f'r$ and fr are the frequencies of r in respect to the two series $1, 2, \ldots j$, $1, 2, \ldots i$, then when r belongs to the range whose number is λ, and the residue of λ in respect to k is (λ), it will be found that $f'r - kfr = (\lambda)$.

By way of example suppose $k=3$, then writing

$$\vartheta j = \left(Tj + T\frac{j}{2} - 2T\frac{j}{3}\right) + \left(T\frac{j}{4} + T\frac{j}{5} - 2T\frac{j}{6}\right) + \ldots$$

it may readily be proved that according as $j=3i$, or $3i+1$, or $3i+2$, ϑj will equal $3i^2$, or $3i^2+3i+1$, or $3i^2+6i+3$, and similarly if

$$\vartheta j = \left(\vartheta j + \vartheta\frac{j}{2} \ldots + \vartheta\frac{j}{k-1} - k\vartheta\frac{j}{k}\right) + \left(\vartheta\frac{j}{k+1} \ldots + \vartheta\frac{j}{2k-1} - k\vartheta\frac{j}{2k}\right) + \ldots$$

then ϑj according as $j=ki$, or $ki+1$, ... or $ki+(k-1)$ will have k distinct and perfectly determinate values of which the first will be $\frac{k-1}{2k}j^2$ and the last $\frac{k-1}{2k}(j+1)^2$.

More general formulæ still may be obtained by supposing

$$j = ki + r, \quad j' = k'i + r',$$

where k, k' are relative prime numbers and r, r' less than k, k' respectively.

Let $P = kk'i + R$, R being less than kk' and congruent to r in respect to modulus k and to r' in respect to modulus k', then if we divide P into harmonic

ranges and call fr, $f'r$ the frequencies of r in respect to the two series $1, 2, \ldots j$; $1, 2, \ldots j'$, and call ν the number of the range to which r belongs, and δ, δ' the residues of ν in respect to k, k' respectively, it will be found that $kf'r - k'fr = \delta' - \delta$.

Take as an example $i = 20$, $j = 41$, $j' = 62$, so that $k = 2$, $k' = 3$, then $P = 125$ and for

$$\begin{aligned} \nu &= 0,\ 1,\ 2,\ 3,\ 4,\ 5, \\ \delta &= 0,\ 1,\ 0,\ 1,\ 0,\ 1, \\ \delta' &= 0,\ 1,\ 2,\ 0,\ 1,\ 2, \\ \delta' - \delta &= 0,\ 0,\ 2,\ \bar{1},\ 1,\ 1, \end{aligned}$$

the harmonic ranges of P beginning with Range No. 2 will be seen to be

$125 - 63$, $62 - 42$, $41 - 32$, $31 - 24$, $23 - 21$, $20 - 18$, $17 - 16$, $15 - 14$, 13, etc.,

and the corresponding frequencies of the numbers in those ranges in regard to the series $1, 2, \ldots 41$, $1, 2, \ldots 62$, will be seen to be

$1, 0;\ 1, 1;\ 2, 1;\ 2, 1;\ 3, 2;\ 3, 2;\ 4, 2;\ 4, 3;\ \ldots$ respectively,

and we have

$$2.1 - 3.0 = 2,\ 2.1 - 3.1 = \bar{1},\ 2.2 - 3.1 = 1,\ 2.2 - 3.1 = 1,\ 2.3 - 3.2 = 0,$$
$$2.3 - 3.2 = 0,\ 2.4 - 3.2 = 2,\ 2.4 - 3.3 = \bar{1}\ \ldots,$$

in which the recurring period is as it ought to be, $2, \bar{1}, 1, 1, 0, 0$.

By means of this division a still wider latitude could be won were it worth while, for the expression of the functional equation to the sum-totient. Another statement and further extensions of the theory are contained in a Note intended for publication in the *Comptes Rendus* of the Institute of France. I may add that I have had a table constructed of the values of Tx for all values of x up to 500 inclusive, and that Tx is always intermediate within this range between $3/\pi^2 x^2$ and $3/\pi^2 (x+1)^2$—a very noteworthy result: and which I have little doubt remains true for all values of x.

85.

PROOF OF A WELL-KNOWN DEVELOPMENT OF A CONTINUED PRODUCT IN A SERIES.

[*Johns Hopkins University Circulars*, II. (1883), p. 46.]

To prove that the general term in the development in a series of powers of a of the reciprocal of

$$(1-a)(1-ax)\dots(1-ax^i)$$

(say of Fx) is

$$(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i}) \div (1-x)(1-x^2)\dots(1-x^i) \,.\, a^j$$

say $X_j a^j$, I proceed as follows.

I call the coefficient of a^j in the development, J_x, and show that every linear factor of X_j is contained in J_x.

Any such factor is a primitive factor of $x^r - 1$, where r is any integer such that

$$E\frac{i+j}{r} - E\frac{i}{r} - E\frac{j}{r} = 1,$$

and it is unrepeated.

Let $x = \rho$, and let the negative minimum residue of $i+1$ in respect to r be $-\delta$.

Then $F\rho$ is equal to the product of δ linear functions of a divided by a power of $(1-a^r)$, and consequently the only powers of a (say a^θ) which appear in its development will be those for which the residue of θ in respect to r, is $0, 1, 2, \dots \delta$, and consequently

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 0.$$

Hence a^j will not appear therein: so that J_x vanishes when any factor of X_j is zero, and consequently since every such factor is unrepeated, J_x contains X_j.

But J_x is obviously of the degree ij in x, and X_j which is the sum of the j-ary homogeneous products of $1, x, x^2, \ldots, x^i$ is of the same degree. Hence the two functions of x can only differ by a constant factor. On making $x=1$, Fx becomes $(1-a)^{-(i+1)}$; so that X_j becomes

$$\frac{(j+1)(j+2)\ldots(j+i)}{2\ldots i}$$

and J_x becomes the product of vanishing fractions

$$\frac{1-x^{j+1}}{1-x}, \frac{1-x^{j+2}}{1-x^2}, \ldots \frac{1-x^{j+i}}{1-x^i}, \text{ that is, } (j+1), \frac{j+2}{2}, \ldots \frac{j+i}{i}.$$

Hence $X_j = J_x$. Q.E.D.

The expansion of

$$(1-ax)(1-ax^2)\ldots(1-ax^i)$$

in terms of powers of a may be verified in like manner.

It is not without interest to observe (if the remark has not been made before) how this development is connected by the principle of correspondence with the preceding one.

Throwing out by multiplication the factor $(1-a)$ in the denominator of Fx we obtain the reciprocal of

$$(1-ax)(1-ax^2)\ldots(1-ax^i),$$

say $\dfrac{1}{Gx}$, under the form

$$1+\ldots+\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i-1})}{(1-x)(1-x^2)\ldots(1-x^{i-1})}x^j a^j+\ldots.$$

Consequently the number of ways in which n can be divided into exactly j parts $1, 2, \ldots i$ (repetitions admissible) is the coefficient of x^n in the expansion according to ascending powers in x of the above multiplier of a^j.

But if any such partition be arranged in ascending order, and $0, 1, 2, \ldots (j-1)$ be added (each to each) to its components, it is converted into a partition without repetitions, and by a converse process of subtraction each such partition is convertible into one of the former, but in which either repetition or non-repetition is allowable. Hence the free partitions of $n-\dfrac{j^2-j}{2}$ into j parts limited not to exceed $i-j+1$, have a one-to-one correspondence with the unrepetitional partitions of n into j parts limited not to exceed i, and must be equal to them in number. Hence the coefficient of a^j in $G(-x)$ may be deduced from that of a^j in $(Gx)^{-1}$ by multiplying the latter

by $x^{\frac{1}{2}(j^2-j)}$ and changing i into $i-j+1$. Hence the general term in $G(-x)$ will be

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^i)}{(1-x)(1-x^2)\dots(1-x^{i-j})}x^{\frac{j^2+j}{2}}a^j$$

which is right.

When $i=\infty$ each of these developments (like a multitude of others, including the Theta-functions) may be obtained intuitively by the graphical method of points given in my communication to the Johns Hopkins Scientific Association at its last meeting; it remains a desideratum to apply the same method to the above two developments, or either of them, for the case of i*.

In the Ferrers, Franklin, Durfee-Sylvester and other conjugate systems of partitions, the partible number is the same for the corresponding partitions; in this last example (and the like will be shown to be the case in the graphical development of the Theta-function, and its generalizations), the one-to-one correspondence is between partitions of two different numbers.

* Not so—the result derived springs from the immediate application of a general logical principle as will hereafter be shown.

86.

ON A NEW THEOREM IN PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), p. 70.]

IT is a well-known theorem that the number of partitions of n into odd numbers is equal to the number of its partitions into unequal numbers. This equality was seen by Euler to result from the identity

$$(1+x)(1+x^2)(1+x^3)\ldots = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}.$$

It may also be proved easily by the method of correspondence. For if we call the partitions of n into odd numbers (repeated or not) the U, and into unequal numbers the V system, any V will be of the form $[V_1, V_2, V_3, \ldots]$, where V_i is of the form

$$q_i \,.\, 2^{a_i},\quad q_i \,.\, 2^{a_i'},\quad q_i \,.\, 2^{a_i''},\quad \ldots,$$

each q being an odd number and all the q's unlike.

Hence writing

$$2^{a_i} + 2^{a_i'} + 2^{a_i''} \ldots = A_i,$$

V is transformable into $A_1q_1,\ A_2q_2,\ A_3q_3,\ \ldots$, which is a member of the U system.

And conversely any U as $A_1q_1,\ A_2q_2,\ A_3q_3,\ \ldots$, will be transformable into a V by decomposing each U into a sum of products of its largest odd divisor into distinct powers of 2 which can be effected in one and only one manner; so that there is a one-to-one correspondence between the U's and V's, and the number of the one set is therefore the same as the number of the other. The theorem which is now to be explained is, so to say, a differentiation (in the Herbert Spencer sense) of this theorem.

It regards the U and V systems each broken up into classes and affirms the equality between the numbers of U's in any class and of the V's in the homonymous class. The proof of this by an analytic identity remains to be

discovered—it is effected without great difficulty by the method of correspondence: but what is very worthy of notice is that the V which corresponds to a U, in the more refined construction about to be explained, is in general (and it may be universally) different from the V which corresponds to it, when the preceding method of conjugation is adopted.

Every U which contains i distinct parts is said to be a U of the ith class, and every V which contains i distinct sequences (not running together) of consecutive numbers is said to be a V of the ith class—and my theorem may be expressed by saying that there exists a one-to-one correspondence (and therefore equality of content) between the U's of any class and the V's of the same class. I ought perhaps rather to say that a correspondence can be *instituted* than that a correspondence *exists*, for the fact that two absolutely unlike bonds of correspondence connect the totality of the U and that of the V system seems to indicate that such correspondence should rather be regarded as something put into the two systems by the human intelligence than an absolute property inherent in the relation between the two. Kant makes a similar remark upon the elementary conceptions (such as the circle), which form the groundwork of geometry.

As an example of the numerical part of the theorem consider the 3rd class of the U's and V's for $n = 16$.

The U's of this class will be

$$11.3.1^2;\ 9.5.1^2;\ 9.3^2.1;\ 9.3.1^4;\ 7.5.1^4;\ 7.3^2.1^3;$$
$$7.3.1^6;\ 5^2.3.1^3;\ 5.3^3.1^2;\ 5.3^2.1^5;\ 5.3.1^8;$$

and the V's which are somewhat more difficult to calculate by an exhaustive process will be found to be

$$1.6.9;\ 1.2.5.8;\ 2.6.8;\ 1.5.10;\ 1.2.4.9;\ 2.5.9;$$
$$1.4.11;\ 1.3.4.8;\ 3.5.8;\ 2.4.10;\ 1.3.12.$$

So again of the 4th class there is only one U and one V, namely, $1.3.5.7$, which is common to the two systems—and of the first class owing to 16 containing only one odd divisor, namely, unity, there is also but one U and one V, namely, the undivided 16 for each alike. In general for the first class the number of U's is obviously the number of odd divisors of the partible number n and the number of single sequences is easily seen to be the same. Thus, for example, for 15 there exist the sequences $1.2.3.4.5$; $4.5.6$; 7.8; 15; and for 9 the sequences $2.3.4$; 4.5; 9.

I will now indicate the mode of proof, the particulars of which will be found set out in full in the forthcoming number of the *American Journal of Mathematics* [Vol. IV. of this Reprint].

The partible number n being given, I take any U belonging to it and form two graphs, one whose rows represent the major halves of each part

of U and the other its minor halves [$q+1$ is the major and q the minor half of $2q+1$]. I then dissect each of these graphs into its component angles and take the content of each; it is easily seen that beginning with the major and passing from it to the minor and back again to the major and so on continually in alternate succession, the readings will form a continually decreasing series of numbers whose sum will be the same as of the parts of the U, and thus U will be transformed into V. The number of parts in V, if we agree to consider that number as always *even* by supplying a zero at the end if it should happen to be *actually* odd, will be $2i$ where i is the number of points in the side of the Durfee-square appertaining to the major graph.

Conversely, if any V be given containing or made to contain $2i$ parts, it is easy to construct a system of $2i$ *linear* equations between the contents of the first i lines and the first i columns of an assumed U having a Durfee-square containing i^2 points, which shall transform into the given V, and to prove that these contents will be all of them greater than $2i$: hence *one* and *only one* U corresponds to a V, and consequently there is a one-to-one correspondence between the entire U and entire V systems. It remains to show that any U_i (a U of the ith class) by the prescribed process of transformation becomes a V_i (a V of the ith class).

This is effected as follows: suppose the first exterior angle to be removed simultaneously from a given major graph and its accompanying minor: begin with supposing that U_i becomes V_j: i is the number of unequal lines in either graph and it is easily proved that $i-j$ remains unaltered by the contraction of the graphs in the manner above indicated: that is, it can be shown that the effect of the contractions is to diminish i and j simultaneously each of them by 0, each of them by 1, or each of them by 2.

Continuing this process of *stripping* the graphs of their outside angles we must come at last to a graph consisting of one line and one column or of only one line, or only one column, or only a point. In the first of these four cases i and j are each equal to 2, and in the last three each equal to 1, hence $i-j$ is always zero and every U_i corresponds to a V_i. This establishes the very remarkable theorem that was to be proved.

87.

NOTE ON THE GRAPHICAL METHOD IN PARTITIONS.

[*Johns Hopkins University Circulars,* II. (1883), pp. 70, 71.]

It is well I think to draw attention to the fact that the graphical method introduces two new processes into Arithmetic as elementary and fundamental as those contained in the well-known four rules—which may be called Transversion and Apocopation.

Transversion is the operation of passing from a partition to its conjugate or transverse, and is identical with that which borrowing from the vernacular of the American Stock Exchange I have elsewhere denominated "calling."

The elements of a partition may be regarded as *Sellers* each holding a certain number of shares in the same stock. On the numbers 1, 2, 3 ... being successively called out each seller who holds at least that number of shares declares himself, and the number of those so responding each time being set down, a new partition is formed with numbers whose sum is identical with the total number of shares on sale.

The discovery of this process is due to Dr Ferrers, who informs me that he himself never published it but left it to me to do so in his name in the *London and Edinburgh Philosophical Magazine* for 1853*. I may mention that I have never missed an opportunity of expressing my sense of the great importance of the discovery and bringing it under the notice of my pupils, to one of whom, Mr Durfee (Fellow of this University), is due the discovery (after the lapse of 30 years) which leads to the second process, namely, Apocopation, which institutes a fixed relation between any partition and its transverse.

Apocopation is a process applied to a partition whose parts are arranged in descending order and consists in cutting off from its beginning, all those terms whose magnitude exceeds the number which denotes their place (reckoning from the highest term) in the arrangement. We have then

[* Vol. I. of this Reprint, p. 597; Vol. II., p. 120.]

this important theorem—*The number of terms subject to apocopation is the same for any partition and its transverse.*

Scaling or *co-summating* a partition consists in adding together each-to-each the apocopated terms in a partition and in its transverse, and diminishing these sums by the several numbers 1, 3, 5 .... In this way a new partition is formed which may be called the associate-sum of the original partition, so that to every partition there is a transverse and an associate-sum; and the content of each of these three partitions is identical.

The process of *scaling* or co-summation may be indefinitely continued and it is a curious question to determine how often the scaling process must be continued in order for a given partition to be eventually converted into a single term after which of course it remains unaltered by any further application of the process—this problem is naturally suggested by the practice of scaling and rescaling an inconveniently large public debt which is sometimes practised in the Old World and is not unknown in the New; but the analogy fails in this respect that in the one case the amount of the debt has a tendency to converge to zero, whereas in the other the content of the partition remains constant throughout.

The passage from a partition into odd numbers to the corresponding partition into unequal numbers, is effected by a co-summation operated simultaneously but independently upon two partitions, one of which has for its parts the major-halves and the other the minor-halves of the parts of the given partition.

88.

AN INSTANTANEOUS GRAPHICAL PROOF OF EULER'S THEOREM ON THE PARTITIONS OF PENTAGONAL AND NON-PENTAGONAL NUMBERS.

[*Johns Hopkins University Circulars*, II. (1883), p. 71.]

I START with the product

$$(1 + ax)(1 + ax^2)(1 + ax^3) \dots ;$$

the coefficient of $x^n a^j$ in its development in a series according to powers of x and a is the number of partitions of n into j unequal parts; each such partition may be represented by a regular graph and these graphs classified according to the magnitude of the Durfee-square which they contain. Calling the side of any such square θ, two cases arise, namely, the vertical side of the square may either be completely covered or one point in it be left exposed: in the former case any number of the points in the base of the square, in the latter case not more than the first $\theta - 1$ points can be covered.

The first case contributes to the total number of partitions of n into j unequal parts the number of ways of distributing $n - \theta^2$ between two groups, one consisting of θ unequal parts unlimited, the other of j unequal parts not exceeding θ in magnitude.

The second case contributes the number of ways of distributing $n - \theta^2$ between two groups consisting one of $\theta - 1$ unequal parts unlimited, the other of $j - \theta$ unequal parts not exceeding $\theta - 1$ in magnitude.

Hence remembering that the number of ways of partitioning any number ν into θ parts is the coefficient of x^ν in

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1 - x \,.\, 1 - x^2 \dots},$$

it is easily seen to follow that

$$(1+ax)(1+ax^2)(1+ax^3)\ldots$$

must be equal to the sum of the two series

$$1+\frac{1+xa}{1-x}x^2a\ldots+\frac{(1+xa)(1+x^2a)\ldots(1+x^\theta a)}{1-x.1-x^2\ldots1-x^\theta}x^{\theta^2+\frac{\theta^2+\theta}{2}}a^\theta+\ldots$$

and
$$xa+\ldots+\frac{1+xa.1+x^2a\ldots1+x^{\theta-1}a}{1-x.1-x^2\ldots1-x^{\theta-1}}x^{\theta^2+\frac{\theta^2-\theta}{2}}a^\theta+\ldots;$$

on making $a=-1$ there results

$$(1-x)(1-x^2)(1-x^3)\ldots=1-x-x^2\ldots+(-)^\theta\left(x^{\frac{3\theta^2-\theta}{2}}+x^{\frac{3\theta^2+\theta}{2}}\right)+\ldots$$

which is the theorem to be proved.

In the Appendix or Exodion to a forthcoming paper in the *American Journal of Mathematics* [Vol. IV. of this Reprint] I give a proof by the method of correspondence of Jacobi's generalization of the above theorem, namely:

$$(1\pm x^{n-m})(1\pm x^{n+m})(1-x^{2n})(1\pm x^{3n-m})(1\pm x^{3n+m})(1-x^{4n})\ldots$$
$$=\sum_{-\infty}^{+\infty}(\pm)^i x^{ni^2+mi}.$$

89.

ON FAREY SERIES.

[*Johns Hopkins University Circulars*, II. (1883), p. 143.]

THE ordinary Farey Series is a succession of proper vulgar fractions arranged in order of magnitude, whereof the denominator does not exceed a given amount. The theory may be generalized and simplified by considering the terms of each fraction as the coordinates of a node in a "réseau." If a *simple* and *anautotomic* closed boundary drawn on a tessellation be called a scroll, and any node within it be assumed as origin, a radius of indefinite length rotating about that point as centre, will pass through a series of nearest nodes to it in succession, all lying within the scroll—and the coupled coordinates to those successive points, say (p, q), (p', q'), ... will form a certain series which in general but not universally will satisfy the equation $pq' - p'q = 1$ or $= -1$ according as the order of magnitude is descending or ascending. The character of the series may be termed Farey if this law is satisfied throughout the entire succession and otherwise Non-Farey. The character obviously can only depend on the form, magnitude, position, and aspect of the scroll and the position of the assumed centre. The author of the paper showed that the position of the centre was indifferent except that it must be taken at some node within the scroll, and that the scroll might undergo uniform expansion and contraction about any internal node (and consequently also translation along any line of nodes) without change of character. Application of these principles was made to a triangular or rectangular scroll (including Mr Glaisher's extension of the theory of ordinary Farey Series to a two-fold constant limit), to the case potentially indicated by Dirichlet where the scroll is formed by two asymptotes to, and the branch of a hyperbola, and two other cases: the theory is deduced without the use

of continued fractions or indeterminate linear equations or any other algebraical process whatever, from the well-known fact that all elementary triangles in a reticulation are of equal area and from the not very recondite theorem that a triangle may be divided into four equal and similar triangles by straight lines joining the bisections of its three sides; and with the aid of a solid reticulation may be extended to triplets and so on indefinitely. It will be found fully set forth in Note H, interact, part 2, Vol. v., No. 4, *American Journal of Mathematics* [Vol. IV. of this Reprint].

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